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DOMINATING VERTEX COVERS: THE VERTEX-EDGE DOMINATION PROBLEM

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Abstract

The *vertex-edge domination number* of a graph, $\gamma_{ve}(G)$, is defined to be the cardinality of a smallest set D such that there exists a vertex cover C of G such that each vertex in C is dominated by a vertex in D . This is motivated by the problem of determining how many guards are needed in a graph so that a searchlight can be shone down each edge by a guard either incident to that edge or at most distance one from a vertex incident to the edge. Our main result is that for any cubic graph G with n vertices, $\gamma_{ve}(G) \leq 9n/26$. We also show that it is *NP*-hard to decide if $\gamma_{ve}(G) = \gamma(G)$ for bipartite graph G .

Keywords: cubic graph, dominating set, vertex cover, vertex-edge dominating set.

2010 Mathematics Subject Classification: 05C69.

1. INTRODUCTION

Let $G = (V, E)$ be an undirected graph with n vertices. A *dominating set* of graph G is a set $D \subseteq V$ such that for each $u \in V \setminus D$, there exists an $x \in D$ adjacent to u . A vertex u is said to *dominate* a vertex v if either $u = v$ or u is adjacent to v . The minimum cardinality amongst all dominating sets of G is the *domination number*, denoted $\gamma(G)$. A *vertex cover* of graph G is a set $C \subseteq V$ such that for each edge $uv \in E$, at least one of u, v is an element of C . The minimum cardinality amongst all vertex covers of G is the *vertex cover number*, denoted $\tau'(G)$.

A number of recent papers have studied problems associated with defending or searching a finite, undirected graph $G = (V, E)$. These problems sometimes refer to *protecting* the graph with guards. A variety of graph protection problems and models have been considered in the literature of late, see the survey [5]. In the usual protection model, each attack in a sequence of attacks is defended by a mobile guard that is sent to the attacked vertex from a neighboring vertex or, in the case when edges are attacked, by sending a guard across the attacked edge (as introduced in [4]). A dominating set can then be viewed as a static positioning of guards which protect the vertices of the graph, while a vertex cover can be viewed a static positioning of guards which protect the edges of the graph.

A number of other papers have considered so-called *searchlight* problems which, inspired by the famous art gallery problem, attempt to use searchlights to find an intruder in a graph or a polygon. See for example [2] and [12]. In this paper, we study a variation on the searchlight problem. We shall consider the problem in which the guards, each of whom holds a searchlight, must shine a searchlight down some edge (where they think there might be an intruder). The problem is formally defined below and was initially defined by Peters in [10]. The problem was also studied in [1, 7–9, 11].

We now define what one may informally think of as a vertex-cover-dominating-set, or what is called a *vertex-edge dominating set*, for simplicity. The parameter $\gamma_{ve}(G)$ is called the *vertex-edge domination number* of G (see [10]) and is defined to be equal to the cardinality of a smallest set D such that there exists a vertex cover C of G such that each vertex in C is dominated by a vertex in D . Alternatively, a set D is a vertex-edge dominating set if and only if the set of vertices not dominated by D form an independent set.

We shall say that an edge uv is *protected* if there is a guard on u, v , or any neighbor of u, v . As examples, observe that $\gamma_{ve}(P_4) = 1$ and $\gamma_{ve}(C_5) = 2$. It is clear that $\tau'(G) \geq \gamma(G) \geq \gamma_{ve}(G)$ for any graph G without isolated vertices.

Informally, we wish to place guards on the vertices of a graph so that any edge is “close” to any guard; that is, each edge is incident to a vertex with a guard or incident with a vertex adjacent to a vertex with a guard. Following the

art gallery metaphor, one may suppose that an alarm is triggered on edge uv . A guard must be able to quickly view uv to determine whether there is an intruder on the edge or a false alarm. Thus, if guards occupy the vertices of a vertex-edge dominating set and an alarm is triggered on edge uv , there is a guard nearby: on an endpoint of u, v or on a vertex adjacent to u or v . Such a guard can shine a flashlight down incident edge uv to check for an intruder or move to one of u, v and shine a flashlight down incident edge uv . As a simple example, consider the graph G shown in Figure 1 with a guard located on vertex y . Suppose an alarm is triggered on some edge e of G . If e is incident with y , the guard simply shines a flashlight down edge e . Otherwise, the guard moves to x or z and shines a flashlight down edge e .

With respect to the formal definition of the vertex-edge domination number, observe that $C = \{x, z\}$ is a vertex cover of graph G shown in Figure 1. It is clear that $D = \{y\}$ is a set of minimum cardinality such that each vertex of C is dominated by D . Thus, $\gamma_{ve}(G) = 1$.

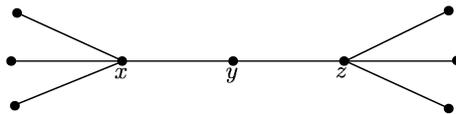


Figure 1. A graph G with $\gamma_{ve}(G) = 1$.

Upper bounds on the vertex-edge domination number of graphs of order n were presented in [1] for non-trivial connected graphs (upper bound of $\gamma_{ve}(G) \leq n/2$) and connected C_5 -free graphs (upper bound of $\gamma_{ve}(G) \leq n/3$).

In this paper, we present results on the vertex-edge domination number of some graphs. Our main result is shown in Section 2: $\gamma_{ve}(G) \leq 9n/26$ for any cubic graph G with n vertices. In Section 3, we show that it is *NP*-hard to determine whether a bipartite graph, B , satisfies $\gamma_{ve}(B) = \gamma(B)$. We start with a simple result.

Proposition 1. *Let G be a connected graph of order at least 2. Then $\gamma_{ve}(G) = \tau'(G)$ if and only if $\tau'(G) = 1$.*

Proof. As G is a connected graph of order at least 2 we have $\tau'(G) \geq 1$. If $\tau'(G) = 1$, then the proposition follows, as $\tau'(G) \geq \gamma_{ve}(G)$ for all G .

Now suppose $\tau'(G) > 1$. Let C be a minimum vertex cover of G . We construct a vertex-edge dominating set D with fewer vertices than C . Initially, let $D = C$. If any two vertices in C are adjacent, then one of them can be removed from D . So suppose no two vertices in C are adjacent. If there exist two vertices in C that are distance two apart, then these two vertices can be replaced in D by the vertex that lies on the path of length two between them. If there are no such

vertices of distance two apart in C , then it follows that the closest pair of vertices in C are distance at least three apart and thus C cannot be a vertex cover, as there is an edge on the shortest path between any two vertices in C that is not covered by any vertex in C . ■

2. CUBIC GRAPHS

Kostochka and Stocker proved that the domination number of a cubic graph with n vertices is at most $5n/14$, see [6]. There exists a cubic graph on 14 vertices where the domination number is 5, so the bound is tight. Thus, trivially, for any cubic graph G , $\gamma_{ve}(G) \leq \gamma(G) \leq 5n/14 \approx 0.35714n$. In this section, we prove our main result, that for any cubic graph G , $\gamma_{ve}(G) \leq 9n/26 \approx 0.34615n$.

In Section 2.1, we define a useful class of hypergraphs and state two useful hypergraph results. In Section 2.2, we state and prove our main result, Theorem 4.

2.1. Main result on hypergraphs from [3]

For the hypergraph H , let $n(H)$ denote the number of vertices in H , $m(H)$ denote the number of edges in H and $e_i(H)$ denote the number of edges in H of size i . For hypergraph H with the vertex set V , a smallest subset of V that contains vertices from every edge is called a *transversal* and its cardinality is denoted by $\tau(H)$.

In order to state the main result from [3], we need to define a particular class of hypergraphs \mathcal{B} . Let \mathcal{B} be the class of *bad hypergraphs* defined as exactly those that can be generated using the operations (A)–(D) below.

- (A) Let H_2 be the hypergraph with two vertices $\{x, y\}$ and one edge $\{x, y\}$ and let H_2 belong to \mathcal{B} .
- (B) Given any $B' \in \mathcal{B}$ containing a 2-edge $\{u, v\}$, define B as follows. Let $V(B) = V(B') \cup \{x, y\}$ and let $E(B) = E(B') \cup \{\{u, v, x\}, \{u, v, y\}, \{x, y\}\} \setminus \{u, v\}$. Now add B to \mathcal{B} .
- (C) Given any $B' \in \mathcal{B}$ containing a 3-edge $\{u, v, w\}$, define B as follows. Let $V(B) = V(B') \cup \{x, y\}$ and let

$$E(B) = E(B') \cup \{\{u, v, w, x\}, \{u, v, w, y\}, \{x, y\}\} \setminus \{u, v, w\}.$$

Now add B to \mathcal{B} .

- (D) Given any $B_1, B_2 \in \mathcal{B}$, such that B_i contains a 2-edge $\{u_i, v_i\}$, for $i = 1, 2$, define B as follows.

Let $V(B) = V(B_1) \cup V(B_2) \cup \{x\}$ and let $E(B) = E(B_1) \cup E(B_2) \cup \{\{u_1, v_1, x\}, \{u_2, v_2, x\}, \{u_1, v_1, u_2, v_2\}\} \setminus \{\{u_1, v_1\}, \{u_2, v_2\}\}$. Now add B to \mathcal{B} .

Definition 1. For any hypergraph H , let $b(H)$ denote the number of connected components in H that belong to \mathcal{B} . Further, let $b^1(H)$ denote the maximum number of vertex disjoint subhypergraphs in H which are isomorphic to hypergraphs in \mathcal{B} and which are intersected by exactly one other edge in H .

Theorem 2 [3]. *If H is a hypergraph whose all edges have size 2, 3, or 4, and $\Delta(H) \leq 3$, then*

$$24\tau(H) \leq 6n(H) + 4e_4(H) + 6e_3(H) + 10e_2(H) + 2b(H) + b^1(H).$$

Using Theorem 2, we can prove the following result, which is implicit in [3]; therefore we include a short proof for completeness.

Theorem 3. *Let H be a hypergraph whose all edges have size 3 or 4, and $\Delta(H) \leq 3$ and every 4-edge contains a vertex that does not belong to any 3-edge. Then $12\tau(H) \leq 3n(H) + 2e_4(H) + 3e_3(H)$.*

Proof. Assume that $R \in \mathcal{B}$ and that R contains no 2-edge. In this case we note that the last operation carried out in the construction of R is operation (D) (see Subsection 2.1), as operations (A)–(C) all create 2-edges. Therefore there exist five vertices $\{u_1, v_1, u_2, v_2, x\}$ in R where $\{\{u_1, v_1, x\}, \{u_2, v_2, x\}, \{u_1, v_1, u_2, v_2\}\} \subseteq E(R)$. However then R is not a subgraph of H as the edge $\{u_1, v_1, u_2, v_2\}$ contains no vertex that does not belong to a 3-edge. Therefore $b(H) = b^1(H) = 0$ and by Theorem 2 we have the following, which completes the proof of the theorem.

$$\begin{aligned} 24\tau(H) &\leq 6n(H) + 4e_4(H) + 6e_3(H) + 10e_2(H) + 2b(H) + b^1(H) \\ &= 6n(H) + 4e_4(H) + 6e_3(H). \quad \blacksquare \end{aligned}$$

2.2. Upper bound for cubic graphs

The bound that we shall present in Theorem 4 cannot be improved to anything below $n/3$, due to the graph in Figure 2. We leave it as an open problem to either find larger connected cubic graphs with $\gamma_{ve}(G) = n/3$ or show that the graph in Figure 2 is the only one; for instance, it does not appear easy to combine copies of the graph in Figure 2 in some way to arrive at another such example.

The *open neighborhood* of a vertex $v \in V(G)$ is $N(v) = \{u \in V \mid uv \in E(G)\}$ and its *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$.

Theorem 4. *If G is a cubic graph, then $\gamma_{ve}(G) \leq 9n/26$.*

Proof. Let S be a maximal independent set in G and assume that $|S| = (5/14 - \varepsilon_1)n$, where $n = |V(G)|$ (ε_1 may be positive or negative). Let T be the set of all vertices in $\bar{S} = V(G) \setminus S$ that have exactly one neighbor in S and let $\varepsilon_2 = (|S| - |T|)/n$. We will now prove the following two claims.

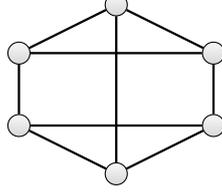


Figure 2. A 6-vertex cubic graph with vertex-edge domination number equal to $n/3$.

Claim A. $\gamma_{ve}(G) \leq |S| - \varepsilon_2 n/4 = \left(\frac{5}{14} - \varepsilon_1 - \frac{\varepsilon_2}{4}\right) n$.

Proof. Let U be a maximal subset of S such that $S \setminus U$ dominates \bar{S} . As \bar{S} is a vertex cover of G , we note that $\gamma_{ve}(G) \leq |S \setminus U|$.

We will now show that $|U| \geq \varepsilon_2 n/4$, which will complete the proof of Claim A. Clearly this is true if $\varepsilon_2 \leq 0$, so assume that $\varepsilon_2 > 0$. For the sake of contradiction assume that $|U| < \varepsilon_2 n/4$ and let T' be the set of all vertices not in T that have a unique neighbor in $S \setminus U$; note that $T' \subseteq N(U)$. As G is cubic, we must have $|T'| \leq 3|U|$, which implies the following inequality.

$$|S \setminus U| = |S| - |U| \geq (|T| + \varepsilon_2 n) - |U| > (|T| + 4|U|) - |U| = |T| + 3|U| \geq |T| + |T'|.$$

As $|T| + |T'| < |S \setminus U|$, we note that some vertex in $s \in S \setminus U$ is not adjacent to a vertex in $T \cup T'$ (as each vertex in $T \cup T'$ is adjacent to at most one vertex in $S \setminus U$). This is a contradiction to the maximality of U , as s could have been added to U . This completes the proof of Claim A. \square

Claim B. $\frac{12}{14} \gamma_{ve}(G) \leq \left(\frac{4}{14} + \varepsilon_1 + \frac{2\varepsilon_2}{14}\right) n$.

Proof. We will first construct a 4-uniform hypergraph H as follows. Let $V(H) = V(G)$ and for every vertex $s \in \bar{S}$ add $N_G[s]$ as a hyperedge in H . This completes the definition of H . As G is cubic, we note that H is 4-uniform with $n = |V(G)|$ vertices and $m_H = |\bar{S}|$ edges.

Note that $\Delta(H) \leq 3$ as for all $x \in V(G)$ at least one vertex in $N[x]$ belongs to S and therefore at most three vertices from $N[x]$ belongs to \bar{S} (which are the vertices that give rise to edges containing x). Furthermore, no 4-edge in H has all its vertices in S .

Let $Q_1 \subseteq V(H)$ be all degree one vertices in H . Note that every vertex in Q_1 belongs to \bar{S} and it has all its neighbors in S . Let H' be the hypergraph obtained from H by deleting all vertices in Q_1 (by deleting a vertex v , we mean deleting v and shrinking every edge, e , containing v such that it contains the vertex set $V(e) \setminus \{v\}$ instead of $V(e)$). Note that all edges in H' have size three or four and if e is a 3-edge, then all vertices in e belong to S . As no 4-edge is completely contained in S we note that every 4-edge contains a vertex (in \bar{S}) which does not

belong to any 3-edge. Therefore the following holds by Theorem 3.

$$12\tau(H') \leq 3n(H') + 2e_4(H') + 3e_3(H') \leq 3(n - |Q_1|) + 2(m_H - |Q_1|) + 3|Q_1|.$$

Next, as $m_H = n - |S|$, this implies the following

$$12\tau(H') \leq 5n - 2|S| - 2|Q_1|.$$

We will first show that $\gamma_{ve}(G) \leq \tau(H')$ and then evaluate $5n - 2|S| - 2|Q_1|$. Let R be a transversal in H' with $|R| = \tau(H')$. As R contains a vertex from $N[y]$ for all $y \in \bar{S}$, we note that R dominates all vertices in \bar{S} . As \bar{S} is a vertex cover of G , we get that $\gamma_{ve}(G) \leq |R| = \tau(H')$ as desired.

We will now evaluate $5n - 2|S| - 2|Q_1|$. Let Q_2 be the vertices in \bar{S} of degree 2 in H and let Q_3 be the vertices in \bar{S} of degree 3 in H . In G the vertices in Q_1 have 3 neighbors in S , the vertices in Q_2 have 2 neighbors in S , and the vertices in Q_3 have 1 neighbor in S . By double counting the number of edges between S and \bar{S} we get the following

$$3|S| = 3|Q_1| + 2|Q_2| + 1|Q_3|.$$

Recall that $Q_3 = T$ and $|S| - |T| = \varepsilon_2 n$ (and therefore $|S| - \varepsilon_2 n = |T|$), and thus we obtain the following

$$3|S| = 3|Q_1| + 2|Q_2| + (|S| - \varepsilon_2 n).$$

As $Q_1 \cup Q_2 = \bar{S} \setminus T$ we also note that the following holds

$$|Q_1| + |Q_2| = |\bar{S}| - |T| \leq (n - |S|) - (|S| - \varepsilon_2 n).$$

Next, the above two equations can be rewritten as follows

$$3|Q_1| + 2|Q_2| = 2|S| + \varepsilon_2 n \quad 2|Q_1| + 2|Q_2| = 2n - 4|S| + 2\varepsilon_2 n.$$

Subtracting the second equation from the first, one obtains the following

$$|Q_1| = 6|S| - 2n - \varepsilon_2 n.$$

Now since $|S| = (5n/14 - \varepsilon_1)$, we get the following equality

$$\begin{aligned} 5n - 2|S| - 2|Q_1| &= 5n - 2|S| - 2(6|S| - 2n - \varepsilon_2 n) = 9n - 14|S| + 2\varepsilon_2 n \\ &= 9n - 14(5/14 - \varepsilon_1)n + 2\varepsilon_2 n = n(4 + 14\varepsilon_1 + 2\varepsilon_2). \end{aligned}$$

Therefore $12\tau(H') \leq n(4 + 14\varepsilon_1 + 2\varepsilon_2)$, which completes the proof of Claim B (by dividing both sides by 14). \square

Adding the results in Claim A and Claim B, we get the following inequality

$$\gamma_{ve}(G) + \frac{12}{14}\gamma_{ve}(G) \leq \left(\frac{5}{14} - \varepsilon_1 - \frac{\varepsilon_2}{4}\right)n + \left(\frac{4}{14} + \varepsilon_1 + \frac{2\varepsilon_2}{14}\right)n$$

which implies

$$\frac{26}{14}\gamma_{ve}(G) \leq \left(\frac{9}{14} - \frac{7\varepsilon_2 - 4\varepsilon_2}{28}\right)n.$$

Therefore if $\varepsilon_2 \geq 0$, then we have $\gamma_{ve}(G) \leq 9n/26$, as desired. If $\varepsilon_2 < 0$, then we note that S is a dominating set in G and therefore $\gamma_{ve}(G) \leq |S| = (5/14 - \varepsilon_1)n$. Combining this with Claim B results in the following inequality

$$\gamma_{ve}(G) + \frac{12}{14}\gamma_{ve}(G) \leq \left(\frac{5}{14} - \varepsilon_1\right)n + \left(\frac{4}{14} + \varepsilon_1 + \frac{2\varepsilon_2}{14}\right)n.$$

Analogously to above this implies the following

$$\frac{26}{14}\gamma_{ve}(G) \leq \left(\frac{9}{14} + \frac{2\varepsilon_2}{28}\right)n.$$

This again implies $\gamma_{ve}(G) \leq 9n/26$, as desired. ■

Following the example shown in Figure 2, we leave open the following question.

Question 1. *Is it true that for any cubic graph G of order n , $\gamma_{ve}(G) \leq n/3$?*

In fact, a stronger open problem was stated in [1]. Namely, is it true that $\gamma_{ve}(G) \leq n/3$ for all connected graphs of order $n \geq 6$?

3. NP-HARDNESS

Recall that a support vertex in a tree is a vertex that is adjacent to a leaf in the tree. The trees, T , satisfying $\gamma_{ve}(T) = \gamma(T)$ were characterized by Theorem 32 of [9]. This result states that $\gamma_{ve}(T) = \gamma(T)$ if and only if T has an efficient dominating set S such that each vertex of S is a support vertex of T . A simple corollary of the result in [9] is the following.

Corollary 5. *We can decide if $\gamma_{ve}(T) = \gamma(T)$ in polynomial time for all trees T .*

We now consider the case when we want to decide whether $\gamma_{ve}(G) = \gamma(G)$ for bipartite graphs G .

Theorem 6. *It is NP-hard to decide whether $\gamma_{ve}(G) = \gamma(G)$ for a bipartite graph G .*

Proof. Recall that if $H = (V, E)$ is a hypergraph, then we denote the cardinality of a smallest subset of V that contains vertices from every edge (called a *transversal*) by $\tau(H)$.

We will reduce from the *NP*-hard problem of deciding whether a 3-uniform hypergraph, H , has a transversal of size at most k . That is, the hypergraph $H = (V, E)$, where V is the vertex set of H and each edge $e \in E$ is a set containing three vertices. We then want to decide whether there is a subset, $X \subseteq V$, of size at most k that contains at least one vertex of every edge of H .

The idea is to construct a graph, G , such that $\gamma_{ve}(G) < \gamma(G)$ if and only if $\tau(H) \leq k$. Start the construction of graph G with vertex set V . To this, for each edge $e \in E$, we add the vertex set $V_e = \{v_i^e \mid i = 1, 2, \dots, k\}$ and the edges from each vertex in V_e to the three vertices in V that belong to e . Then we add the vertices $W = \{w_1, w_2, \dots, w_k\}$ and for all $i = 1, 2, \dots, k$ add all edges from w_i to v_i^e for all $e \in E$. Finally, we add the two new vertices x and y and all edges from x to $V \cup \{y\}$. This completes the construction of G .

We will show that $\tau(H) \leq k$ if and only if $\gamma_{ve}(G) < \gamma(G)$. Let $S_i = w_i \cup \{v_i^e \mid e \in E\}$. Note that $\gamma(G) = k + 1$, as any dominating set in G must contain at least one vertex from each S_i (in order to dominate w_i) and a vertex from $\{x, y\}$ (in order to dominate y) and $W \cup \{x\}$ is a dominating set in G .

If $\tau(H) \leq k$, then let T be a transversal of H of size $\tau(H)$. Note that $T \subseteq V$ and T is a vertex-edge-dominating set in G (as the only vertices not dominated by T in G are $\{w_1, w_2, \dots, w_k, y\}$ which form an independent set). Therefore $\gamma_{ve}(G) \leq |T| \leq k < k + 1 = \gamma(G)$.

Now assume that $\gamma_{ve}(G) < \gamma(G)$. For the sake of contradiction assume that $\tau(H) > k$. Let Q be a vertex-edge dominating set in G of size $\gamma_{ve}(G)$. As $|Q| = \gamma_{ve}(G) < \gamma(G) = k + 1$ and $\tau(H) > k$ we note that $Q \cap V$ is not a transversal in H . Therefore some edge $e \in E$ is not covered by $Q \cap V$. Due to the edge $w_i v_i^e$, we note that Q must contain at least one vertex from each S_i , $i = 1, 2, \dots, k$. As $|Q| \leq k$, we therefore note that the edge xy is not covered by Q , a contradiction. Therefore $\tau(H) > k$, which completes the proof. ■

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