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On upper transversals in 3-uniform hypergraphs

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Abstract

A set S of vertices in a hypergraph H is a transversal if it has a nonempty intersection with every edge of H . The upper transversal number $\Upsilon(H)$ of H is the maximum cardinality of a minimal transversal in H . We show that if H is a connected 3-uniform hypergraph of order n , then $\Upsilon(H) > 1.4855\sqrt[3]{n} - 2$. For n sufficiently large, we construct infinitely many connected 3-uniform hypergraphs, H , of order n satisfying $\Upsilon(H) < 2.5199\sqrt[3]{n}$. We conjecture that $\sup_{n \rightarrow \infty} \left(\inf \frac{\Upsilon(H)}{\sqrt[3]{n}} \right) = \sqrt[3]{16}$, where the infimum is taken over all connected 3-uniform hypergraphs H of order n .

Mathematics Subject Classifications: 05C88, 05C89

1 Introduction

In this paper, we continue the study of transversals in hypergraphs. Hypergraphs are systems of sets which are conceived as natural extensions of graphs. A *hypergraph* $H = (V(H), E(H))$ is a finite set $V(H)$ of elements, called *vertices*, together with a finite multiset $E(H)$ of nonempty subsets of $V(H)$, called *hyperedges* or simply *edges*. A *k-edge* in H is an edge of size k . The hypergraph H is *k-uniform* if every edge of H is a k -edge. Every 2-uniform hypergraph is a graph. Thus graphs are special hypergraphs. The *degree* of a vertex v in H , denoted by $d_H(v)$, is the number of edges of H which contain v . The

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minimum and maximum degrees among the vertices of H is denoted by $\delta(H)$ and $\Delta(H)$, respectively.

A subset T of vertices in a hypergraph H is a *transversal* (also called *hitting set* or *vertex cover* or *blocking set* in many papers) if T has a nonempty intersection with every edge of H . A vertex *hits* or *covers* an edge if it belongs to that edge. The *transversal number* $\tau(H)$ of H is the minimum size of a transversal in H , while the *upper transversal number* $\Upsilon(H)$ of H is the maximum cardinality of a minimal transversal in H . In hypergraph theory the concept of transversal is fundamental and well studied. The major monograph [1] of hypergraph theory gives a detailed introduction to this topic. Transversals in hypergraphs are well studied in the literature (see, for example, [3, 4, 11, 12, 13] for recent results and further references).

A set S of vertices in a graph G is a *dominating set* of G if each vertex in $V(G) \setminus S$ has a neighbor in S . A set is *independent* if no two vertices in it are adjacent. An *independent dominating set* of G is a set that is both dominating and independent in G . The *independent domination number* of G , denoted by $i(G)$, is the minimum cardinality of an independent dominating set. Domination is well studied in graph theory and we refer the reader to the monographs [9, 10] which detail and survey many results on the topic. A survey of known results on independent domination in graphs can be found in [8].

2 Main Results

We have two immediate aims in this paper. First to provide a sharp lower bound on the upper transversal number of graphs. Secondly to present a lower bound on the upper transversal number of 3-uniform hypergraphs, and to show that this bound is a sense asymptotically best possible. More precisely, we prove the following results, where we use the notation $n_H = |V(H)|$ to denote the *order* of H . Proofs of Theorem 1 and Theorem 2 are given in Section 3 and Section 4, respectively.

Theorem 1. *If H is a connected graph with $\delta(H) \geq \delta$, then*

$$\Upsilon(H) \geq 2\sqrt{\delta n_H} - 2\delta,$$

and this bound is sharp.

Theorem 2. *If H is a connected 3-uniform hypergraph, then*

$$\Upsilon(H) > \sqrt[3]{\frac{n_H}{0.305}} - 2 > 1.4855\sqrt[3]{n_H} - 2.$$

Further, there exist infinitely many connected 3-uniform hypergraphs H of sufficiently large order n_H satisfying

$$\Upsilon(H) < \sqrt[3]{16 \cdot n_H} < 2.52\sqrt[3]{n_H}.$$

3 Proof of Theorem 1

Recall that a transversal in a graph is a set of vertices covering all the edges of the graph, where a vertex covers an edge if it is incident with it. Theorem 1 can be restated as follows.

Theorem 1. *If G is a connected graph of order n with $\delta(G) \geq \delta$, then $\Upsilon(G) \geq 2\sqrt{\delta n} - 2\delta$, and this bound is sharp.*

In order to prove Theorem 1, we first establish a relationship between the upper transversal number and independent domination number of a graph.

Theorem 3. *If G is an isolate-free graph on n vertices, then $i(G) + \Upsilon(G) = n$.*

Proof. Let G be an isolate-free graph. Let S be an independent dominating set in G of minimum cardinality, and so $|S| = i(G)$. Let $T = V(G) \setminus S$ and note that T is a transversal in G as S is an independent set. Since every vertex in T has a neighbor in S , we furthermore note that T is a minimal transversal, which implies that $\Upsilon(G) \geq |T| = n - |S| = n - i(G)$.

Conversely, let T be a minimal transversal in G of maximum cardinality, and so $|T| = \Upsilon(G)$. Let $S = V(G) \setminus T$ and note that S is an independent set as T is a transversal. Since T is a minimal transversal, every vertex in T has a neighbor in S , implying that S is an independent dominating set in G . Therefore, $i(G) \leq |S| = n - |T| = n - \Upsilon(G)$. Consequently, $i(G) + \Upsilon(G) = n$. \square

Favaron [5] was the first to prove the following upper bound on the independent domination of a graph with no isolated vertex: If G is an isolate-free graph of order n , then $i(G) \leq n + 2 - 2\sqrt{n}$. We remark that this result also follows from a result due to Bollobás and Cockayne [2] (and was also proved in [6]). Sun and Wang [14] proved the following more general result, which was originally posed as a conjecture by Favaron [5] and was proved for $\delta = 2$ by Glebov and Kostochka [7].

Theorem 4. ([14]) *If G is a graph of order n with $\delta(G) \geq \delta$, then $i(G) \leq n + 2\delta - 2\sqrt{\delta n}$.*

Theorem 1 is an immediate consequence of Theorem 3 and Theorem 4. That this bound is sharp, follows from a result of Favaron [5] who showed that for every positive integer δ , the bound in Theorem 4 is attained for infinitely many graphs. The same graphs achieve equality in the bound for Theorem 1. For example, for $c \geq 2$, let G_c be the connected graph constructed as follows. Let F_c be the complete graph of order c , and so $F_c \cong K_c$. For every vertex v of F_c , add $c - 1$ new vertices v_1, \dots, v_{c-1} and add the $c - 1$ edges vv_i for all $i \in [c - 1]$. Let $G = G_c$ denote the resulting graph of order $n = c^2$. We note that all the new vertices added to F_c have degree 1 in G_c . Every transversal in G_c must contain all except possibly one vertex of F_c in order to cover all the edges of F_c . If a minimal transversal in G_c contains exactly $c - 1$ vertices of F_c , say all vertices of F_c except for v , then the transversal contains exactly $c - 1$ vertices not in F_c , namely v_1, \dots, v_{c-1} , in order to cover the edges vv_i for all $i \in [c - 1]$. Such a minimal transversal therefore has

size exactly $2(c - 1)$. If a minimal transversal in G_c contains all c vertices of F_c , then it contains no other vertex of G_c and therefore has size exactly c . Therefore, the connected graph G of order $n = c^2$ satisfies $\Upsilon(G) = \max\{2(c - 1), c\} = 2(c - 1) = 2\sqrt{n} - 2$, noting that $c \geq 2$. Thus, the bound of Theorem 1 when $\delta = 1$ is sharp. For every $\delta \geq 2$ one can similarly show that Theorem 1 is tight.

As a special case of Theorem 1, if H is a connected 2-uniform hypergraph of order $n \geq 2$, then $\Upsilon(H) \geq 2\sqrt{n} - 2$. When $n \geq 3$, we observe that $2\sqrt{n} - 2 \geq \sqrt{\frac{1}{2}n}$. Further, we observe that when $n = 2$, $\Upsilon(H) = 1 = \sqrt{\frac{1}{2}n}$. Thus, as an immediate consequence of Theorem 1 we observe that if H is a connected graph, then $\Upsilon(H) \geq \sqrt{\frac{1}{2}n_H}$.

4 Proof of Theorem 2

We first prove the lower bound in Theorem 2.

Theorem 5. *If H is a connected 3-uniform hypergraph, then $\Upsilon(H) > \sqrt[3]{\frac{n_H}{0.305}} - 2$.*

Proof. Let H be a connected 3-uniform hypergraph of order n_H and let T be a minimal transversal of maximum size. Let $T = \{t_1, t_2, \dots, t_c\}$, and so $\Upsilon(H) = |T| = c$. For all i and j where $1 \leq i < j \leq c$ and for all $k \in [c]$, define $Z_{i,j}$, E_k and Y_k as follows.

$$\begin{aligned} Z_{i,j} &= \{v \in V(H) \setminus T \mid \{t_i, t_j, v\} \in E(H)\} \\ E_k &= \{e \in E(H) \mid V(e) \cap T = \{t_k\}\} \\ Y_k &= V(E_k) \setminus \{t_k\}. \end{aligned}$$

We note that $Y_k \subseteq V(H) \setminus T$ for each $k \in [c]$. Let $Q \subseteq V(T)$ be a minimum set of vertices in T that covers all edges that are completely within T (i.e., all edges e with $V(e) \subseteq V(T)$). Possibly, $Q = \emptyset$. Let $q = |Q|$. Renaming vertices of T if necessary, we may assume that $Q = \{t_1, \dots, t_q\}$. Let

$$\begin{aligned} \mathcal{I} &= \{(i, j) \mid 1 \leq i < j \leq c\} \\ \mathcal{I}_q &= \{(i, j) \mid 1 \leq i \leq q \text{ and } i < j \leq c\} \\ \mathcal{I}_{>q} &= \{(i, j) \mid q + 1 \leq i < j \leq c\}. \end{aligned}$$

We note that $\mathcal{I} = \mathcal{I}_q \cup \mathcal{I}_{>q}$. We proceed further with the following claims.

Claim 6. $|Z_{i,j}| \leq c - q$ for all $(i, j) \in \mathcal{I}$.

Proof. Suppose, to the contrary, that $|Z_{i,j}| > c - q$ for some i and j , where $1 \leq i < j \leq c$. Let $R = V(H) \setminus \{t_i, t_j\}$. Clearly, R is a transversal in H as it contains all vertices in H except two and H is 3-uniform. Let R' be obtained from R by removing vertices until we get a minimal transversal in H . We note that $Z_{i,j} \subseteq R'$ since each vertex $z \in Z_{i,j}$ is needed in order to cover the edge $\{t_i, t_j, z\}$. Further, R' contains at least q vertices from T in order to cover the edges that are contained entirely within T . Hence, $\Upsilon(H) \geq |R'| \geq |Z_{i,j}| + q > c$, contradicting the fact that $\Upsilon(H) = c$. \square

Claim 7. $\left| \bigcup_{(i,j) \in \mathcal{I}_{>q}} Z_{i,j} \right| \leq c - q.$

Proof. Suppose, to the contrary, that

$$\left| \bigcup_{(i,j) \in \mathcal{I}_{>q}} Z_{i,j} \right| > c - q.$$

Let $R = V(H) \setminus \{t_{q+1}, \dots, t_c\}$. By definition of the set Q , every edge of H intersects $\{t_{q+1}, \dots, t_c\}$ in at most two vertices, implying that R is a transversal in H . Let R' be obtained from R by removing vertices from R until we get a minimal transversal in H . We note that

$$\bigcup_{(i,j) \in \mathcal{I}_{>q}} Z_{i,j} \subseteq R'$$

since each vertex $z \in Z_{i,j}$ where $q + 1 \leq i < j \leq c$ is needed in order to cover the edge $\{t_i, t_j, z\}$. Further, R' contains at least q vertices from Q in order to cover the edges that are contained entirely within T . Hence,

$$\Upsilon(H) \geq |R'| \geq \left| \bigcup_{(i,j) \in \mathcal{I}_{>q}} Z_{i,j} \right| + q > c,$$

contradicting the fact that $\Upsilon(H) = c$. □

Claim 8. $\left| \bigcup_{(i,j) \in \mathcal{I}} Z_{i,j} \right| \leq \left(\binom{c}{2} - \binom{c-q}{2} \right) (c - q) + (c - q).$

Proof. As observed earlier, $\mathcal{I} = \mathcal{I}_q \cup \mathcal{I}_{>q}$. By Claim 6, $|Z_{i,j}| \leq c - q$ for all $(i, j) \in \mathcal{I}_q$. Since there are $\binom{c}{2} - \binom{c-q}{2}$ pairs $(i, j) \in \mathcal{I}_q$ where $1 \leq i \leq q$ and $i < j \leq c$, we note by Claim 6 and Claim 7 that

$$\left| \bigcup_{(i,j) \in \mathcal{I}} Z_{i,j} \right| \leq \left| \bigcup_{(i,j) \in \mathcal{I}_q} Z_{i,j} \right| + \left| \bigcup_{(i,j) \in \mathcal{I}_{>q}} Z_{i,j} \right| \leq \left(\binom{c}{2} - \binom{c-q}{2} \right) (c - q) + (c - q). \quad \square$$

Claim 9. $|Y_i| \leq \left(\frac{c-q}{2} + 1\right)^2$ for all $i \in [c]$.

Proof. Suppose, to the contrary, that $|Y_i| > \left(\frac{c-q}{2} + 1\right)^2$ for some $i \in [c]$. Let H' be the graph with vertex set $V(H') = Y_i$ and with edge set $E(H') = \{e \setminus \{t_i\} \mid e \in E_i\}$. By Theorem 1, there is a minimal transversal T' in H' , such that $|T'| \geq 2(\sqrt{|Y_i|} - 1)$. Let $R' = T' \cup (T \setminus \{t_i\})$ and note that R' is a transversal in H . Let R'' be obtained from R' by removing vertices from R' until we get a minimal transversal in H . In order to cover the edges E_i , we must have $T' \subseteq R''$, noting that T' is a minimal transversal in H' . Further, R'' contains at least q vertices from $T \setminus \{t_i\}$ in order to cover the edges that are contained entirely within T . Therefore,

$$\Upsilon(H) \geq |R''| \geq |T'| + q \geq 2(\sqrt{|Y_i|} - 1) + q > 2 \left(\sqrt{\left(\frac{c-q}{2} + 1\right)^2} - 1 \right) + q = c$$

contradicting the fact that $\Upsilon(H) = c$. □

Since T is a transversal in H , we note that

$$V(H) = T \cup \left(\bigcup_{(i,j) \in \mathcal{I}} Z_{i,j} \right) \cup \left(\bigcup_{i=1}^c Y_i \right). \quad (1)$$

Let β be defined such that $(c - q) = \beta c$. We note that $0 \leq \beta \leq 1$. By Equation (1) and by Claim 8 and 9, we therefore get the following.

$$\begin{aligned} n_H &\leq |T| + \sum_{i,j} |Z_{i,j}| + \sum_i |Y_i| \\ &\leq c + \left(\binom{c}{2} - \binom{c-q}{2} \right) (c - q) + (c - q) + c \left(\frac{c-q}{2} + 1 \right)^2 \\ &\leq c + \left(\frac{c(c-1)}{2} - \frac{\beta c(\beta c-1)}{2} \right) \beta c + \beta c + c \left(\frac{\beta c}{2} + 1 \right)^2 \\ &= c + \frac{1}{2} (c^2 - c - \beta^2 c^2 + \beta c) \beta c + \beta c + c \left(\frac{\beta^2 c^2}{4} + \beta c + 1 \right) \\ &= c^3 \left(\frac{\beta}{2} - \frac{\beta^3}{2} + \frac{\beta^2}{4} \right) + c^2 \left(\frac{-\beta}{2} + \frac{\beta^2}{2} + \beta \right) + c(2 + \beta) \\ &= \frac{c^3}{4} (-2\beta^3 + \beta^2 + 2\beta) + c^2 \left(\frac{\beta}{2} + \frac{\beta^2}{2} \right) + c(2 + \beta). \end{aligned}$$

Let

$$f(\beta) = -2\beta^3 + \beta^2 + 2\beta.$$

The maximum value of $f(\beta)$ when $0 \leq \beta \leq 1$ is obtained when $\beta = (1 + \sqrt{13})/6$, noting that $0 = f'(\beta) = -6\beta^2 + 2\beta + 2$ implies $\beta = (1 \pm \sqrt{13})/6$. Therefore, $f(\beta) \leq f\left(\frac{1+\sqrt{13}}{6}\right) < 1.22$ for all $0 \leq \beta \leq 1$, implying by our earlier observations that

$$\begin{aligned} n_H &\leq \frac{c^3}{4} (-2\beta^3 + \beta^2 + 2\beta) + c^2 \left(\frac{\beta}{2} + \frac{\beta^2}{2} \right) + c(2 + \beta) \\ &< \frac{c^3}{4} (1.22) + c^2 \left(\frac{1}{2} + \frac{1}{2} \right) + c(2 + 1) \\ &= 0.305c^3 + c^2 + 3c \\ &< 0.305(c + 2)^3, \end{aligned}$$

and so $\Upsilon(H) = c > \sqrt[3]{\frac{n_H}{0.305}} - 2$. This completes the proof of Theorem 5. □

We remark that $\sqrt[3]{\frac{1}{0.305}} > 1.48559$, and so as a consequence of Theorem 5, if H is a connected 3-uniform hypergraph of order $n \geq 3$, then $\Upsilon(H) > 1.4855\sqrt[3]{n} - 2$. When $n \geq 17$, we observe that $1.4855\sqrt[3]{n} - 2 > \sqrt[3]{\frac{1}{3}n}$. Further, we observe that when $n = 3$, $\Upsilon(H) = 1 = \sqrt[3]{\frac{1}{3}n}$, while for $4 \leq n \leq 16$, $\Upsilon(H) \geq 2 > \sqrt[3]{\frac{1}{3}n}$. Thus, as an immediate consequence of Theorem 5, we observe that if H is a connected 3-uniform hypergraph,

then $\Upsilon(H) \geq \sqrt[3]{\frac{n_H}{3}}$. We show next that the lower bound in Theorem 5 is asymptotically best possible.

Proposition 10. *For all $n \geq 3$, there exists a connected 3-uniform hypergraph $H = H_n$ of order $n_H = \frac{1}{2}(n^3 - n^2 + 2n)$ such that*

$$\Upsilon(H) = \sqrt[3]{16(1 - \epsilon_n) \cdot n_H} \quad \text{where} \quad \epsilon_n = \frac{2n^2 - n + 1}{n^3 - n^2 + 2n}.$$

Proof. For each $n \geq 3$, let H_n be the connected 3-uniform hypergraph constructed as follows. Let F_n be the complete 3-uniform hypergraph on n vertices, and so F_n has $\binom{n}{3}$ hyperedges corresponding to the 3-element subsets of $V(F_n)$. Thus, every set of three vertices in F_n belongs to a 3-edge of F_n . Let $S = V(F_n)$. For every pair of vertices, $\{x, y\}$, in S add n new vertices, $v_1^{xy}, v_2^{xy}, \dots, v_n^{xy}$ to F_n and add the n hyperedges $\{x, y, v_1^{xy}\}, \{x, y, v_2^{xy}\}, \dots, \{x, y, v_n^{xy}\}$. Let $H = H_n$ denote the resulting hypergraph of order

$$n_H = \binom{n}{2}n + |V(F_n)| = \frac{n^2(n-1)}{2} + n = \frac{1}{2}(n^3 - n^2 + 2n).$$

We note that all the new vertices added to F_n have degree 1 in H_n . Every transversal in H_n must contain all n vertices in F_n , except for possibly two vertices in order to cover all the edges in F_n . Every minimal transversal in H_n contains either exactly $n - 1$ vertices of S (and no other vertex in H_n) or exactly $n - 2$ vertices of S , say all vertices of S except for the vertices x and y , and exactly n vertices not in S , namely $v_1^{xy}, v_2^{xy}, \dots, v_n^{xy}$, in order to cover the edges $\{x, y, v_1^{xy}\}, \{x, y, v_2^{xy}\}, \dots, \{x, y, v_n^{xy}\}$, implying that

$$\Upsilon(H_n) = (|S| - 2) + n = 2(n - 1).$$

Therefore, letting $\epsilon_n = \frac{2n^2 - n + 1}{n^3 - n^2 + 2n}$, we note that the connected 3-uniform hypergraph $H = H_n$ satisfies

$$\begin{aligned} \Upsilon(H) &= 2(n - 1) \\ &= \sqrt[3]{8 \cdot (n - 1)^3} \\ &= \sqrt[3]{16 \cdot \frac{(n - 1)^3}{2n_H} \cdot n_H} \\ &= \sqrt[3]{16 \left(\frac{n^3 - 3n^2 + 3n - 1}{n^3 - n^2 + 2n} \right) \cdot n_H} \\ &= \sqrt[3]{16 \left(1 - \frac{2n^2 - n + 1}{n^3 - n^2 + 2n} \right) \cdot n_H} \\ &= \sqrt[3]{16(1 - \epsilon_n) \cdot n_H}. \quad \square \end{aligned}$$

Using the notation introduced in the statement of Proposition 10, we note that $\frac{2}{3} = \epsilon_3 > \epsilon_4 > \epsilon_5 > \dots$ and that $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$. In particular, we note that given any

$\epsilon > 0$, we can choose n sufficiently large so that $\epsilon_n < \epsilon$, implying by Proposition 10 that the connected 3-uniform hypergraph $H = H_n$ satisfies

$$\sqrt[3]{16(1 - \epsilon) \cdot n_H} < \Upsilon(H) < \sqrt[3]{16 \cdot n_H}$$

5 Closing Conjectures

We pose the following conjecture. As observed earlier, Conjecture 11 is true for $k \in \{2, 3\}$. However, we have yet to settle the conjecture for $k \geq 4$.

Conjecture 11. For $k \geq 2$, if H is a connected k -uniform hypergraph then $\Upsilon(H) \geq \sqrt[k]{\frac{n_H}{k}}$.

Let \mathcal{H}_n denote the class of all connected 3-uniform hypergraphs of order n . As observed earlier, Proposition 10 implies that for n sufficiently large there exist hypergraphs $H \in \mathcal{H}_n$ such that

$$\frac{\Upsilon(H)}{\sqrt[3]{n}} > \sqrt[3]{16(1 - \epsilon)}$$

for any given $\epsilon > 0$. We close with the following conjecture that we have yet to settle.

Conjecture 12. $\sup_{n \rightarrow \infty} \left(\inf_{H \in \mathcal{H}_n} \frac{\Upsilon(H)}{\sqrt[3]{n}} \right) = \sqrt[3]{16}$.

References

- [1] C. Berge, *Hypergraphs – Combinatorics of Finite Sets*. North-Holland, 1989.
- [2] B. Bollobás and E. J. Cockayne. Graph-theoretic parameters concerning domination, independence, and irredundance. *J. Graph Theory*, 3:241–249, 1979.
- [3] Cs. Bujtás, M. A. Henning, and Zs. Tuza. Transversals and domination in uniform hypergraphs. *European J. Combin.*, 33:62–71, 2012.
- [4] Cs. Bujtás, M. A. Henning, Zs. Tuza, and A. Yeo. Total transversals and total domination in uniform hypergraphs. *Electron. J. Combin.*, 21(2) #P2.24, 2014.
- [5] O. Favaron. Two relations between the parameters of independence and irredundance. *Discrete Math.*, 70:17–20, 1988.
- [6] J. Gimbel and P. D. Vestergaard. Inequalities for total matchings of graphs. *Ars Combin.*, 39:109–119, 1995.
- [7] N. I. Glebov and A. V. Kostochka. On the independent domination number of graphs with given minimum degree. *Discrete Math.*, 118:261–266, 1998.
- [8] W. Goddard and M. A. Henning. Independent domination in graphs: A survey and recent results. *Discrete Math.*, 313:839–854, 2013.

- [9] T. W. Haynes, S. T. Hedetniemi, P. J. Slater (eds). *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc. New York, 1998.
- [10] T. W. Haynes, S. T. Hedetniemi, P. J. Slater (eds). *Domination in Graphs: Advanced Topics*, Marcel Dekker, Inc. New York, 1998.
- [11] M. A. Henning and C. Löwenstein. A characterization of the hypergraphs that achieve equality in the Chvátal-McDiarmid Theorem. *Discrete Math.*, 323:69–75, 2014.
- [12] M. A. Henning and A. Yeo. Hypergraphs with large transversal number. *Discrete Math.*, 313:959–966, 2013.
- [13] Z. Lonc and K. Warno. Minimum size transversals in uniform hypergraphs. *Discrete Math.*, 313:2798–2815, 2013.
- [14] L. Sun and J. Wang. An upper bound for the independent domination number. *J. Combin. Theory, Ser. B*, 76:240–246, 1999.