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# Relativistic Massive Vector Condensation

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At sufficiently high chemical potential massive relativistic spin one fields condense. This phenomenon leads to the spontaneous breaking of rotational invariance while linking it to the breaking of internal symmetries. We study the relevant features of the phase transition and the properties of the generated Goldstone excitations. The interplay between the internal symmetry of the vector fields and their Lorentz properties is studied. We predict that new phases set in when vectors condense in Quantum Chromodynamics like theories.

## I. INTRODUCTION

The Quantum chromodynamics (QCD) phase diagram as a function of temperature, chemical potential and the number of light flavors is very rich and highly structured. QCD should behave as a color superconductor for a sufficiently large quark chemical potential [1]. This leads to new phenomenological applications associated with the description of quark stars, neutron star interiors and the physics near the core of collapsing stars [1–3]. Much is known about the phase structure of QCD at nonzero temperature through a combination of perturbation theory and lattice simulations while the phase structure at nonzero chemical potential and for large numbers of flavors has been less extensively explored [4]. Standard importance sampling methods employed in lattice simulations fail at nonzero chemical potential for  $N_c = 3$  since the fermionic determinant is complex. For  $N_c = 2$ , the situation is very different since the quarks are in a pseudoreal representation of the gauge group and lattice simulations can be performed [5–19].

At non zero chemical potential Lorentz invariance is explicitly broken down to the rotational subgroup  $SO(3)$  and higher spin fields can condense, thus potentially enriching the phase diagram structure of QCD and QCD-like theories. Indeed in [20] it has been suggested that some of the lightest massive vectors at non zero baryon chemical potential may condense. Recent lattice simulations seem to support these predictions [21]. In the context of the Electroweak theory vector condensation in the presence of a strong external magnetic field has also been studied in [22].

In this paper we explore the condensation of relativistic massive vectors when introducing a non zero chemical potential via an effective Lagrangian approach. In order for the vectors to couple to the chemical potential they must transform under a given global symmetry group. In view of the possible physical applications we consider the vectors to belong to the adjoint representation of the  $SU(2)$  group.

To set the stage for the more complicated case of the vector field we first consider a scalar field in the adjoint representation of  $SU(2)$  at non zero chemical potential. We choose the chemical potential to lie along one of the generator's directions explicitly breaking  $SU(2)$  to  $U(1) \times Z_2$ . We then investigate the dispersion relations of the 3 fields as function of the chemical potential and show the existence of a second order phase transition when the chemical potential equals the zero density mass of the scalar field. In the broken phase we observe the emergence of one gapless mode associated to the spontaneous breaking of the  $U(1)$  symmetry possessing linear dispersion relations. This is consistent with the Chadha-Nielsen [23] counting rule for Goldstone's bosons when Lorentz invariance is not assumed.

The situation becomes more involved when turning to the spin-one massive field case. When the vector field condenses the rotational  $SO(3)$  invariance breaks spontaneously together with part of the flavor (i.e. global) symmetry of the theory. The condensate locks together space and flavor symmetries. Following the standard mean-field approach we identify our order parameter with the vector field itself.

In order to elucidate the mechanism we construct the simplest effective Lagrangian at zero chemical potential. This consists of a standard kinetic type term and a tree level potential constructed as a series expansion in the order parameter. We truncate our potential at the fourth order in the fields while retaining all the terms allowed by Lorentz

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and  $SU(2)$  symmetry. The introduction of the potential term allows us to describe the phase transition in some detail while our results will be general.

We prove that the number of gapless states emerging when the flavor $\times$ rotational symmetry breaks spontaneously is insensitive to the choice of the coefficients in the effective potential. However the momentum dependence of the associated dispersion relations is heavily affected by the choice of the potential. The latter is related to the differences between the symmetries of the potential with respect to the ones of the kinetic type terms. The full vector Lagrangian (i.e. kinetic – potential) always possesses the  $Z_2 \times U(1) \times SO(3)$  (flavor $\times$ rotational) symmetry which at high enough chemical potential breaks spontaneously to  $Z_2 \times SO(2)$ . The gapless modes are linked to the 3 broken generators. One state is an  $SO(2)$  scalar while the other two states constitute a vector of  $SO(2)$ . The scalar Goldstone possesses linear dispersion relations independently of the choice of the vector potential and it resembles the gapless mode of the scalar theory. However the  $SO(2)$  vector state can have either quadratic or linear dispersion relations. In particular (as summarized in the Table I) the quadratic ones occur when the potential parameters are such that it possesses an enhanced  $SO(6)$  symmetry. When condensing the vector breaks the  $SO(6)$  symmetry of the potential down to  $SO(5)$  while the kinetic-type term still has only a  $U(1) \times SO(3)$  invariance. In this case the potential has 5 flat directions (i.e. 5 null curvatures) and we would count in the spectrum 2-gapless vectors and an  $SO(2)$  scalar. However the reduced symmetry of the kinetic term prevents the emergence of two independent gapless vectors while turning the linear dispersion relations of one vector state into quadratic ones. This result is also in agreement with the Chadha-Nielsen counting scheme. However since the Goldstone modes with quadratic dispersion relations must be counted twice relatively to the ones with linear dispersion relations we saturate the number of broken generators associated with the symmetries of just the potential term which is larger than the ones associated with the symmetries of the full Lagrangian. It is certainly important to investigate, in the future, if the choice of the parameter space leading to an  $SO(6)$  symmetry for the potential term is stable against quantum corrections.

Our model can be related immediately to the physics of 2 colors QCD which has been partially analyzed in [20] and is within reach of present lattice studies. Here we suggest that vector condensation can be the leading mechanism for a superfluid transition for 2 color QCD with one flavor. Indeed in this case the global symmetry is  $SU(2)$  which remains intact at zero chemical potential and no Goldstone bosons emerge. The massive spectrum of the theory still contains an  $SU(2)$  spin-one massive multiplet. When turning on a non zero chemical baryon potential the vector condensation can lead to a novel superfluid phase transition. Clearly our model can also be applied to QCD with 3 color at non zero isospin chemical potential [24].

The paper is structured as follows. In section II we first consider a toy model for a scalar field in the regular representation of  $SU(2)$  at non zero chemical potential and investigate all of the relevant features of the model. We then propose a simple effective Lagrangian for vectors which allows us to uncover the main features related to the condensation of relativistic massive spin-one fields. We conclude and point out some physical applications in section III. In particular we suggest that new phases should set in for QCD like theories. Finally we extend our simple model to  $D - 1$  number of space dimensions (with Euclidean  $SO(D)$  Lorentz symmetry).

## II. VECTOR CONDENSATION AT NONZERO CHEMICAL POTENTIAL

### A. Toy Model: Scalar Theory

In order to illustrate and disentangle the main problems related to the vector condensation we first consider the case of a scalar field in the adjoint representation of an  $SU(2)$  global symmetry. This will allow us to show the feature which only depends on the global symmetries first. The toy model Lagrangian is:

$$\mathcal{L} = \text{Tr} [\partial_\mu \phi \partial^\mu \phi] - m^2 \text{Tr} [\phi^2] - \lambda (\text{Tr} [\phi^2])^2, \quad (1)$$

with  $\phi = \phi^a T^a$  and  $a = 1, 2, 3$  and  $T^a = \tau^a/2$ .  $\phi^a$  is a real field and  $\tau^a$  the standard Pauli matrices.

We introduce the chemical potential associated with the  $T^3$  generator. We shall see that our choice turns out to be particularly relevant when considering 2 colors QCD and one flavor at non zero quark chemical potential. The effects of the chemical potential are included by generalizing the covariant derivative as follow:

$$D_\mu \phi = \partial_\mu \phi - i\mu \delta_{\mu 0} [T^3, \phi]. \quad (2)$$

$\mu$  is the chemical potential. Expanding the kinetic term we have that the Lagrangian reads in components  $\phi = (\varphi^\pm = (\phi^1 \mp i\phi^2)/\sqrt{2}, \psi = \phi^3)$ :

$$\mathcal{L} = \frac{1}{2}(\partial\psi)^2 + |\partial\varphi|^2 + \mu \varphi^{(+)*} i\partial_0 \varphi^{(+)} - \mu \varphi^{(-)*} i\partial_0 \varphi^{(-)} - (m^2 - \mu^2) |\varphi|^2 - \frac{m^2}{2} \psi^2 - \frac{\lambda}{4} (|\varphi|^2 + \psi^2)^2. \quad (3)$$

Our choice of the chemical potential explicitly breaks the  $SU(2)$  'flavor'-symmetry to a  $U(1)$  acting on the complex field  $\phi$  and a discrete  $Z_2$  symmetry for the neutral field  $\psi$ . For  $\mu \leq m$ , the model has a unique vacuum at  $|\phi| = \psi = 0$  and the effect of the chemical potential is simply to split the energies of the charged states as follows:

$$E_{\varphi^\pm}(\vec{p}) = \mp\mu + \sqrt{m^2 + \vec{p}^2}, \quad E_\psi(\vec{p}) = \sqrt{m^2 + \vec{p}^2}. \quad (4)$$

The mass-gaps are defined as the energy of each state evaluated at zero momentum. Expanding the dispersion relations for small momenta gives:

$$E_{\varphi^\pm}(\vec{p}) = (m \mp \mu) + \frac{\vec{p}^2}{2m} + \mathcal{O}(\vec{p}^4). \quad (5)$$

This leads to the presence of a gapless excitation with a *nonrelativistic* type of dispersion relation at  $\mu = m$ . Note that due to the loss of Lorentz-covariance the mass-gaps do not coincide with the zero density masses. The latter are defined as the curvature of the potential evaluated on the vacuum and are  $m_{\varphi^\pm}^2 = m^2 - \mu^2$ ,  $m_\psi^2 = m^2$ . Since the  $\psi$  field is not affected at all by the chemical potential it does not condense and its mass-gap is  $m_\psi = m$  for  $\mu \leq m$ .

For  $\mu > m$  the classical potential is unstable when  $\phi = 0$  and we choose the global minimum for

$$\langle \varphi^1 \rangle = \sqrt{\frac{\mu^2 - m^2}{\lambda}}, \quad (6)$$

breaking the  $U(1) \times Z_2$  symmetry to  $Z_2$ . Computing the curvature of the potential on the new vacuum for all the fields yields:

$$m_\psi^2 = \mu^2, \quad m_{\phi^1}^2 = 2(\mu^2 - m^2), \quad m_{\phi^2}^2 = 0. \quad (7)$$

Note that in this phase the  $\psi$  mass is given precisely by the chemical potential. This is so since  $\psi$  is not charged under the chemical potential. After diagonalizing the quadratic terms we obtain the following dispersion relations:

$$E_\pm(\vec{p}) = \sqrt{\vec{p}^2 + \bar{\mu}^2 \pm \sqrt{\bar{\mu}^4 + 4\vec{p}^2\bar{\mu}^2}}, \quad (8a)$$

with  $\bar{\mu}^2 = 3\mu^2 - m^2$ . Note that with our choice of the vev  $\varphi^\pm$  are no longer eigenstates. We see that a gapless mode persists and the massive one has the mass gap  $\sqrt{2}\bar{\mu}$ . At small momenta the dispersion relation of the gapless (Goldstone-) mode is

$$E_-^2(\vec{p}) \approx \frac{\mu^2 - m^2}{\bar{\mu}^2} |\vec{p}|^2 + \frac{2\mu^4}{\bar{\mu}^6} |\vec{p}|^4. \quad (9a)$$

Hence in the broken phase we obtain the standard superfluid Goldstone boson with an energy depending *linearly* on the momentum,  $E_- \sim v_s(\mu) |\vec{p}|$ . However the *velocity*  $v_s(\mu)$  vanishes at the transition point  $v_s(\mu = m) = 0$  turning the dispersion relation of the gapless excitation into quadratic ones (i.e.  $E_- \sim |\vec{p}|^2/2m$ ). This behavior can be better understood by rewriting the velocity as function of  $m_{\phi^1}^2$  as follows:

$$v_s^2 = \frac{m_{\phi^1}^2}{m_{\phi^1}^2 + 4\mu^2}. \quad (10)$$

We observe that the superfluid velocity depends on the curvature of the potential in the direction orthogonal to the gapless excitation. At the second order phase transition point this curvature is zero, reflecting the conformal invariance of the potential for the  $\varphi$ -fields. In the absence of Lorentz breaking we have two massless modes at this point both with linear dispersion relations. At nonzero chemical potential only one mode is gapless while the information about the conformal invariance is now inherited by the quadratic dispersion law. We shall see in the next section, that this behavior is quite general, i.e. whenever the potential has more symmetries than the derivative terms, this information is encoded nontrivially into the momentum dependence of the dispersion relations. Furthermore we can now smoothly connect the dispersion relations of each state when traversing the different regions of the phase diagram. Our result is in agreement with the Chada-Nielsen [23] counting scheme as well as with recent studies related to the physics of color superconductivity [25,26].

## B. The Vector Story

There are different ways to introduce vector fields at the level of the effective Lagrangian (for example the hidden local gauge symmetry of Ref. [27], or the antisymmetric tensor field of Ref. [28]) and they are all equivalent at tree-level. We choose to introduce the massive vector fields following the method outlined in [29–32]. This method also allows a straightforward generalization of our model to an arbitrary number of space dimensions. We adopt the following model Lagrangian for describing relativistic spin one fields (in  $3 + 1$  dimensions) belonging to the regular representation of  $SU(2)$

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + \frac{m^2}{2}A_\mu^a A^{a\mu} - \frac{\lambda}{4}(A_\mu^a A^{a\mu})^2 + \frac{\lambda'}{4}(A_\mu^a A^{a\nu})^2, \quad (11)$$

with  $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a$ ,  $a = 1, 2, 3$  and metric convention  $\eta^{\mu\nu} = \text{diag}(+, -, -, -)$ . Here  $m^2$  is the tree level mass term and  $\lambda$  and  $\lambda'$  are positive dimensionless coefficients with  $\lambda > \lambda'$ . Other possible choices of the parameters do not guarantee stability of the potential and will not be considered in the following. This Lagrangian is not renormalizable, contrary to the scalar theory presented above. However it possesses the same global symmetries as the toy model.

The effect of a nonzero chemical potential associated to a given conserved charge - related to the generator (say  $B$ ) - can be readily included [20] by modifying the derivatives acting on the vector fields:

$$\partial_\nu A_\rho \rightarrow \partial_\nu A_\rho - i[B_\nu, A_\rho], \quad (12)$$

with  $B_\nu = \mu \delta_{\nu 0} B \equiv V_\nu B$  where  $V = (\mu, \vec{0})$ . The vector kinetic term modifies according to:

$$\text{Tr}[F_{\rho\nu} F^{\rho\nu}] \rightarrow \text{Tr}[F_{\rho\nu} F^{\rho\nu}] - 4i\text{Tr}[F_{\rho\nu}[B^\rho, A^\nu]] - 2\text{Tr}[[B_\rho, A_\nu][B^\rho, A^\nu] - [B_\rho, A_\nu][B^\nu, A^\rho]]. \quad (13)$$

The terms due to the kinetic term, after integration by parts, yields [20]

$$\mathcal{L}_{kinetic} = \frac{1}{2}A_\rho^a \left\{ \delta_{ab}[g^{\rho\nu}\square - \partial^\rho\partial^\nu] - 4i\gamma_{ab}\left[g^{\rho\nu}V \cdot \partial - \frac{V^\rho\partial^\nu + V^\nu\partial^\rho}{2}\right] + 2\chi_{ab}[V \cdot V g^{\rho\nu} - V^\rho V^\nu] \right\} A_\nu^b \quad (14)$$

with

$$\gamma_{ab} = \text{Tr}[T^a[B, T^b]], \quad \chi_{ab} = \text{Tr}[[B, T^a][B, T^b]]. \quad (15)$$

For  $B = T^3$  we have

$$\gamma_{ab} = -\frac{i}{2}\epsilon^{ab3}, \quad \chi_{11} = \chi_{22} = -\frac{1}{2}, \quad \chi_{33} = 0. \quad (16)$$

The chemical potential induces a ‘‘magnetic-type’’ mass term for the vectors at tree-level. The similarities with respect to the scalar toy model as well as the symmetries of the vect Lagrangian are more easily understood using the following Euclidean notation:

$$\varphi_M^a = (A_M^1, A_M^2), \quad \psi_M = A_M^3, \quad (17)$$

with  $A_M = (iA_0, \vec{A})$  and metric signature  $(+, +, +, +)$ . In these variables the potential reads:

$$V_{vector} = \frac{m^2}{2}[|\vec{\varphi}_0|^2 + \psi_M^2] + \frac{m^2 - \mu^2}{2}|\vec{\varphi}_I|^2 + \frac{\lambda}{4}[|\vec{\varphi}_M|^2 + \psi_M^2]^2 - \frac{\lambda'}{4}[\vec{\varphi}_M \cdot \vec{\varphi}_N + \psi_M \psi_N]^2 \quad (18)$$

with  $I = X, Y, Z$  while  $M, N = 0, X, Y, Z$  and repeated indices are summed over. At zero chemical potential  $V_{vector}$  is invariant under the  $SO(4)$  Lorentz transformations while only the  $SO(3)$  symmetry is manifest at non zero  $\mu$ . Clearly the  $SU(2)$  symmetry is also explicitly broken, at non zero chemical potential, to  $U(1)$  as for the scalar model.

For the reader’s convenience we summarize the quadratic term in the fields Lagrangian in the new variables.

$$\begin{aligned} \mathcal{L}_{kinetic} = & \frac{1}{2}\psi_M[\delta_{MN}\square_E - \partial_M\partial_N]\psi_N + \frac{1}{2}\varphi_M^a[\delta_{MN}\square_E - \partial_M\partial_N]\varphi_N^a - \varphi_M^a\epsilon^{ab3}\left[\delta_{MN}V \cdot \partial - \frac{V_M\partial_N + V_N\partial_M}{2}\right]\varphi_N^b \\ & - \varphi_M^a[V \cdot V\delta_{MN} - V_M V_N]\varphi_N^a \end{aligned} \quad (19)$$

According to the value of the chemical potential we distinguish two phases.

1. *The Symmetric Phase:  $\mu \leq m$*

Here the  $SO(4)$  Lorentz and  $SU(2)$  symmetries are explicitly broken to  $SO(3)$  and  $U(1)$  respectively by the presence of the chemical potential but no condensation happens (i.e.  $\langle \vec{\varphi}_M \rangle = \langle \psi_M \rangle = 0$ ). The masses at zero chemical potential are:

$$M_{\varphi_0^a}^2 = M_{\psi_M}^2 = m^2, \quad M_{\varphi_i^a}^2 = m^2 - \mu^2. \quad (20)$$

The dispersion relations obtained by diagonalizing the 12 (4 space-time  $\times$  3  $SU(2)$  states) by 12 quadratic eigenvalue problem leads to 3 physical vectors (i.e. each of the following states has 3 components) with the following dispersion relations:

$$E_{\varphi_{\mp}} = \pm\mu + \sqrt{\vec{p}^2 + m^2}, \quad E_{\psi} = \sqrt{\vec{p}^2 + m^2}. \quad (21)$$

This shows that when approaching  $\mu = m$  the 3 physical components associated with  $E_{\varphi^-}$  will become massless signaling an instability. Indeed we now show that for higher values of the chemical potential a vector type condensation sets in.

2. *The Spin-Flavor Broken Phase:  $\mu > M$*

In this phase the global minimum of the potential is for  $\langle \varphi_0^a \rangle = \langle \psi_M \rangle = 0$ , while we can choose the vev to lie in the (Spin-Flavor) direction:

$$\langle \varphi_X^1 \rangle = \sqrt{\frac{\mu^2 - m^2}{\lambda - \lambda'}}. \quad (22)$$

We have a manifold of equivalent vacua which are obtained rotating the chosen one under a  $Z_2 \times U(1) \times SO(3)$  transformation. The choice of the vacuum partially locks together the Lorentz group and the internal symmetry while leaving unbroken only the subgroup  $Z_2 \times SO(2)$ . Two generators associated to the Lorentz rotations are now spontaneously broken together with the  $U(1)$  generator.

To compute the full dispersion relations of the theory we first need to provide the curvatures evaluated on the vacuum:

$\psi$  Sector

$$M_{\psi_0}^2 = M_{\psi_Y}^2 = M_{\psi_Z}^2 = m^2 + \lambda \frac{\mu^2 - m^2}{\lambda - \lambda'}, \quad M_{\psi_X}^2 = \mu^2, \quad (23)$$

$\varphi^1$  Sector

$$M_{\varphi_0^1}^2 = \mu^2, \quad M_{\varphi_X^1}^2 = 2(\mu^2 - m^2), \quad M_{\varphi_Y^1}^2 = M_{\varphi_Z^1}^2 = 0, \quad (24)$$

$\varphi^2$  Sector

$$M_{\varphi_0^2}^2 = m^2 + \lambda' \frac{\mu^2 - m^2}{\lambda - \lambda'}, \quad M_{\varphi_X^2}^2 = 0, \quad M_{\varphi_Y^2}^2 = M_{\varphi_Z^2}^2 = \lambda' \frac{\mu^2 - m^2}{\lambda - \lambda'}. \quad (25)$$

In general three states have null curvature (specifically  $M_{\varphi_X^2}^2 = M_{\varphi_Y^2}^2 = M_{\varphi_Z^2}^2 = 0$ ) however for the special case  $\lambda' = 0$  we have 5 zero curvature states. To explain this behavior we note that for  $\lambda' = 0$  the potential possesses  $SO(6)$  global symmetry which breaks to  $SO(5)$  when the vector field condenses. The associated 5 states would correspond to the ordinary Goldstone modes in the absence of an explicit Lorentz breaking. Using these curvatures we compute the dispersion relations. Four of the  $\varphi$  states have (2 per each sign) the following dispersion relations:

$$E_{\varphi_{\mp}}^2 = \vec{p}^2 + \Delta^2 \mp \sqrt{4\mu^2 \vec{p}^2 + \Delta^4} \quad (26)$$

with

$$\Delta^2 = 2\mu^2 + \frac{\lambda'}{2} \frac{(\mu^2 - m^2)}{\lambda - \lambda'}, \quad (27)$$

and to each sign we associate a vector (with two components) with respect to  $SO(2)$ . In the limit of small momenta:

$$\begin{aligned} E_{\varphi_V^-}^2 &= \frac{\lambda'}{2\Delta^2} \frac{\mu^2 - m^2}{(\lambda - \lambda')} \bar{p}^2 + \frac{2\mu^4}{\Delta^6} |\bar{p}|^4 + \mathcal{O}(p^6) = v_{\varphi_V^-}^2 \bar{p}^2 + \dots, \\ E_{\varphi_V^+}^2 &= 2\Delta^2 + \left[ 1 + \frac{2\mu^2}{\Delta^2} \right] \bar{p}^2 - \frac{2\mu^4}{\Delta^6} |\bar{p}|^4 + \mathcal{O}(p^6). \end{aligned} \quad (28)$$

For the last two  $\varphi$  physical modes (scalars of  $SO(2)$ ) we show directly the dispersion relations as a momentum expansion.

$$E_{\varphi_S^-}^2 = \frac{\mu^2 - m^2}{3\mu^2 - m^2} \bar{p}^2 + \mathcal{O}(p^4) = v_{\varphi_S^-}^2 \bar{p}^2 + \dots, \quad E_{\varphi_S^+}^2 = 2(3\mu^2 - m^2) + \gamma \bar{p}^2 + \mathcal{O}(p^4), \quad (29)$$

where the subscripts  $V, S$  stand for vector and scalar respectively, and

$$\gamma = \frac{(3 + 8c)\mu^6 + (7 - 16c)m^2\mu^4 + 2(5c - 4)m^4\mu^2 - 2(c - 1)m^6}{\mu^2(3\mu^2 - m^2)(c\mu^2 - (c - 1)m^2)} > 0, \quad (30)$$

and  $c = \lambda/(\lambda - \lambda')$ . Finally for completeness we show the three  $\psi$  type state dispersion relations:

$$E_{\psi_I} = \sqrt{\bar{p}^2 + M_{\psi_I}^2}, \quad (31)$$

with  $M_{\psi_I}$  defined in eq. (23). As for the scalar toy model, the velocities of our gapless modes can be expressed directly in terms of the curvatures in the directions orthogonal to the gapless modes as follows:

$$v_{\varphi_V^-}^2 = \frac{M_{\varphi_V^-}^2}{M_{\varphi_V^-}^2 + 4\mu^2}, \quad v_{\varphi_S^-}^2 = \frac{M_{\varphi_S^-}^2}{M_{\varphi_S^-}^2 + 4\mu^2}. \quad (32)$$

We find that the number of gapless states emerging when the flavor $\times$ rotational symmetry spontaneously breaks is insensitive to the choice of the coefficients in the effective potential. However the momentum dependence depends crucially on the symmetries of the potential. The full vector Lagrangian (i.e. kinetic – potential) possesses the  $Z_2 \times U(1) \times SO(3)$  (flavor $\times$ rotational) symmetry which breaks, at non zero chemical potential, when the vector condenses to  $Z_2 \times SO(2)$ . The gapless modes are associated to the 3 broken generators. One state is a scalar of  $SO(2)$  while the other two fields constitute a vector of  $SO(2)$ . The scalar Goldstone possesses linear dispersion relations independently of the choice of the vector potential. This is the Goldstone associated to the  $U(1)$  breaking of the previously investigated scalar toy model. However for the dispersion relations of the  $SO(2)$  vector state we have either quadratic or linear dispersion relations. The quadratic ones occur when the potential parameters are such that it possesses an enhanced  $SO(6)$  symmetry. When condensing the vector breaks the potential symmetries  $SO(6)$  to  $SO(5)$  while the kinetic term (i.e. the derivative part of the Lagrangian) is still invariant under the continuous  $U(1) \times SO(3)$  only. In this case the potential has 5 flat directions (i.e. 5 null curvatures (occurring for  $\lambda' = 0$ )) and we would count in the spectrum 2-massless vectors and a scalar of  $SO(2)$ . However the reduced symmetry of the kinetic term prevents the emergence of another gapless vector type mode while turning the linear dispersion relations of one of the vector states into quadratic ones. As for the toy model case, at the second order phase transition point, all the velocities of the gapless modes vanish regardless of the value of the coupling constants. Again the information of the symmetries (conformal and/or global) of the potential, at nonzero chemical potential, are nicely transferred from the curvature to the momentum dependence of the dispersion relations via the velocities of the gapless modes. Our result is in agreement with the Chadha-Nielsen counting scheme.

### III. CONCLUSIONS AND PHYSICAL APPLICATIONS

We summarized our findings in the Table I and conclude our work by considering some possible physical applications of massive relativistic vector condensation at high chemical potential.

## A. 2 Color QCD

Two color QCD at non zero chemical potential recently attracted a great flurry of interest since lattice simulations can be reliably performed at non zero quark chemical potential. In [20] vector condensation at non zero quark chemical potential has been predicted for two color QCD with an even numbers of flavors. The predictions are supported [21] by lattice studies. Here we confirmed the previous results and provided a more detailed study of the vector condensation phenomenon at non zero chemical potential.

An interesting direct application of our results is for QCD with two colors and one light flavor. This theory has global quantum symmetry group  $SU(2)$ . The extra classical axial  $U_A(1)$  symmetry is broken by the Adler-Bell-Jackiw anomaly. At zero chemical potential Lorentz invariance cannot be broken and the simplest bilinear condensate (see Ref. [20] for conventions) is of the type:

$$\epsilon^{c_1 c_2} \epsilon^{\alpha \beta} Q_{\alpha, c_1}^I Q_{\beta, c_2}^J E_{IJ} , \quad (33)$$

with  $E = 2iT^2$  the antisymmetric matrix in the flavor space and  $Q_{\alpha c}^I$  is a Weyl spinor with  $\alpha = 1, 2$  the spinorial index,  $c = 1, 2$  the color and  $I = 1, 2$  the flavor indices. The generators of  $SU(2)$  in the fundamental representation are  $T^a = \tau^a/2$  with  $\tau^a$  the standard Pauli matrices. It is easy to check that  $SU(2)$  remains unbroken since we have  $T^{aT} E + E T^a = 0$  for all of the  $SU(2)$  generators. So we have no Goldstone bosons at all in the theory. However we do have physical vectors (i.e. spin 1 fields) and massive scalar states at zero density. These states should be comparable in mass (recall the  $\eta'$  versus the octet of ordinary vector bosons in 3 color QCD). The massive spin one fields (i.e. the hadrons) must transform according to the regular representation of  $SU(2)$  and carry non trivial baryon number which is proportional to  $T^3$ .

Our analysis can be immediately used to predict that when turning on a non zero baryon chemical potential a superfluid phase transition will set in for  $\mu$  larger then the mass of the vector fields. This is so since we always have at least one gapless mode with linear dispersion relations. We also predict the spontaneous breaking of rotational invariance and the emergence of a vector state (i.e. doublet of  $SO(2)$ ) with either linear or quadratic dispersion relations. It is straightforward to extend and apply our analysis to the case of different number of flavors and to ordinary QCD at finite isospin chemical potential. [20].

## B. Extra-dimensions

Finally we extend our model to an  $SU(2)$  charged vector in  $D - 1$  number of space dimensions (with  $SO(D)$  Euclidean Lorentz symmetry) with an associated non zero chemical potential. In this case the chemical potential first breaks explicitly  $SO(D)$  to the spatial  $SO(D - 1)$ . At sufficiently high chemical potential  $SO(D - 1)$  breaks spontaneously to  $SO(D - 2)$ . The gapless states consist of a scalar field under the  $SO(D - 2)$  transformations with linear dispersion relations and one vector of  $SO(D - 2)$  with either linear or quadratic dispersion relations.

	$\lambda > \lambda'$	$\lambda' = 0$
Lagrangian	$Z_2 \times U(1) \times SO(3) \longrightarrow Z_2 \times SO(2)$	$Z_2 \times U(1) \times SO(3) \longrightarrow Z_2 \times SO(2)$
Potential	$Z_2 \times U(1) \times SO(3) \longrightarrow Z_2 \times SO(2)$	$Z_2 \times SO(6) \longrightarrow Z_2 \times SO(5)$
# of broken generators:		
Lagrangian	3	3
Potential	3	5
spectrum of Goldstone modes:		
Type I ( $E \propto  \vec{p} $ )	1 SO(2) Scalar + 1 SO(2) Vector	1 SO(2) Scalar
Type II ( $E \propto  \vec{p} ^2$ )	–	1 SO(2) Vector

TABLE I. Summary of the relevant symmetries and spectrum of the Goldstone modes for the parameter choices  $\lambda > \lambda'$  and  $\lambda' = 0$  of the fourth-order potential. We also indicate in the first row the symmetries (after the arrows) which are not broken by the vector condensate.

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