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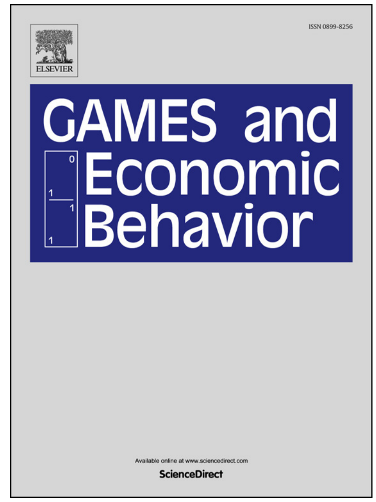

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THE PROBLEM OF MULTIPLE COMMONS: A MARKET DESIGN APPROACH

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ABSTRACT. There are several locations, each of which is endowed with a resource that is specific to that location. Each agent will go to a single location and harvest some of the resource there. Several agents may go to each location, and thereby cause a congestion effect that is modeled in a general way. We assign harvesting rights based on preferences alone, though we later extend the model to accommodate private endowments of money. We find the best allocation rule in the class of rules that are strategy-proof, anonymous, and that satisfy a weak continuity property. For a special class of preferences, we also find an ascending mechanism, similar to an auction, that implements the rule. The rule coincides with a special psuedo-market equilibrium, wherein agents buy their desired resource with tokens distributed by the social planner.

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1. Introduction

This paper studies a novel allocation problem that combines elements of the assignment game of Shapley & Shubik (1971) and the congestion game of Rosenthal (1973). Like an assignment game, each agent is to be matched to a single, discrete object, and consume some divisible commodity. Unlike the classical assignment problem, and similar to the problem of Andersson & Svensson (2014), the amount of divisible commodity to be consumed is bounded both from above and below. Furthermore, several agents can be matched to a single object and must then share the commodity associated with it, which introduces the question of congestion.

One example is the assignment of patients to a general practitioner (GP) in countries with socialized medicine. In Canada, Denmark, and Norway, each citizen is assigned to a GP who acts as gatekeeper to the broader health system. It is costly to change one's assigned GP, incentivizing long-term relationships.[1] Naturally one has preferences over doctors that are intrinsic to the doctor herself, such as the distance from one's home, but the patient would also take into account the relative accessibility of the doctor. When services are needed, how quickly can one get an appointment? Conditional on getting an appointment, what will be the quality of the attention one receives? Anecdotally, some physicians will consider only one issue per visit, so as to move through their queue of patients quickly. Thus, indirectly, preferences depend also on the number of other patients assigned to one's GP. In sum, when choosing a physician, patients will balance factors intrinsic to the doctor as well as factors related to congestion. [2]

While this is far from the classical example of a commons—a field upon which any shepherd may graze his sheep—it is in fact a commons problem, as socialized medicine has created an environment in which medical resources are effectively non-excludable. It is, however, different from the classical problem in several important respects: 1) each doctor is a different commons, among which consumers must choose, 2) the congestion effect is abstract, 3) the price for service is fixed (via taxes) and 4) there is a maximal number of patients a doctor can be assigned before risking accusations of negligence.

Our approach is to apply a *pseudo-market* in which an *ad hoc numeraire* commodity is given to participants for use as a tool to express their preferences. Unit-prices are announced, one for each available resource, and agents will then choose a site and exchange their *numeraire* for the

[1] Absent of mitigating circumstances, one can choose a GP once per year in Denmark.

[2] Obviously these two will be correlated, but not perfectly, if only because of distance considerations. Thus the problem cannot be reduced to one dimension.

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resource available there. In the case of GP assignment, we imagine that most patients can be treated equally, and so each is given the same quantity of *numeraire*. Patients can then be presented with a simpler menu, specifying the share of services available at each doctor. The benefit of using the *numeraire* is that it allows for generalization to the case when some payment is required for use of the commons, which we take up in Section §5.

The *pseudo-market* approach is of course not novel. Among others, it has been used by Hylland & Zeckhauser (1979) and Budish (2011). However, in both these cases, it is used to make a discrete problem continuous, whereas the choice of harvesting location remains discrete in our model; agents will not spread their *numeraire* among the various discrete options. Moreover, unlike these papers, our concern is incentives in *finite* economies.

Once the pseudo-market is established, for special classes of preferences we can run a simultaneous ascending auction, *a la* Demange et al. (1986), to arrive at the allocation. We might expect that the other important results of the assignment game literature—the order structure on equilibrium prices, the incentive compatibility of the auction—also extend. We show that there are analogues of these results, and for general preferences, but they do not extend exactly. Moreover, they do not emerge from an embedding of our model into any extant model. In particular, while equilibrium prices typically form a lattice in classical results, they only form a *lower* lattice here. The incentive properties of the lattice minimum point hold, as with the previous literature. The ascending auction we describe terminates precisely at the lattice-minimal price, and so is incentive compatible.

We find that the lattice-minimal pseudo-market equilibrium (MPE) is the uniquely most efficient way of deterministically allocating the (divisible) resources while giving agents dominant strategy incentives and avoiding discrimination based on non-economic factors. By "avoiding discrimination" we mean that the procedure is *anonymous* in the sense relevant to our model. In short, each agent's name does not affect her allocation; her preferences are the only relevant data. The MPE is also asymptotically efficient: as the number of agents increases, the per-capita loss of efficiency goes to zero.

While the MPE is most efficient given incentive and anonymity constraints, it is not in fact efficient. This loss is due to the usual reason: information frictions. We can see, via isomorphism with the classical assignment problem in Euclidean space, that no probabilistic mechanism will be efficient and satisfy these constraints either. We make this explicit in Section §6. Welfare comparisons between lotteries and deterministic rules, are beyond the scope of this paper.

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2. Model

There are a finite set S of sites and a finite set N of agents. No more than $c_s \in \mathbb{N}$ agents may consume at site s. Assume $\sum_{s \in S} c_s \geq |N|$. The quantity of commodity available at site s is a positive, non-decreasing function f_s of the number of agents who consume there. A classical, rival resource is described by the constant vector-valued function $f \equiv \mathbf{e} \in \mathbb{R}^S$, causing the per-capita supply of commodity to decrease hyperbolically. We call this special case **extractive**. More generally, we assume each f_s satisfies:

Congestion: The per-capita supply at s, ${f_s(n)}/{n}$, is strictly decreasing.

Each agent $i \in N$ is assigned to a single site $s \in S$ and given a non-negative quantity x_i of that site's commodity. Thus a typical consumption bundle is a pair $(x_i, s) \in \mathbb{R}_+ \times S$. A typical preference relation is denoted R_i with symmetric and asymmetric parts I_i and P_i, respectively. A preference relation R_i is **increasing** if for each for each site $s \in S$, and each pair $x_i,\ y_i \in \mathbb{R}$ satisfying $x_i > y_i$, $(x_i, s)\ P_i\ (y_i, s)$. The set of continuous, increasing preferences is denoted $\overline{\mathcal{R}}$.

A function $\alpha : N \to S$ is called a **site assignment**.[3] An **allocation** is a pair $(x, \alpha) \in \mathbb{R}^N_+ \times S^N$, i's bundle being given by $(x_i, \alpha(i))$. As is standard, α^{-1} denotes the pre-image. Agents are self-centered and so we extend their preferences to the space of allocations in the usual way: $(x, \alpha)\ R_i\ (y, \beta)$ if and only if $(x_i, \alpha(i))\ R_i\ (y_i, \beta(i))$. An allocation (x, α) is **feasible** if each site distributes no more commodity than it produces, and hosts no more agents than its capacity. Formally, for each site $s \in S$,

(Endowment Constraint) $$\sum_{i \in \alpha^{-1}(s)} x_i \leq f_s\left(\left|\alpha^{-1}(s)\right|\right),$$

(Capacity Constraint) $$\left|\alpha^{-1}(s)\right| \leq c_s.$$

Note that the Endowment Constraint implies that the commodity at a given site need not be exhausted. For applications to waiting-time problems, this means that the manager of the commons must be able to artificially increase the wait faced by an individual. Such a loss of efficiency is generally necessary when incentive constraints are present; in the presence of money, it takes the form of a budget imbalance.

Let $\mathcal{R} \subseteq \overline{\mathcal{R}}$ and $\Phi : \mathcal{R}^N \rightrightarrows \mathbb{R}^N \times S^N$. If, for each $R \in \mathcal{R}^N$,

[3]For each $i \in N$, there is $s \in S$ such that $\alpha(i) = s$. Our notation follows Alkan et al. (1991), distinguishing an arbitrary site s from an element of a site assignment, $\alpha(i)$.

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- $\Phi(R)$ is a singleton, then it is **single-valued**,
- $\Phi(R) \neq \emptyset$, then it is **non-empty-valued**,
- $\Phi(R)$ is a singleton in welfare terms, then it is **essentially single valued**.[4]

A **rule** is a non-empty-valued correspondence $\Phi : \mathcal{R}^N \rightrightarrows \mathbb{R}^N \times S^N$ that is essentially single-valued. Given a rule Φ, a **selection** from Φ is a function $\varphi : \mathcal{R}^N \to \mathbb{R}^N \times S$ such that for each $R \in \mathcal{R}^N$, $\varphi(R) \in \Phi(R)$. We write $\varphi \in \Phi$ to indicate that φ is a selection from Φ. Since rules are non-empty-valued, we conflate the function $\varphi : \mathcal{R}^N \to \mathbb{R}^N \times S^N$ with the single-valued rule $\Phi : \mathcal{R}^N \rightrightarrows \mathbb{R}^N \times S^N$ defined by the equation $\Phi(R) = \{\varphi(R)\}$.

The properties of rules that we study are more easily understood when defined for their selections. Therefore, all properties are defined on single-valued rules and extended as follows: Given a property **P** defined for single-valued rules, rule Φ satisfies **P** only if each selection $\varphi \in \Phi$ satisfies **P**.

3. Pseudo-market Equilibrium.

A fairly general phenomenon in resource allocation is that, even absent money, outcomes can be stated as equilibria of a pseudo-market: a market-like institution that generates prices in terms of an *ad hoc* monetary instrument and uses these to allocate resources (Shapley & Scarf (1974); Hylland & Zeckhauser (1979); He et al. (2015)). Our rule can be stated thus, and with the additional properties that prices are anonymous and each agent has the same budget. We do not impose *envy-freeness* when characterizing the rule, but He et al. (2015) show that these latter characteristics of a pseudo-market imply it, and thus the rule has stronger fairness properties than we ask for to begin with. Finally, pseudo-markets are of independent interest because of their similarity with the real economy.

3.1. **Definition.** The manager of the commons endows each agent $i \in N$ with w units of the numeraire, which has no value outside the allocation mechanism. We define the **demand correspondence** $\boldsymbol{D}$ for each preference relation $R_i \in \overline{\mathcal{R}}$ and price vector $p \in \overline{\mathbb{R}}_+^S$, by[5]

$$D(R_i, p) := \{(x_i, s) : p_s x_i \leq w \text{ and } (y_i, t)\ P_i\ (x_i, s) \implies p_t y_i > w\}.$$

Denote by D_S the projection of the demand correspondence onto the set of sites. The **site demand** $\boldsymbol{D_S(R_i, p)}$ indicates which site assignments are considered best by preference relation R_i at prices p.

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[4]For each pair (x, α) and (y, β) in $\varphi(R)$, and each agent $i \in N$, $(x, \alpha)\ I_i\ (y, \beta)$.

[5]Let $\overline{\mathbb{R}}_+$ denote $\mathbb{R}_+ \cup \{\infty\}$.

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To understand the interaction between demand and supply, assume for this paragraph that $f \equiv \mathbf{1} \in \mathbb{R}^S$ for all sites, $w = 1$, and $c = (|N|, \dots, |N|)$ (Capacity Constraint is not binding). Given price vector $p \in \mathbb{R}^S_+$, the maximum quantity of commodity available to an agent at site $s \in S$ is $\frac{1}{p_s}$. Since the *numeraire* has no value outside the mechanism, and since the agent will eventually be assigned to a single site only, if $(x_i, s) \in D_i(R_i, p)$, $x_i = \frac{1}{p_s}$. To satisfy the Endowment Constraint at this site while respecting demands, the number of agents assigned to it, n_s, must satisfy $\frac{n_s}{p_s} \leq 1$. Thus, given this price, we can imagine there is a supply of $\lfloor p_s \rfloor$ discrete virtual units of site s commodity (with the remainder discarded). The problem is thus transformed into finding a one-to-one matching between the agents and $|S|$ types of objects, each of which may have several identical copies. Raising prices thus induces a larger supply of these virtual units; however, each unit yields less welfare to the consumer. The challenge we face then is to induce sufficiently many units to satisfy demands, but to retain as high a possible welfare value for each individual unit. Since we want to achieve this for general preferences, a simple, closed-form calculation is not available to us.

The fact that some "remainder" commodity might be discarded suggests that our notion of price equilibrium cannot impose exact material balance. This is a necessary relaxation: an implication of Theorem 1 below is that anonymous pricing is incompatible with the allocation of all resources.

Equilibrium: An equilibrium is a list $(p, x, \alpha) \in \overline{\mathbb{R}}^S_+ \times \mathbb{R}^N \times S^N$ such that (x, α) is feasible and, for each $i \in N$, $(x_i, \alpha(i)) \in D(R_i, p)$.

If price vector p admits an equilibrium allocation then we call it an **equilibrium price vector**. The set of equilibrium price vectors for profile R is $\mathbb{P}(R)$, where it may be the case that $\mathbb{P}(R) = \emptyset$.

3.2. **Existence.** Equilibrium need not exist, and when it does, it is typically not unique. For the applications we have in mind, existence is not an issue, whereas selection is a non-trivial problem. Nonetheless we must first make the former assertion formal, so in this subsection we introduce two simple properties, each of which, on its own, guarantees that equilibrium exists. Except when otherwise noted, the reader is free to assume his or her preferred property. Let us begin, however, with an example of an economy for which no equilibrium exists.

Example 1. Let $N = \{1, 2\}$, $S = \{a, b\}$, $c = (1, 2)$, and $f \equiv \mathbf{1}$. Assume $R_1 = R_2 =: R_0$, which satisfies $(0, a)\ P_0\ (1, b)$. Going to a is overwhelmingly preferred to b. Thus, we would try to set p_a high and p_b low. In particular, assuming $w = 1$, we can set $p_a = \infty$ and $p_b = 1$, in which case the agent will get no commodity if he chooses to go to a and the entire endowment at b if he chooses

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to go there. They both prefer to stand at a and consume nothing, but since the capacity at a is 1, this is not feasible. Clearly no price adjustment can remedy this.

Two easy ways to solve the problem become apparent: i) assume the sites are valued only for their commodity or ii) assume the capacity constraints do not bind. Either of these will suffice.

The following assumption is a restriction on the preference domain, $\mathcal{R}$.

Zero-Commodity Indifference: For each $R_i \in \mathcal{R}$, and each pair of sites s and s', $(0, s)\ I_i\ (0, s')$.

This preference restriction says that if an agent will not consume any resource, regardless of where she goes, then she does not care which site she is assigned. We consider this appropriate for natural resource applications and many others, like computer time sharing, where there is no value to simply occupying a location. Assuming $\mathcal{R}$ satisfies *zero-commodity indifference*, we can find a finite equilibrium price by embedding our model into that of Alkan et al. (1991). First, map the consumption space from $\mathbb{R}_+ \times S$ to $\mathbb{R} \times S$ via the logarithm. Zero Commodity Indifference and continuity guarantee the satisfaction of condition (2) in Alkan et al. (1991). Then make imaginary copies of the sites so as to respect the capacity constraints. The result of Alkan et al. (1991) than guarantees a finite price such that the allocation (x, α) that results satisfies $\sum_{i \in N} x_i = M$, for $M > 0$. This allocation is not guaranteed to respect the Endowment Constraint, which is now that $\sum_{i \in \alpha^{-1}(s)} e^{x_i} \leq f\left(\left|\alpha^{-1}(s)\right|\right)$; however, it is valid for any $M > 0$. Thus, we may lower M, and continuity of preferences, together with *zero-commodity indifference*, implies the existence of the price we seek. Typically, there will be a much lower equilibrium price than the one found by this embedding, and in many cases, no simple embedding to previous work will find the price we shall recommend as our solution concept.

It is worthwhile here to introduce a particular class of preferences that satisfy *zero-commodity indifference*. We say a preference relation $R_0 \in \overline{\mathcal{R}}$ is **linear** if there is a positive vector $v_0 \in \mathbb{R}^S_{++}$ such that the function $(x_0, s) \mapsto v_{0s} x_0$, represents R_0. Let $\mathcal{L} \subset \overline{\mathcal{R}}$ be the set of linear preference relations.

The alternative assumption is to restrict the environment.

Sufficient Capacity: For each site s, $c_s \geq |N|$.

This environment restriction eliminates the capacity constraint. This may be compelling in the case of club membership, or computer time sharing, where an agent's presence takes no physical space; only their usage taxes the resource. The sufficient capacity condition guarantees that price vector $p = |N| \left({}^{1}/{}_{f_s(1)w} \right)_{s \in S}$ admits an equilibrium with each agent spending his entire endowment w

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of numeraire to get $f_s(1)/|N|$ of some commodity. Again, we will demonstrate how to find a much lower price vector.

When the economy satisfies *sufficient capacity*, it is easy for us to consider preferences for which there is some value in going to a site, even when no resource is harvested. Consider, for example, a set of mutually exclusive clubs, each of which has money it divides among its members. We may find reason to consider preferences $R_0 \in \overline{\mathcal{R}}$ for which there is a vector $v \in \mathbb{R}^S$ such that the function $(x, s) \mapsto v_s + x$ represents R_0. In this case, v_s represents the value of membership in group $s \in S$ and x is the amount of money the agent receives. We call such preferences **translation invariant** and collect them in set $\mathcal{Q} \subseteq \overline{\mathcal{R}}$.

Summary. If the setting satisfies *either* zero commodity indifference *or* sufficient capacity, equilibrium in finite prices always exists. Henceforth, if we do not make clear which of these conditions hold, it means that either is acceptable.

Remark 1. We mention briefly one further assumption that can guarantee the existence of equilibrium. We shall *not* impose this assumption as it cannot guarantee that equilibrium prices will be finite. If there is a special site $\emptyset \in S$ with capacity $|N|$ and the property that, for each agent $i \in N$ and each site $s \in S$, $(0, \emptyset)\ R_i\ (0, s)$, then the problem of Example 1 is solved by having one of the agents consume at $\emptyset$. Clearly site $\emptyset$ is an outside option, of sorts, but because we do not have money or transferrable utility in this model, we cannot truly price the value of participation. More importantly, we cannot use this site to find entry fees for the other sites: in Example 1, either the entry fee to site a precludes both agents from entering, or, if it leaves any *numeraire* for them to spend, they still both demand $(0, a)$. In Section §5, where money is brought back in to the model, this is possible, but yields a more complicated price space (not a lower lattice) and so is left to future work.[6]

3.3. **Auction Implementation of Equilibrium.** We now discuss how equilibrium might be calculated in practice via an auction-like mechanism for the class of linear preferences, $\mathcal{L} \subseteq \overline{\mathcal{R}}$. It will be clear from our exposition how the procedure can be adapted for the translation invariant preferences $\mathcal{Q}$. Our main results of the paper, namely the existence and characterization of an optimal psuedo-market equilibrium, hold for general preferences. Focusing on a restricted class of preferences allows us to adapt the Exact Auction Mechanism of Demange et al. (1986). For more general

[6]I thank an anonymous referee for the suggestion to consider two-part tariffs. Work in that direction improved Lemma 6.

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preferences, we might consider adapting the Directional Demand procedure of Alkan (1992), but the mechanism is significantly more complicated and so beyond the scope of this paper. It also asks the agents to reveal, at certain key points of the procedure, information that is equivalent to the gradient of the utility function. Maintaining simplicity allows us to use the traditional procedure wherein the auctioneer calls out a price and the participants merely report their demand at that price.

An auction is a type of game form used to implement allocation rules that are based on prices. The messages of an auction are demand schedules. In this model, a demand schedule for agent i is a function $D_i : \overline{\mathbb{R}}_+^S \to 2^S$ such that there exists $R_i \in \overline{\mathcal{R}}$ satisfying $D_i(\cdot) = D_S(R_i, \cdot)$. The set of demand schedules is $\mathcal{D}$. It is desirable that, rather than reporting their entire demand schedule, agents instead report their demands in response to a smaller list of prices. Typically, a price $q \in \mathbb{R}^S$ is announced and the reported demands, $(D_i(q))_{i\in N}$, then determine the next price asked, thus making the auction game an extensive form.

Since the auction proceeds in discrete time, with a discrete jump in price for each period, we consider a discretized version of the model. Let $\delta > 1$ be near 1, and let $\mathcal{L}^\delta \subseteq \mathcal{L}$ be the set of linear preferences v_0 such that, for each $s \in S$, v_{0s} is an integer power of δ. Recall that since linear preferences are represented by the utility function $(x_0, s) \mapsto v_{0s}x_0$, and since preferences are ordinal, taking the logarithmic transformation yields an equivalent model. Thus, we can study the utility function $(x_0, s) \mapsto \log_\delta v_{0s} + \log_\delta x_0$. The following procedure is derived from this view. We propose a price dynamic $p(\cdot)$ whose increment satisfies, for each time $\tau \in \mathbb{N}$, each demand profile $D \in \mathcal{D}^N$, and each site $s \in S$, either $p_s(\tau + 1) = p_s(\tau)$ or $\delta p_s(\tau)$.

Fix $D \in \mathcal{D}^N$. For each price vector $q \in \mathbb{R}^S$, we construct a bipartite graph $\Gamma(q)$ in which we seek to match agents with "copies" of the sites. Given prices p, let $\tilde{c}_s(p) \in \mathbb{Z}$ denote the greatest integer n satisfying

$$\frac{w}{p_s} \leq \frac{f_s(n)}{n}.$$

The **implied capacity of site** s **given prices** p, defined as $\mathsf{c}_s(p) := \min\{c_s, \tilde{c}_s(p)\}$, is the largest number of agents that can be assigned to s under price p_s. Note that if $p_s f_s(1) < w$ then $\mathsf{c}_s(p) = 0$. For each $s \in S$, construct the set $s^\star := \{s_1^\star, \ldots, s_{\mathsf{c}_s(q)}^\star\}$. Let $S^\star := \cup_{s\in S} s^\star$. The two vertex sets of $\Gamma(p)$ are N and $S^\star$. The edge $(i, s_k^\star) \in \Gamma(p)$ if and only if $s \in D_i(p)$. Clearly, if we can find a matching on this graph, then we can find an equilibrium in the economy. Recall that Hall's theorem tells us a

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matching exists if and only if, for each subset $N' \subseteq N$, $\left|\left\{s_k^\star : \exists i \in N',\ (i, s_k^\star) \in \Gamma(q)\right\}\right| \geq |N'|$. Given the way we have constructed the graph, we can simplify this inequality to $\sum_{s \in \cup_{i \in N'} D_i(q)} \left|s^\star\right| \geq |N'|$.

A set of sites $S' \subseteq S$ is **overdemanded at** q if the number of agents who require S' to be satisfied exceeds the number of units available. Formally, $\sum_{s \in S'} \left|s^\star\right| < |\{i \in N : D_i(q) \subseteq S'\}|$. Let $N' \subseteq N$ and $S' = \cup_{i \in N'} D_i(q)$. If there are no overdemanded sets, then

$$\sum_{s \in \cup_{i \in N'} D_i(q)} \left|s^\star\right| = \sum_{s \in S'} \left|s^\star\right| \geq |\{i \in N : D_i(q) \subseteq S'\}| \geq |N'|,$$

and so Hall's theorem guarantees a matching in the graph, and *a fortiori*, an equilibrium. Thus, we should try to find a price path that progressively eliminates overdemanded sets.

Note, however, that until equilibrium is found, all sites are overdemanded, and thus it is clear that simply raising the price of *all* overdemanded sites will not succeed. The standard procedure (Demange et al., 1986; Andersson et al., 2013; Andersson & Erlanson, 2013), which continues to work here, is to raise the prices of a set of items that is not merely overdemanded but minimally so: the set contains no proper subsets that are also overdemanded. A site $s \in S$ is **minimally overdemanded (MOD)** if it is a member of a minimally overdemanded set. Define

$$\Delta_s(D, q) := \begin{cases} \delta & s \text{ is minimally overdemanded at } q \\ 1 & \text{otherwise.} \end{cases}$$

Let $p : \mathcal{D}^N \times \mathbb{R} \to \mathbb{R}^S$ be given by the equation

$$p(D, \tau + 1) = \Delta(D, p(\tau))\, p(D, \tau). \tag{1}$$

Note that the logarithmic transform of preferences are quasilinear in the logarithm of the commodity. The quantity of commodity, moreover, takes the form w/p_s. Thus, by incrementing the price exponentially, we have constructed a problem with effectively quasilinear preferences. It is easy to verify, using techniques developed below, that if $R \in \mathcal{L}^\delta$, then there is $p \in \mathbb{P}(R)$ that conforms to the logarithmic grid induced by $\mathcal{L}^\delta$. Thus, we let $\mathbb{P}^\delta(R) \subseteq \mathbb{P}(R)$ denote the set of such prices.

Before presenting the general result, we work through a simple example.

Example 2. There are three consumers and two sites. Each site has the same production function:

$$f(n) := n \exp\left(\frac{4 - n}{3}\right).$$

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	Agents 1 and 2			Agent 3			Implied Capacities		
$6\hat{p}$	$\hat{v}_1(1)+\hat{w}-\frac{\hat{p}}{6}$	$\hat{v}_1(2)+\hat{w}-\frac{\hat{p}}{6}$	$D_S(v_1,p)$	$\hat{v}_2(1)+\hat{w}-\frac{\hat{p}}{6}$	$\hat{v}_2(2)+\hat{w}-\frac{\hat{p}}{6}$	$D_S(v_3,p)$	$\mathsf{c}_1(p)$	$\mathsf{c}_2(p)$	MODs
(6,6)	$\frac{2}{6}+2-1=\frac{8}{6}$	$\frac{1}{6}+2-1=\frac{7}{6}$	$\{1\}$	$\frac{1}{6}+2-1=\frac{7}{6}$	$\frac{1}{6}+2-1=\frac{7}{6}$	$\{1,2\}$	1	1	1
(7,6)	$\frac{2}{6}+2-\frac{7}{6}=\frac{7}{6}$	$\frac{1}{6}+2-1=\frac{7}{6}$	$\{1,2\}$	$\frac{1}{6}+2-\frac{7}{6}=1$	$\frac{1}{6}+2-1=\frac{7}{6}$	$\{2\}$	1	1	1, 2
(8,7)	$\frac{2}{6}+2-\frac{8}{6}=1$	$\frac{1}{6}+2-\frac{7}{6}=1$	$\{1,2\}$	$\frac{1}{6}+2-\frac{8}{6}=\frac{5}{6}$	$\frac{1}{6}+2-\frac{7}{6}=1$	$\{2\}$	2	1	$\emptyset$

Table 1. An Example Auction

Consumers 1 and 2 have $v_1 = v_2 = (e^{2/6}, e^{1/6})$. Consumer 3 has $v_3 = (e^{1/6}, e^{1/6})$. The price space is $\{e^{\hat{p}/6} : \hat{p} \in \mathbb{Z}\}$, and throughout this example $\hat{x} := \log x$. Table 1 shows an execution of our auction. We distribute $w = e^2$ units of numeraire to each agent and open the bidding at price vector (e, e). The auction terminates at prices $(e^{8/6}, e^{7/6})$. The resource at site 1 is completely consumed, whereas the resource at site 2 is used at the rate $\frac{e^{5/6}}{e} \approx .86$.

The following proposition shows the convergence of the procedure, conditional on a low-enough initial price.

Proposition 1. *Assume $\mathbb{P}^\delta(R)$ contains $p \gneq p(0)$. Assume also that each $p_s(0)$ is an integer power of δ. Assume agents report truthfully: for each $i \in N$, $D_i(\cdot) = D(R_i, \cdot)$. Then, the auction terminates at a finite equilibrium price that is lower, in the vector partial order, than the others in $\mathbb{P}^\delta(R)$.*

Proof. Let $q \in \mathbb{P}^\delta(R)$.

Claim 1. For each $\tau \in \mathbb{R}_+$, $p(D, \tau) \leq q$.

Proof. Let $q' \leq q$ and let $S^{eq} := \{s \in S : q'_s = q_s\}$. Let $N^{eq} := \{i \in N : D_i(q') \cap S^{eq} \neq \emptyset\}$. If $s \in S^{eq}$ and $s \in D_i(q')$ then $s \in D_i(q)$. In fact,

$$\text{if } D_i(q') \cap S^{eq} \neq \emptyset, \text{ then } D_i(q) \subseteq D_i(q') \cap S^{eq}, \tag{2}$$

as the welfare attainable via any site $s' \notin S^{eq}$ must have decreased. Suppose, to arrive at a contradiction, that $S' \subseteq S^{eq}$ were overdemanded at q'. Then line 2 yields that $D_i(q') \subseteq S'$ implies $D_i(q) \subseteq S'$ and we derive the following inequality:

$$|\{i \in N :\ D_i(q) \subseteq S'\}| \geq \left|\{i \in N :\ D_i(q') \subseteq S'\}\right| > \sum_{s \in S'} \left|s^\star\right|.$$

This, via Hall's theorem, implies that q is not an equilibrium price, a contradiction. Conclude that no subset of S^{eq} is overdemanded at q'.

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We must also show that each $S' \subseteq S^{eq}$ does not belong to any minimally overdemanded set at q'. Let $\hat{S} \subseteq S$ and $\hat{N} := \left\{i \in N : \ D_i(q') \subseteq \hat{S}\right\}$. Since q is an equilibrium price,

$$\left|\hat{N} \cap N^{eq}\right| \leq \sum_{s \in \cup_{i \in \hat{N} \cap N^{eq}} D_i(q)} \left|s^\star\right|$$

(by line 2)
$$\leq \sum_{s \in S^{eq} \cap \left(\cup_{i \in \hat{N} \cap N^{eq}} D_i(q')\right)} \left|s^\star\right|$$
$$\leq \sum_{s \in S^{eq} \cap \hat{S}} \left|s^\star\right|.$$

If $\hat{S}$ is overdemanded,

$$\begin{aligned} \left|\hat{N}\right| &> \sum_{s \in \hat{S}} \left|s^\star\right| \\ &= \sum_{s \in \hat{S} \setminus S^{eq}} \left|s^\star\right| + \sum_{s \in S^{eq} \cap \hat{S}} \left|s^\star\right| \\ &\geq \sum_{s \in \hat{S} \setminus S^{eq}} \left|s^\star\right| + \left|\hat{N} \cap N^{eq}\right|. \end{aligned}$$

We deduce that $\left|\hat{N} \setminus N^{eq}\right| > \sum_{s \in \hat{S} \setminus S^{eq}} \left|s^\star\right|$ and conclude that $\hat{S} \setminus S^{eq}$ is overdemanded. Recalling from the previous paragraph that $\hat{S} \cap S^{eq}$ is not overdemanded, we therefore have that no element $s \in S^{eq}$ is part of any minimally overdemanded set. □

By construction, the auction will proceed until an equilibrium is reached. Since each $q \in \mathbb{P}^\delta(R)$ is an upper bound for $p(D, \cdot)$, there will be a time t for which S^{eq} becomes non-empty, after which the prices of these objects remain constant until $S^{eq} = S$. It follows that the price to which $p(D, \cdot)$ converges is the least element of $\mathbb{P}^\delta(R)$, in the vector partial order. ■

Our auction has identified the unique minimal vector in $\mathbb{P}^\delta(R)$, so we have accomplished our heuristic goal of maximizing welfare. In Section 3.4 we show that, in fact, there is a minimal element of the continuous price space $\mathbb{P}(R)$, which we henceforth denote $p^*(R)$. Theorem 2 below implies that when $R \in \mathcal{L}^\delta$, then $p^*(R)$ is also the minimal element of $\mathbb{P}^\delta(R)$, so the auction procedure calculates the ideal minimal price when preferences are discrete-linear.

In Section §4, for general preferences, we demonstrate that giving agents an equilibrium allocation with respect to $p^*(R)$ is essentially the *only* way to achieve our goal. However, to make this

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more clear, we ought to abstract from procedural questions and focus on the outcome alone. Let $\mathcal{A}^*(R)$ be the set of site assignments α such that for some $x \in \mathbb{R}^N$, $(p^*(R), x, \alpha)$ is an equilibrium.

The Min-Price Rules: Given parameters $c \in \mathbb{Z}_+^S$ and $(f_s)_{s \in S} : \mathbb{N} \to \mathbb{R}$, let $\mathcal{R} \subseteq \overline{\mathcal{R}}$ satisfy, for each $R \in \mathcal{R}^N$, $\mathbb{P}(R) \neq \emptyset$. Let $R \in \mathcal{R}^N$ and let (x, α) and (y, β) be two equilibria supported by price $p^*(R)$. Then (x, α) and (y, β) are Pareto indifferent for profile R. It follows that the mapping $F^* : \mathcal{R}^N \rightrightarrows \mathbb{R}_+^S \times S^N$ given for each profile $R \in \mathcal{R}^N$ by

$$F^*(R) = \left\{ (x, \alpha) : \alpha \in \mathcal{A}^*(R),\ x_i = \frac{w}{p^*_{\alpha(i)}(R)} \right\}$$

is a rule. We call F^* the *min-price rule on* $\mathcal{R}^N$.

3.4. **The Structure of Equilibrium Prices.** We said above that when equilibrium prices exist, they are typically not unique. We overcomplicate the following simple example to better make the connection between special cases of our model and traditional selling mechanisms with quasilinear preferences:

Example 3. Let $N = \{1, 2\}$, $S = \{a, b\}$ and $f \equiv (10, 10)$. Assume sufficient capacity. Let $R \in \mathcal{Q}$, so for each agent $i \in N$ there is a vector $v \in \mathbb{R}^S$ such that the mapping $(x, s) \mapsto v_s + x$ represents R_i. Assume then that $v_1 = (1, 1)$ and $v_2 = (2, 1)$. Clearly, agent two will be indifferent between bundles of the form (x, a) and $(x + 1, b)$, subject to both being feasible. Thus, $((x + 1, b), (x, a))$ is an equilibrium allocation if $0 < x < 10$, and for ε sufficiently small, so is $((x + 1 + \varepsilon, b), (x + \varepsilon, a))$. If a and b were discrete objects and the first component were money, this would be the familiar observation that equilibrium prices in the assignment game or auction mechanisms (ignoring individual rationality) are parallel shifts of each other. The reader can verify that the formula this parallel shifting induces for our psuedo-market prices is $\left(p_1, \frac{wp_1}{w+p_1}\right)$ (ignoring boundary issues).

For discrete linear preferences, we have already identified the correct selection from $\mathbb{P}^\delta(R)$. However, for general preferences, the set of equilibrium prices has a global structure: we show in this section that there is a unique minimal price $p^*(R)$ so long as $\mathbb{P}(R)$ is non-empty.

Given two elements, p and $p' \in \mathbb{R}^S$, let $p \wedge p'$ be the component-wise minimum of p and p'. That is, for each s, $(p \wedge p')_s := \min\{p_s, p'_s\}$. If $A \subset \mathbb{R}^k$, and if for each pair $\{p, p'\} \subseteq A$, $p \wedge p' \in A$, then we say the pair $(A, \wedge)$ is a **lower semi-lattice**.

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In the following theorem, we do *not* assume equilibrium exists. Accordingly, neither zero commodity indifference nor sufficient capacity are assumed. If $\mathbb{P}(R)$ is empty, the theorem holds vacuously.

Theorem 1. *For each $R \in \mathcal{R}^N$, $(\mathbb{P}(R), \wedge)$ is a lower semi-lattice.*

The proof of the theorem is in Appendix Section §A. We have the immediate corollary:

Corollary 1. *F^* is well-defined on any domain $\mathcal{R} \subseteq \overline{\mathcal{R}}$ where $\mathbb{P}$ is non-empty.*

3.4.1. *Discussion: Why lattice results of previous work do not apply here.* Lattice structures have been observed in many discrete assignment models, with and without the presence of divisible resources. For objects-and-money, Demange and Gale's (1985) model of one-to-one matching is readily adapted to the case where one side is not agents but objects. We show here that previous techniques cannot successfully be applied this model, even for the special case of extractive production ($f \equiv \mathbf{e} \in \mathbb{R}^S$).

The most stark difference between our model and previous work is that we do *not* in fact have an upper lattice, only a lower lattice. The following example illustrates why.

Example 4. Let $S := \{a, b, c\}$, $N := \{1, 2\}$, $f \equiv (1, 4, 4)$ and $c := (2, 2, 2)$. Let $w = 1$ and let preferences be linear, represented by vectors v_1 and $v_2 \in \mathbb{R}^S$. Let $v_1 := v_2 := (2, 1, 1)$. First, observe that $p := (1, \frac{1}{2}, 1)$ and $q := (1, 1, \frac{1}{2})$ are both equilibrium prices. In particular, $2 \cdot {}^{w}/_{p_b} = 4 \leq e_b$, and

$$v_1(b)\frac{w}{p_b} = 2 = v_1(a)\frac{w}{p_a}.$$

However, the component-wise minimum of p and q is $(1, 1, 1)$, and, for each $i \in N$, $D(R_i, (1, 1, 1)) = \{(1, a)\}$.

Nonetheless, this leaves open the question of whether or not our lower lattice result is a corollary of previous work. When confronted with our problem, one might first attempt to solve it by, *ex ante*, making "copies" of the sites. In fact, this is precisely the technique that we used in showing that the auction converged. However, the auction price path is, except for the terminal price, entirely outside the equilibrium set. To show the lattice property of $\mathbb{P}$, this approach fails.

Example 5. Let $S := \{a, b\}$, $N := \{1, 2, 3\}$, $f \equiv (1, 1)$, and $c := (3, 2)$. Let $R_0 \in \overline{\mathcal{R}}^N$ satisfy $({}^1/_3, a)\ P_0\ (1, b)$. Clearly, the min-price equilibrium of the profile (R_0, R_0, R_0) has all agents consuming at site a. Moreover, it must have all agents consuming the same quantity. Thus, to find this equilibrium

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via the "making copies" method, we must make three copies of site a, $\{a^1, a^2, a^3\}$, and give each a^k endowment $e_a/3$.

Consider a preference relation R'_0 that is indifferent between consumption at sites a and b: for each $x \in \mathbb{R}$, $(x, a)\ I'_0\ (x, b)$. It is easy to verify that the minimal equilibrium price for profile (R'_0, R'_0, R'_0) is $(2, 2)$. With this price, at equilibrium, an agent goes to site a and consumes $1/2$. However, we deduced above that, by making *ex ante* copies, the most he could consume is $1/3$.

Thus, making *ex-ante* copies sometimes prevents us from finding the truly minimal price, and so a new approach is necessary. However, we could perhaps have a simpler proof by the proper use of copies. We would begin with two equilibrium prices p and q and attempt to show that $p \wedge q$ is an equilibrium price. The Demange and Gale proof begins with the following *Decomposition Lemma*.

Lemma (Demange and Gale)**.** *Let (p, x, α) and (q, y, β) be equilibria for economy $R \in \mathcal{R}^N$. Define the following sets:*

$$N^p := \{i \in N : (x, \alpha)\ P_i\ (y, \beta)\} \qquad S^p := \{s \in S : q_s < p_s\}$$

$$N^q := \{i \in N : (y, \beta)\ P_i\ (x, \alpha)\} \qquad S^q := \{s \in S : p_s < q_s\}$$

Then $\alpha(N^p) \subseteq S^p$, $\beta(N^p) \subseteq S^p$, $\alpha(N^q) \subseteq S^q$ and $\beta(N^q) \subseteq S^q$.

In fact, this lemma fails in our class of models, as we show with the following example.[7]

Example 6. The environment is the same as in Example 5, except we set $f \equiv (\lambda, 1)$, with $\lambda > 1$. Assuming $w = 1$, we may be sure that, for each site $s \in S$, we need $\lfloor (p \wedge q)_s \rfloor$ copies to find an equilibrium supported by $p \wedge q$.

Let $R \in \mathcal{L}^N$ be a profile of linear preferences with preference coefficients $v_1 = (1, 1/2)$, $v_2 = (1/2, \lambda)$, and $v_3 = (1, 1)$. Let $p = (4, 2)$ and $q = (2\lambda^{-1}, 4)$. Clearly, $D_S(R_1, q) = \{a\}$, and since

$$v_{1,a} \cdot \frac{w}{p_a} = 1 \cdot \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = v_{1,b} \cdot \frac{w}{p_b},$$

$D_S(R_1, p) = \{a, b\}$. Symmetrically, $D_S(R_2, p) = \{b\}$ and, since

$$v_{2,b} \cdot \frac{w}{q_b} = \lambda \cdot \frac{1}{4} = \frac{1}{2} \cdot \frac{\lambda}{2} = v_{2,a} \cdot \frac{w}{q_a},$$

[7]Our statement of the Lemma implies an embedding in which sites have preferences over the price of their commodity, with higher prices better. Alternatively, we could imbue sites with preferences over how much of their commodity is distributed, with less being better. The example below remains a counterexample in this case.

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$D_S(R_2, p) = \{a, b\}$. Two site assignments α and β are given in the following table:

	1	2	3
α	a	b	b
β	a	b	a

It is clear that α is an equilibrium assignment for price p and β is an equilibrium assignment for price q. The Decomposition Lemma is violated as $3 \in N^q$, but $\alpha(3) = b \in S^q$ and $\beta(3) = a \in S^p$.

3.5. **Seeking Min-price Equilibria: Necessary and Sufficient Conditions.** If the resource produced at a site is completely distributed, we say the site is **exhausted**. Theorem 2 below implies that, at any min-price equilibrium, at least one site is exhausted. What prevents all sites from being exhausted are the preference relations of those consuming at exhausted sites. To see why, consider a simple example.

Example 7. Let $S := \{a, b\}$, $N := \{1, 2, 3\}$, $f \equiv (1, 1)$, and $c := (3, 2)$. For the pseudo-market, give each agent $w = 1$ unit of numeraire. Let $R \in \mathcal{L}^N$ be a profile of linear preferences given by vectors $v_1 = (3, 1)$, $v_2 = (1, 1)$, and $v_3 = (1, 3)$. The minimal equilibrium price vector for this profile is $(2, 2)$ and note that $D_S(R_2, (2, 2)) = \{a, b\}$. Consider site assignment $\alpha \in S^N$ given by $\alpha(1) = \alpha(2) = a$ and $\alpha(3) = b$. Site a is exhausted, while agent 3 consumes only half of site b's endowment. We would therefore like to lower the price of site b. For $\varepsilon > 0$, consider price vector $(2, 2 - \varepsilon)$. At this price, agents 1 and 3 still choose to consume at sites a and b respectively. However, agent 2 is no longer indifferent; $D_S(R_2, (2, 2 - \varepsilon)) = \{a\}$. Then the consumption of site a's resource is $2\left(\frac{w}{2-\varepsilon}\right) = \frac{2}{2-\varepsilon} > 1 = f_a(2)$, violating feasibility.

The tension in the previous example can be transmitted via chains of indifference in economies of arbitrary size. These chains decide the equilibrium price list. Let $(x, \alpha) \in F^*(R)$. If there is a statement of the form

$$\left(x_{i^1}, \alpha(i^1)\right) I_{i^1} \left(x_{i^2}, \alpha(i^2)\right) I_{i^2} \cdots I_{i^n} \left(x_i, \alpha(i)\right),$$

then site $\alpha(i)$ is **blocked via indifference by site** $\alpha(i^1)$. Theorem 2 below shows that each site is blocked via indifference by an exhausted site, where we include in this terminology the case that the site itself is exhausted.

Only if $w\mathsf{c}_s(p) = p_s f_s(\mathsf{c}_s(p))$ is it possible for s to be exhausted at p. We will refer to such sites as having **endowment-divisible value**. Given $p \in \mathbb{P}(R)$, say that $\alpha \in N^S$ is **balanced** if, 1) for each site s having endowment-divisible value, $\left|\alpha^{-1}(s)\right| \geq \mathsf{c}_s(p) - \mathbb{1}[c_s \geq \tilde{c}_s(p)]$, and 2) for every other site t, $\left|\alpha^{-1}(t)\right| = \mathsf{c}_t(p)$. An equilibrium is balanced if its associated site-assignment is balanced.

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Recall that either zero-commodity indifference or sufficient capacity is sufficient for F^* to always be non-empty-valued. The following result, however, regards a single economy $R \in \overline{\mathcal{R}}$ only, in which case $F^*(R)$ may be non-empty despite neither of the conditions holding.

Theorem 2. *If $F^*(R)$ is non-empty, then it contains a balanced equilibrium. At each $(x, \alpha) \in F^*(R)$, whether balanced or not, each site is blocked via indifference by an exhausted site. Finally, if (p, x, α) is a balanced equilibrium for economy $R \in \overline{\mathcal{R}}^N$ such that each site is blocked via indifference by an exhausted site, then $(x, \alpha) \in F^*(R)$.*

3.6. **Welfare and comparative statics of Min-price Equilibria.** As is typical of incentive compatible rules, the outcome of a min-price rule is not always *Pareto efficient*.[8] It is efficient in a limited sense: there is no benefit to *ex post* re-trading among the agents. Unfortunately, for many preference profiles, there will be no allocations in F^* that distribute all of the social endowment. When the economy satisfies *sufficient capacity*, we can calculate the quantity of undistributed resources at balanced equilibria and find that it is small, asymptotically. In particular, the amount discarded of each resource is bounded above by $f_s(k)/k+1$ if k people consume the resource.

Theorem 3. *Assume sufficient capacity. For each profile $R \in \overline{\mathcal{R}}^N$, there is an allocation $(x, \alpha) \in F^*(R)$ such that for each site s,*

$$\sum_{i \in \alpha^{-1}(s)} x_i \geq \left(\frac{\left|\alpha^{-1}(s)\right|}{\left|\alpha^{-1}(s)\right| + 1} \right) f_s\left(\left|\alpha^{-1}(s)\right| \right).$$

Proof. Since we assume *sufficient capacity*, F^* is defined on $\overline{\mathcal{R}}^N$. Let $(x, \alpha) \in F^*(R)$ be such that α is balanced. Let $p = p^*(R)$. If site s has endowment-divisible value at p, then since α is balanced, $\left|\alpha^{-1}(s)\right| + 1 \geq \mathsf{c}_s(p)$. Endowment-divisibility implies $w\mathsf{c}_s(p) = p_s f_s(\mathsf{c}_s(p))$, so $\left|\alpha^{-1}(s)\right| + 1 \geq p_s f_s(\mathsf{c}_s(p)) w^{-1}$. Dividing both sides by $\left|\alpha^{-1}(s)\right|$, and plugging in $\left(\left|\alpha^{-1}(s)\right| w\right) p_s^{-1} = \sum_{i \in \alpha^{-1}(s)} x_i$, gives

$$\frac{\left|\alpha^{-1}(s)\right| + 1}{\left|\alpha^{-1}(s)\right|} \geq \frac{f_s(\mathsf{c}_s(p))}{\sum_{i \in \alpha^{-1}(s)} x_i}. \tag{3}$$

If site s is not endowment-divisible, $\left|\alpha^{-1}(s)\right| = \mathsf{c}_s(p)$ and so

$$\left|\alpha^{-1}(s)\right| + 1 = \mathsf{c}_s(p) + 1 > \frac{p_s f_s(\mathsf{c}_s(p))}{w}.$$

[8]Recall that Vickrey-Clarke-Groves mechanisms are typically not budget balanced.

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Again dividing through by $\left|\alpha^{-1}(s)\right|$ gives line 3. The result then follows from the monotonicity of f_s and the fact that $\left|\alpha^{-1}(s)\right| \leq \mathsf{c}_s(p)$. ■

We now analyze the response of prices to changes in preferences, for which we no longer need to assume *sufficient capacity*. The result is a theorem that provides useful tools for comparative statics. We later use these tools to demonstrate the incentive properties of min-price rules.

For analytical precision, we confine ourselves to a class of preference transformations that represent an unambiguous strengthening or weakening of preference for a given site or set of sites. Let $R_i \in \overline{\mathcal{R}}$ and $S' \subseteq S$. For each $d \in \mathbb{R}$, define $\boldsymbol{R_i^{S',d}}$ so that $R_i^{S',d}\Big|_{\mathbb{R}\times S\setminus S'} = R_i\Big|_{\mathbb{R}\times S\setminus S'}$ and for each $s \in S'$ and each $t \in S \setminus S'$,

$$(x, s)\ R_i\ (y, t) \implies (x - d, s)\ R_i^{S',d}\ (y, t).$$

We say that $R_i^{S',d}$ is a **site-translation through set** S', or an S'**-translation**, of R_i. If $S' = \{s\}$, we simply write s**-translation**. If $d > 0$ we call the translation **positive**. We consider this the positive direction because it represents an increased preference for s relative to other sites. In fact, for any $x_i \in \mathbb{R}$ and any $s \in S$, the lower contour set of $R_i^{s,d}$ at (x_i, s) contains the lower contour set of R_i at (x_i, s).

Note that, generally, $R_i \in \mathcal{R}$ does not guarantee $R_i^{s,d} \in \mathcal{R}$. In particular, when $\mathcal{R}$ satisfies *zero commodity indifference*, the only translations that remain in the domain are the identity translations. However, it is often the case that, if F^* is defined on domain $\mathcal{R}^N$, then it is also defined on $\left(\mathcal{R} \cup \{R_i^{s,d}\}\right)^N$. As this subsection and the next study the properties of F^*, we welcome any expansion of the domain, as it is better to show more general properties. In the next section, when we characterize F^*, the opposite is true: the smaller the domain on which F^* is the unique rule of interest, the more powerful the result.

We collect in a lemma some important properties of p^*.

Lemma 1. *Fix $i \in N$ and $s \in S$. Assume $\mathbb{P}(R)$ is non-empty. Define the function π for each $a \in \mathbb{R}$ by $\pi(a) := p^*\left(R_i^{s,a}, R_{-i}\right)$.*

Property 1: *π is non-decreasing where defined*

Property 2: *If $s \notin D_S\left(R_i, p^*(R)\right)$, then there exists $\bar{d} > 0$ such that for each $d < \bar{d}$, $\pi(d) = \pi(0)$.*

The following lemma, proved in the appendix, is important in showing that groups cannot manipulate min-price rules. In particular, no group of agents can lower the price of *all* the sites at which they consume, though they may be able to lower the prices of *some*.

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Lemma 2. *Let* $N' \subseteq N$, $R \in \overline{\mathcal{R}}^N$, *and* $R'_{N'} \in \mathcal{R}^{N'}$. *Assume* $\mathbb{P}(R)$ *and* $\mathbb{P}(R'_{N'}, R_{N\backslash N'})$ *are non-empty. Let* $\alpha \in \mathcal{A}^*\left(R'_{N'}, R_{N\backslash N'}\right)$ *and* $d := 2\max_{s\in S} f_s(|N|)$. *Then* $p^*\left(\left(R_j^{\alpha(j),d}\right)_{j\in N'}, R_{N\backslash N'}\right)$ *is defined and there is a site* $t \in \alpha(N')$ *such that*

$$p_t^*\left(\left(R_j^{\alpha(j),d}\right)_{j\in N'}, R_{N\backslash N'}\right) \geq p_t^*(R).$$

3.7. **Incentive Compatibility of Min-price Equilibrium Rules.** Let $\mathcal{R}^N$ be a domain on which F^* is defined. The following incentive compatibility property is standard: no group of agents should strictly benefit by reporting false preferences.

Weak Group-strategy-proofness (w-GStP): For each $R \in \mathcal{R}^N$, each group $N' \subseteq N$, and each partial profile of preferences $R'_{N'} := \left(R'_i\right)_{i\in N'} \in \mathcal{R}^{N'}$, there is an agent $k \in N'$ such that

$$\varphi_k(R)\ R_k\ \varphi_k(R'_{N'}, R_{N\backslash N'}).$$

The tools of Lemma 1 and Lemma 2 make intuitive why min-price rules should have such nice incentive properties. Agents have limited influence over the prices of each site, and what influence they do have is the “appropriate” kind. In particular, an agent can only cause the minimal price of a resource to decrease if she abandons consumption there.

Theorem 4. F^* *is weakly group-strategy-proof.*

Proof. In what follows, the preferences of agents $N\backslash N'$ are held constant and therefore we suppress their notation.

Let $\varphi \in F^*$, and $R \in \mathcal{R}^N$. To arrive at a contradiction, assume there is a set $N' \subseteq N$ and a partial profile $R' := \left(R'_j\right)_{j\in N'}$ such that for each $k \in N'$,

$$(x_k, \alpha(k)) := \varphi_k(R') \quad P_k \quad \varphi_k(R).$$

Note that $\varphi(R')$ is an equilibrium for $R^d := \left(R_j'^{\alpha(j),d}\right)_{j\in N'}$ so long as $d > 0$. Therefore $p^*\left(R^d\right) \leq p^*(R')$. Assuming d is sufficiently large, each $k \in N'$, with preferences R_k^d, will feasibly choose only $\alpha(k)$ at prices $p^*\left(R^d\right)$. Thus, for each $k \in N'$, there is $\bar{x}_k$ satisfying $\bar{x}_k \geq x_k$ such that

$$\varphi_k\left(R^d\right) = (\bar{x}_k, \alpha(k)).$$

Therefore, R^d is also a joint manipulation for group N'. Now we apply Lemma 2: there is a site $t \in (\alpha(j))_{j\in N'}$ such that

$$p_t^*\left(R^d\right) \geq p_t^*(R),$$

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a contradiction. ■

Corollary 2. *The auction mechanism of Section 3.3 implements F^* in dominant strategies.*

4. Characterization

In this section we provide a characterization of min-price rules in terms of appealing properties. Our primary focus is on preference elicitation in the presence of a weakened form of *anonymity*. We show that, if $\mathcal{R}$ is sufficiently rich, then F^* is a supercorrespondence of any rule satisfying the following properties. Recall here that we define properties on single-valued rules and say that a correspondence satisfies a property if all its selections do.

The first property is implied by, and is much weaker than, *weak group-strategy-proofness*.

Strategy-proofness (StP): For each $R \in \mathcal{R}^N$, each $i \in N$, and each preference relation $R'_i \in \mathcal{R}$,

$$\varphi_i (R)\ R_i\ \varphi_i (R'_i, R_{-i}) .$$

In this model, as in many others, we may adapt the usual sequential priority procedure to both elicit preferences truthfully and achieve *Pareto efficiency*. The inequity of such rules is extreme and therefore they are inappropriate for the applications we have envisioned. We insist upon a form of *anonymity* that requires an agent's *welfare* depend only on his preferences and the unordered list of preference relations present in the economy. Note that this is weaker than the usual form, which requires an agent's *bundle* depend only on his preferences and the unordered list of preference relations present in the economy.

Welfare Anonymity (W-Anon): Let $R \in \mathcal{R}^N$ and let $\sigma : N \to N$ be a bijection. Let $i \in N$ and $\sigma(j) = i$. Then

$$\varphi_i (R)\ I_i\ \varphi_j \left(R_{\sigma(1)}, R_{\sigma(2)}, \ldots, R_{\sigma(|N|)}\right) .$$

Since the consumption space is compact and preferences continuous, the Hausdorff distance H^Δ between the graphs of any pair R and R' of preference relations generates a metric topology on the space of continuous preference relations. The following property is very mild, as it requires continuity in welfare space only in special cases. Consider a convergent sequence of profiles. Suppose the rule chooses the same allocation for all of the profiles on the sequence. Then in the limit profile, the agents are indifferent between what the rule chooses at the limit and what it has chosen all along the sequence.

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Constant Sequence Welfare Continuity (w-Cont): Let $(R^n)^{n\in\mathbb{N}} \subset \mathcal{R}^N$ be a sequence converging to R. Assume there is an allocation (x, α) such that for each $n \in \mathbb{N}$, $\varphi(R^n) = (x, \alpha)$. Then for each agent $i \in N$, $(x_i, \alpha(i))\ I_i\ \varphi_i(R)$.

In a model as rich as this, we should not expect *StP*, *w-Cont*, and *W-Anon* to identify a single rule. Rather than introduce further properties, however, we study the consequences of a rule being the *most efficient*, in a point-wise sense, in a class of rules. In general, we may define

Strong Undomination in $\mathfrak{C}$: Fix a class of rules $\mathfrak{C}$. Say that φ is *strongly undominated in* $\mathfrak{C}$ if for each $\psi \in \mathfrak{C}$ and each $R \in \mathcal{R}^N$

$$\psi_i(R)\ P_i\ \varphi_i(R) \implies \exists j,\ \varphi_j(R)\ P_j\ \psi_j(R).$$

We require that φ be *strongly undominated* in the class of rules satisfying our previous properties.

Strong Undomination in W-Anon, StP, w-Cont: Rule φ is *strongly undominated* in the class of rules defined on $\mathcal{R}^N$ satisfying *welfare anonymity*, *strategy-proofness*, and *constant sequence continuity*. Henceforth we refer to this property simply as *strong undomination*.

The smaller the set $\mathcal{R}$, the less bite our first three conditions have. If $\mathcal{R}$ is a singleton, then in fact *strategy-proofness* and *constant sequence continuity* are vacuous. Such is clearly not an interesting case. Instead, make the following assumptions about $\mathcal{R}$:

(1) For each $R \in \mathcal{R}^N$, $\mathbb{P}(R) \neq \emptyset$.
(2) For each $R \in \mathcal{R}^N$, there is $R_0^{p^*(R)} \in \mathcal{R}$ such that $D_S\left(R_0^{p^*(R)}, p^*(R)\right) = S$. In general, we use R_0^p when discussing a preference relation with the property that $D_S\left(R_0^p, p\right) = S$.
(3) Let $R \in \mathcal{R}^N$, $i \in N$, and $A \subseteq D(R_i; p^*(R))$. Denote by $R_i\!\uparrow\!(x, s)$ and $R_i\!\downarrow\!(x, s)$ the upper and lower contour sets, respectively, of R_i at (x, s). Let $\mathcal{T}(R_i, A)$ be the set of preference relations $R_i' \in \overline{\mathcal{R}}$ such that, for each $(x, s) \in A$, $R_i'\!\uparrow\!(x, s) \cap R_i\!\downarrow\!(x, s) = A$. Assume A belongs to a single indifference class: $A \subseteq R_i\!\uparrow\!(x, s) \cap R_i\!\downarrow\!(x, s)$, for some (x, s). Then for each $\varepsilon > 0$, there is $R_i^\varepsilon \in \mathcal{T}(R_i, A)$ such that $H^\Delta\left(R_i, \hat{R}_i\right) < \varepsilon$ and $R_i' \in \mathcal{R}$.

Since $\mathcal{R}$ satisfies 1, 2, and 3, we say it is sufficiently rich. Note that the third assumption amounts to ensuring that the domain has sufficiently many Maskin monotonic transformations. The linear domain ($\mathcal{L}$) is sufficiently rich. The translation invariant domain ($\mathcal{Q}$) satisfies all the requirements except 1. Thus if the environment has *sufficient capacity*, $\mathcal{Q}$ is sufficiently rich. However, this is not

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necessary; subsets of $\mathcal{Q}$ for which the preference coefficients are bounded can also be sufficiently rich.

We may now state the characterization, the proof of which is in the appendix.

Theorem 5. *A rule Φ on $\mathcal{R}^N$ is strategy-proof, welfare anonymous, constant sequence continuous, and strongly undominated in these properties if and only if it is a subcorrespondence of F^*.*

This characterization is tight. To show the independence of *constant sequence continuity*, we may adapt the rule used to similar effect in Tierney (2016).

5. Extension: Variable or Private Endowment of Numeraire in the Extractive Case

We now restrict attention to the extractive model and allow for agents to purchase the resource with their own money. Thus, each agent $i \in N$ has an endowment $w_i > 0$ of money. A typical consumption bundle is a triple $(m_i, x_i, s) \in \mathbb{R}^2 \times S$, where m_i is his consumption of money. Notation for a typical preference relation is unchanged. Assume that preferences are monotone and strictly increasing in commodities: if $(m_i', x_i') \geq (m_i, x_i)$ then for each $s \in S$, $(m_i', x_i', s)\ R_i\ (m_i, x_i, s)$, and if $x_i' > x_i$, then $(m_i', x_i', s)\ P_i\ (m_i, x_i, s)$. Assume also that for each $s \in S$, preferences over the set $\mathbb{R}^2 \times \{s\}$ are either strictly convex, or that money has no value to the agent.

Given prices p, agent i's budget set is $B_i(p) := \{(m_i, x_i, s) : \ p_s x_i + m_i \leq w_i\}$. Define vector function z_i^* so that for each $s \in S$, $z_{is}^*(R_i; p) \in \mathbb{R}^2$ is the unique favorite element of R_i from the set $B_{is}(p) := B_i(p) \cap \left(\mathbb{R}^2 \times \{s\}\right)$. Let $z_{is}^*(R_i; p) := (m_{is}^*(R_i; p), x_{is}^*(R_i; p))$, where m_{is}^* is the optimal choice of money and x_{is}^* the optimal choice of resource quantity. The demand correspondence is therefore

$$D_i\,(R_i; p) := \left\{(m_i, x_i, s) : \begin{array}{c} x_i = x_{is}^*(R_i; p),\ m_i = m_{is}^*(R_i; p), \\ \forall t \in S,\ (m_i, x_i, s)\ R_i\ (m_{it}^*(R_i; p), x_{it}^*(R_i; p), t) \end{array}\right\}.$$

Note that since endowments are now personalized, D_i must be indexed by agent. As before, we let D_{iS} denote the projection of the demand correspondence onto S.

An economy can be summarized by the list (N, S, R, w, e), where $w \in \mathbb{R}^N$. For the following theorem, the elements (R, w, e) all remain constant. Thus, we summarize an economy by the pair (N, S). Note that if (p, z, α) is an equilibrium, then for each $i \in N$, $z_i = z_{i\alpha(i)}^*(R_i; p)$. Thus, the pair (p, α) is sufficient to identify an equilibrium. A typical **reduced economy** is a list $(N', S', (R_i)_{i\in N'}, (w_i)_{i\in N'}, (e_s)_{s\in S'})$, which we denote simply by (N', S'). With some abuse of notation, we write $(p \wedge q) \in \mathbb{P}(N', S')$ if the list $(p_s \wedge q_s)_{s\in S'}$ is an equilibrium price for reduced economy (N', S').

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Theorem 6. *For each $R \in \mathcal{R}^N$, $(\mathbb{P}(R), \wedge)$ is a lower semi-lattice.*

Proof. Let $R \in \mathcal{R}^N$. Let p and $q \in \mathbb{P}(R)$. Assume $p,\ q < (\infty, \ldots, \infty)$. Let α_p and α_q be site assignments such that $\left(p, \alpha_p\right)$ and $\left(q, \alpha_q\right)$ are equilibria for (N, S). If $i \in N$ has $D_s\left(R_i, p \wedge q\right) = \emptyset$, then we give him bundle $((w_i, 0), \emptyset)$; he has no effect on the existence of an equilibrium for $p \wedge q$. Thus, it is without loss of generality to assume that, for each $i \in N$, $D_s\left(R_i, p \wedge q\right) \neq \emptyset$.

Partition S and N as follows

$$S^p := \{s : \ p_s < q_s\} \qquad N^p := \{i : \ D_S\left(R_i, p \wedge q\right) \subseteq S^p\}$$
$$S^q := \{s : \ q_s < p_s\} \qquad N^q := \{i : \ D_S\left(R_i, p \wedge q\right) \subseteq S^q\}$$
$$S^= := \{s : \ p_s = q_s\} \qquad N^= := N \setminus \left(N_p \cup N_q\right).$$

We will repeatedly use the following fact: for each $s \in S$ and each pair of price vectors $p',\ q' \in \mathbb{R}^S$, if $p'_s = q'_s$ then $z^*_{is}(R_i; p') = z^*_{is}(R_i; q')$.

Claim 2. $(p \wedge q) \in \mathbb{P}\left(N \setminus N^p, S \setminus S^p\right)$ and $(p \wedge q) \in \mathbb{P}\left(N \setminus N^q, S \setminus S^q\right)$.

Proof. We show the proof of the first statement. The proof of the second is symmetric.

Let $\hat{\alpha} := \alpha_q|_{N \setminus N^p}$, the restriction of α_q to $N \setminus N^p$. Let $i \in N$ be such that $\alpha_q(i) = s \in S^p$. Thus, for each $t \in S$, $\left(z^*_{is}(R_i; q), s\right) R_i \left(z^*_{it}(R_i; q), t\right)$. In particular, for $t \notin S^p$,

$$\begin{aligned}
\left(z^*_{is}(R_i; p \wedge q), s\right) &= \left(z^*_{is}(R_i; p), s\right) \\
&P_i \left(z^*_{is}(R_i; q), s\right) \\
&R_i \left(z^*_{it}(R_i; q), t\right) = \left(z^*_{it}(R_i; p \wedge q), t\right).
\end{aligned}$$

Thus, $i \in N^p$, and it follows that $\alpha_q(N \setminus N^p) \subseteq S \setminus S^p$, so $\hat{\alpha}$ is an admissible site assignment for $(N \setminus N^p, S \setminus S^p)$.

Now let $t \in S \setminus S^p$. Since $\left(q, \alpha_q\right)$ is an equilibrium for (N, S), for each $i \in N \setminus N^p$,

$$\begin{aligned}
\left(z^*_{i\hat{\alpha}_q(i)}(R_i; p \wedge q), \hat{\alpha}(i)\right) &= \left(z^*_{i\hat{\alpha}_q(i)}(R_i; q), \hat{\alpha}(i)\right) \\
R_i &\ \left(z^*_{it}(R_i; q), t\right) \\
&= \left(z^*_{it}(R_i; p \wedge q), t\right).
\end{aligned}$$

Thus $\hat{\alpha}(i)$ is an optimal choice from $S \setminus S^p$ given prices $(p_s \wedge q_s)_{s \in S \setminus S^p}$.

It remains to show that price $p \wedge q$ with assignment $\hat{\alpha}$ yields a feasible allocation for $(N \setminus N^p, S \setminus S^p)$. Since $\left(q, \alpha_q\right)$ is an equilibrium, for each $s \in S$, $\sum_{i \in \alpha_q^{-1}(s)} x^*_{is}(R_i; q) \leq e_s$. For each $s \in S \setminus S^p$, since

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$p_s \wedge q_s = q_s$,

$$\sum_{i \in \hat{\alpha}^{-1}(s)} x^*_{is}(R_i; p \wedge q) \leq \sum_{i \in \alpha_q^{-1}(s)} x^*_{is}(R_i; p \wedge q) = \sum_{i \in \alpha_q^{-1}(s)} x^*_{is}(R_i; q) \leq e_s.$$

Thus the Endowment Constraint is satisfied. Clearly $\left|\hat{\alpha}^{-1}(s)\right| \leq \left|\alpha_q^{-1}(s)\right|$, so the Capacity Constraint is satisfied as well. ■

Claim 3. If $(p \wedge q) \in \mathbb{P}(N \setminus N^q, S \setminus S^q)$ and $(p \wedge q) \in \mathbb{P}(N \setminus N^p, S \setminus S^p)$, then $(p \wedge q) \in \mathbb{P}(N, S)$.

Proof. Let $\hat{\alpha}$ be a site-assignment such that the list $\hat{z} := \left(z^*_{i\hat{\alpha}(i)}(R_i; p \wedge q)\right)_{i \in N \setminus N_p}$ yields an equilibrium, with price vector $p \wedge q$, for economy $(N \setminus N^p, S \setminus S^p)$. Similarly, let $\tilde{\alpha}$ be a site-assignment such that the analogous list, $\tilde{z} := \left(z^*_{i\tilde{\alpha}(i)}(R_i; p \wedge q)\right)_{i \in N \setminus N_q}$, yields an equilibrium, with price vector $p \wedge q$, for economy $(N \setminus N^q, S \setminus S^q)$. Now define (z, α) as follows:

$$\alpha^{-1}(s) := \begin{cases} \hat{\alpha}^{-1}(s) & s \in S_q \\ \tilde{\alpha}^{-1}(s) & s \in S \setminus S_q \end{cases} \qquad z_i := x^*_{i\alpha(i)}(R_i; p \wedge q).$$

Note that if $\alpha(i) = \hat{\alpha}(i)$ then $z_i = \hat{z}_i$, and if $\alpha(i) = \tilde{\alpha}(i)$ then $z_i = \tilde{z}_i$. We first verify that (z, α) gives each agent an optimal choice given prices $p \wedge q$. Let $i \in N_q$. Then $\alpha(i) = \hat{\alpha}(i)$. Since $(p \wedge q, \hat{z}, \hat{\alpha})$ is an equilibrium for $(N \setminus N^p, S \setminus S^p)$, and since $D_{iS}(R_i, p \wedge q) \cap S^p = \emptyset$, we have $(\hat{z}_i, \hat{\alpha}(i)) \in D(R_i, p \wedge q)$. If $i \in N_p$, then $\alpha(i) = \tilde{\alpha}(i)$ and the argument is symmetric with the previous case. Finally, assume $i \in N^=$. Then there exist $\hat{s} \in S \setminus S^p$ and $\tilde{s} \in S \setminus S^q$, not necessarily distinct, such that $\{\hat{s}, \tilde{s}\} \subseteq D_S(R_i, p \wedge q)$. It follows that

$$\begin{aligned} (\hat{z}_i, \hat{\alpha}(i)) &= \left(z^*_{i\hat{\alpha}(i)}(R_i; p \wedge q), \hat{\alpha}(i)\right) \\ &I_i\ (z^*_{i\hat{s}}(R_i; p \wedge q), \hat{s}) \\ &I_i\ (z^*_{i\tilde{s}}(R_i; p \wedge q), \tilde{s}) \\ &I_i\ \left(z^*_{i\tilde{\alpha}(i)}(R_i; p \wedge q), \tilde{\alpha}(i)\right) = (\tilde{z}_i, \tilde{\alpha}(i)). \end{aligned}$$

Thus, $(z_i, \alpha(i)) \in D(R_i, p \wedge q)$.

Feasibility of (z, α) follows immediately from its construction. □

Claims 2 and 3 give the result. ■

Continuity of preferences implies the closedness of $(\mathbb{P}(R), \wedge)$. Since $(\mathbb{P}(R), \wedge)$ is also bounded below by 0, there is a unique minimal price vector $p^*(R)$ for economy R. The correspondence of

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equilibria that result by using $p^*(R)$ as the price for economy R is essentially-single-valued. Thus, we may define a min-price Walrasian rule for this more general model.

6. The Impossibility of Efficient (Probabilistic) Rules

We have so far considered deterministic allocation rules, and found that the best *strategy-proof*, *anonymous* rule is not (ex-ante) efficient. One may wonder then if probabilistic rules can achieve this. We show here that for the extractive model they cannot. We must defer to future work the question of whether any probabilistic rules satisfy our conditions and dominate the MPW rule.

We must first discuss how agents evaluate random variables over the consumption space. To keep things simple, let us assume $S = \{1, 2\}$. Furthermore, assume the analogue of risk neutrality for this model: the domain of linear preferences, $\mathcal{L}$. It follows then that the expected utility of a random variable with distribution $\mu \in \Delta(\mathbb{R} \times S)$ is

$$\begin{aligned} U_i(\mu; v_i) &= \sum_{s\in S} \mu(\mathbb{R} \times \{s\}) \int_0^{e_s} x v_i(s) d\mu(x|S = s) \\ &=: \sum_{s\in S} v_i(s)\mu_s, \end{aligned}$$

where the equality defines notation μ_s for the expected allocation of commodity $s \in S$. Let $\mu :=(\mu_1, \mu_2)$. Then expected utility is measured simply as the inner product $\langle v_i, \mu\rangle$.

The proof of the following theorem is adapted from Cho & Thomson (2012), which studies linear preferences in the classical model of exchange. Our model has more restrictions, but these are easily avoided.

Theorem 7. *Say that a rule treats equals equally if agents with identical preferences are indifferent between each others' bundles. There is no strategy-proof and efficient rule that treats equals equally.*

Note that *equal-treatment of equals* is implied by *anonymity*, and is far weaker.

Proof. Let $N = \{1, 2, 3\}$, $S = \{1, 2\}$, $f \equiv (1, 1)$ and $c = (3, 3)$. Consider a restricted domain: without loss of generality, normalize $v_i(2) = 1$, and *with* loss of generality, restrict $v_i(1) \in (\frac{1}{2}, 2)$. For the remainder of the proof we abuse notation and order preferences $v_i > v_j$ when $v_i(1) > v_j(1)$. Let $\mathcal{M}(R \times S)^N$ be the space of Borel measurable random vectors from an (unspecified) state space to our allocation space. For any random vector X let $\overline{X}$ denote its (component-wise) expectation.

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If $\varphi' : \mathcal{L}^N \to \mathcal{M}(R \times S)^N$ is *strategy-proof* and *efficient*, then it must also satisfy these things on our restricted domain.

Suppose there is a state of the world under $\varphi'(v)$ in which some commodity s were not fully exhausted. If there is i consuming at s at this state, then give him more of the commodity, and he will be happier. If there is no i at s, then all agents must be at $t \neq s$, with some i consuming $x_i \leq 1/3$. However, by our domain restriction, $1 \cdot v_i(s) > \frac{1}{3} \cdot v_i(t)$, so agent i would be happier at s consuming 1. It follows that φ' puts probability one on allocations in which all commodities are exhausted and so, for each $s \in \{1, 2\}$ and each v in the restricted domain, $\sum_{i \in N} \overline{\varphi'_{is}}(v) = 1$.

Next note that, by linearity of preference, every efficient allocation is welfare equivalent to an allocation having only one agent consuming an interior expected bundle $\mu_i := \overline{\varphi'_i}(v) > 0$. Moreover, these bundles are feasible via the following scheme: have the agent i with $\mu_{is} > 0$ and $\mu_{it} = 0$ always go to site s. With probability p have the interior agent j go to site 1. In states where j goes to 1, give her ξ and in states where j goes to 2 give her γ. The reader can verify that the free parameters p, ξ, and γ allow the expected allocations in this class to have the same feasibility constraints as with a classical allocations, with the added restriction that each $i \in N$ has $\mu_{i1} + \mu_{i2} \leq 1$.

The proof now follows easily from the following claims:

Claim 4. Let v^1, v^2, and v^3 be in our restricted domain and be ordered $v^1 \geq v^2 \geq v^3$. Let v be a profile of these three preferences and $\mu = \overline{\varphi'}(v)$. Assume $v_i = v^1$. Letting $x \in \mathbb{R}$ satisfy $\left\langle (x, 0), v^1 \right\rangle = \left\langle \mu_i, v^1 \right\rangle$, point $(x, 0)$ is attainable for i given report (v^2, v^3) or (v^3, v^2) by the other agents. A symmetric statement holds for preferences in the opposite order.

Proof of claim. Let $v' > v^1$. And $\mu' = \overline{\varphi'}(v'_i, v_{-i})$. By strategy-proofness, μ' satisfies inequalities

$$\langle (v, 1), \mu_i \rangle \geq \langle (v, 1), \mu_i \rangle$$
$$\langle -(v', 1), \mu_i \rangle \geq \left\langle -(v', 1), \mu'_i \right\rangle .$$

Let $v_j = v^2$. If $\mu'_{i2} \neq 0$, the following trade is Pareto improvement: move i and k to the horizontal and vertical axes, respectively, while maintaining indifference with μ'. By the slope ordering, an (expected) vector $\tau \gneq 0$ is freed for consumption by j, who is happier. Thus, by *efficiency*, $\mu'_{i2} = 0$, and by *strategy-proofness*, $\mu_{i1} := x' \leq x$. However, if $x' < x$, then for some $v'' > v^1$ with $v'' - v^1$ sufficiently small, $\langle (v'', 1), \mu_i \rangle > \langle (v'', 1), (x', 1) \rangle$. Conclude by the same arguments that $\mu''_i = \varphi'_i((v'', 1), v_{-i})$ has $\mu''_i = (x'', 0)$ with $x'' > x'$, and so at $(v', 1)$, agent i reports $(v'', 1)$. ∎

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Claim 5. Let v^1, v^2, and v^3 have $v^1 > v^2 > v^3$. Let $\mu = \overline{\varphi'}(v)$ and assume $v_j = v^2$. Let $x \in \mathbb{R}$ satisfy $\left\langle (x, 0), v^2 \right\rangle = \left\langle (1/3, 1/3), v^2 \right\rangle$, and $y \in \mathbb{R}$ satisfy $\left\langle (0, y), v^2 \right\rangle = \left\langle (1/3, 1/3), v^2 \right\rangle$. Then $v_i = v^1$ implies $\overline{\varphi'}_i(v) = (x, 0)$ and $v_k = v^3$ imples $\overline{\varphi'}_k(v) = (0, y)$.

Proof of claim. By *equal-treatment of equals* and feasibility, all three expected bundles in $\overline{\varphi'}(v^2, v^2, v^2)$ lie on the hyperplane with normal $(v^2, 1)$ through point $(1/3, 1/3)$. By Claim 4, $\overline{\varphi'_i}(v^1, v_{-i}) = (x, 0)$. *Equal-treatment of equals* and feasibility imply that $\overline{\varphi'_j}(v^1, v_{-i})$ and $\overline{\varphi'_k}(v^1, v_{-i})$ remain on the hyperplane. Thus, Claim 4 again implies that when k declares v^3, $\overline{\varphi'_k}(v^1, v^2, v^3) = (0, y)$ (where the order of preferences in the profile is arbitrary, with abuse of notation). It remains that $\overline{\varphi'_i}(v^1, v^2, v^3) = (x, 0)$, because if i declares v^2, the symmetric argument implies $\overline{\varphi'_i}(v^2, v^2, v^3)$ is on the original hyperplane, and so $(x, 0)$ is attainable for i by the invariance of one's attainable set to one's own report. ∎

The proof is completed via some arithmetic. Let $v_1 = (1/2, 1)$ and $v_3 = (2, 1)$. Claim 5 and feasibility imply, when $v_2 = (2/3, 1)$, agent 2 gets expected bundle $(5/12, 5/18)$, whereas when $v_2 = (5/6, 1)$ she gets $(11/30, 11/36)$. Since

$$\frac{45}{72} = \left\langle \left(\frac{5}{12}, \frac{5}{18}\right), \left(\frac{5}{6}, 1\right) \right\rangle > \left\langle \left(\frac{11}{30}, \frac{11}{36}\right), \left(\frac{5}{6}, 1\right) \right\rangle = \frac{110}{180},$$

we have a violation of *strategy-proofness*. ∎

Efficiency and *strategy-proofness* imply that, even ex-ante, some agents will have to be given priority over others.

7. Conclusion

We have developed a rich and flexible framework for analyzing a novel class of problems. We identified two regimes in which the minimum-price psuedo-market equilibrium exists: When agents are treated equally and when the production functions are extractive. It seems reasonable to expect that some mixture of these two regimes will also be well-behaved—that is, some degree of differential treatment should be feasible when there is general congestion—but discovering the relationship between these two dimensions is beyond our scope here.

Finally, further work on random mechanisms is worthwhile. Even though these cannot solve our problem in a non-discriminatory way, they may increase efficiency. By employing random priority with budgets, *a la* Hashimoto (2016), we could limit *ex-post* inequality, though we would face again the problem of creating *ex-ante* an indivisible-goods problem from our divisible goods

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problem (see Section 3.4.1 for more on this difficulty), and such mechanisms are not generally efficient. Alternatively, it is worth noting the rule from Kovalenkov (2002), designed for the classical exchange setting, which achieves neither fairness nor efficiency, but approximates both in a novel way: The population is divided into two groups, and the Walrasian price of one group is given as the trading price of the other. We have seen in Section §6 that the random model has much in common with the classical allocation model, so such an approach may bear fruit.

Appendix A. Proof of Theorem 1

Our results depend on elementary matroid structure and one non-elementary result, the Edmonds Matroid Partition Theorem, which is itself a generalization of Hall's Marriage Theorem. We define, here, the structures necessary and state the theorem. For a less brief but nonetheless compact treatment, I recommend the chapter in Vohra (2004).

A matroid $\mathcal{M}$ consists of a finite ground set N and a family $\mathcal{I} \subseteq 2^N$ with the following properties: (i) $\mathcal{I} \neq \emptyset$, (ii) if $A \in \mathcal{I}$ and $B \subseteq A$ then $B \in \mathcal{I}$, and (iii) if $\{A, B\} \subseteq \mathcal{I}$, and $|A| > |B|$ then there is $a \in A \setminus B$ such that $\{a\} \cup B \in \mathcal{I}$. Note that properties i-iii are also satisfied by the sets of linearly independent vectors in a finite-dimensional vector space, and by the paths in a graph. In fact, the elements of $\mathcal{I}$ are termed **independent sets**.

Alternatively, one often sees matroids defined as having properties i, ii, and the following: (iii') if $\{A, B\} \subseteq \mathcal{I}$, and both sets are maximal elements of $\mathcal{I}$ in the containment order, then $|A| = |B|$. We leave it to the reader to verify that, given i and ii, iii and iii' are equivalent.

Define the *rank* of a set $A \subseteq N$ as the cardinality of the cardinally-largest independent set contained in A. Thus, by property iii', the structure of a matroid can be captured by its rank function $\rho : 2^\Omega \to \mathbb{Z}$.

Let $(\mathcal{M}_s)_{s\in S} = ((N, \mathcal{I}_s))_{s\in S}$ be a list of matroids all having the same ground set N. Let ρ_s be the rank function of matroid $\mathcal{M}_s$. We say the matroids are **partitionable**, if there is a partition $(I_s)_{s\in S}$ of N such that, for each $s \in S$, $I_s \in \mathcal{I}_s$. The following theorem is proved in Edmonds (1965).

Theorem (Edmonds Matroid Partition Theorem)**.** *A list of matroids $(\mathcal{M}_s)_{s\in S}$, with associated rank functions $(\rho_s)_{s\in S}$ is partitionable if and only if, for each subset $N' \subseteq N$, $|N'| \leq \sum_{s\in S} \rho_s(N')$.*

We now give the proof of Theorem 1.

Proof. For each $s \in S$, each $\hat{p} \in \mathbb{R}^S_+$, let

$$\mathcal{I}_s(\hat{p}) := \{N' \subseteq N : \ s \in D_S(R_i, \hat{p}),\ |N'|\, w \leq \hat{p}_s f_s(|N'|),\ |N'| \leq c_s\}.$$

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This is the family of all sets N' of agents who all demand s, and that can feasibly consume at site s. We shall first show that $\mathcal{M}_s(p \wedge q) := (N, \mathcal{I}_s(p \wedge q))$ is a matroid, whose rank function we denote $\rho_s(\cdot)$. For any $\hat{p} \in \mathbb{R}^S_+$, since f_s is positive, $\emptyset \in \mathcal{I}_s(\hat{p})$. Let $N' \in \mathcal{I}_s(\hat{p})$ and $N'' \subseteq N'$. Since f_s satisfies congestion, we have

$$\frac{w}{\hat{p}_s} \leq \frac{f_s(|N'|)}{|N'|} \leq \frac{f_s(|N''|)}{|N''|},$$

so $N'' \in \mathcal{I}_s(\hat{p})$. Finally, let $N', N'' \in \mathcal{I}_s(\hat{p})$ have $|N'| \geq |N''| + 1$ with $i \in N' \setminus N''$. Since f_s is a function only of the cardinality of the set of agents, and not of their identity, *congestion* again gives

$$\frac{w}{\hat{p}_s} \leq \frac{f_s(|N'|)}{|N'|} \leq \frac{f_s(|N'' \cup \{i\}|)}{|N'' \cup \{i\}|},$$

so $N'' \cup \{i\} \in \mathcal{I}_s(\hat{p})$.

Note that $p \wedge q$ admits an equilibrium if and only if the set N can be partitioned into sets, each of which is a member of a different $\mathcal{I}_s(p \wedge q)$. In other words, $p \wedge q$ admits an equilibrium if and only if the matroids $(\mathcal{M}_s(p \wedge q))_{s \in S}$ are partitionable.

Let $S_p := \{s \in S : p_s = (p \wedge q)_s\}$ and $S_q := \{s \in S : q_s = (p \wedge q)_s\}$. Let $N_p := \{i \in N : D(R_i, p \wedge q) \cap S_q = \emptyset\}$ and $N_q := \left\{i \in N : D(R_i, p \wedge q) \cap S_p = \emptyset\right\}$. For each $N' \subseteq N$, denote by $\mathcal{M}_s(\hat{p}) \setminus N'$ the matroid $\mathcal{M}_s(\hat{p})$ delete N'; this is simply the independent sets with all N' agents removed. We now claim that if $\left(\mathcal{M}_s(p \wedge q) \setminus N_p\right)_{s \in S_q}$ and $\left(\mathcal{M}_s(p \wedge q) \setminus N_q\right)_{s \in S_p}$ are both partitionable then $(\mathcal{M}_s(p \wedge q))_{s \in S}$ is partitionable. For each $N' \subseteq N$,

$$\begin{aligned} \left|N' \setminus N_p\right| + \left|N' \cap N_p\right| &\leq \sum_{s \in S_q} \rho_s\left(N' \setminus N_p\right) + \sum_{s \in S_p} \rho_s\left(N' \cap N_p\right) \\ &= \sum_{s \in S_q} \rho_s\left(N' \setminus N_p\right) + \sum_{s \in S \setminus S_q} \rho_s\left(N' \cap N_p\right) \leq \sum_{s \in S} \rho_s(N'). \end{aligned}$$

The first inequality is by Edmonds' Theorem. The equality is because, for each $s \in S_q$, $\rho(N_p) = 0$. The third, and final, inequality is by monotonicity of the rank function. The resulting inequality, $|N'| \leq \sum_{s \in S} \rho_s(N')$, implies via Edmond's theorem that $(\mathcal{M}_s(p \wedge q))_{s \in S}$ is partitionable, proving the claim.

Therefore, if $(\mathcal{M}_s(p \wedge q))_{s \in S}$ is not partitionable, one of the deleted matroid families is not partitionable. Assume without loss of generality that $\left(\mathcal{M}_s(p \wedge q) \setminus N_p\right)_{s \in S_q}$ is not partitionable, and we shall show this implies that $(\mathcal{M}_s(q))_{s \in S}$ is not partitionable, contradicting the assumption that q is an equilibrium price.

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For each $s \in S$, let $\hat{\rho}_s$ be the rank function of $\mathcal{M}_s(q)$. For $i \in N \setminus N_p$, there exists $s \in D_S(R_i, p \wedge q)$ such that $q_s = (p \wedge q)_s$. Let $t \in D_S(R_i, q)$. Then

$$\left(\frac{w}{(p \wedge q)_t}, t\right) R_i \left(\frac{w}{q_t}, t\right) R_i \left(\frac{w}{q_s}, s\right) I_i \left(\frac{w}{(p \wedge q)_s}, s\right), \tag{4}$$

and so $t \in D_S(R_i, p \wedge q)$. This implies that, for each $s \in S_q$ and each $N' \subseteq N \setminus N_p$, $\rho_s(N') \geq \hat{\rho}_s(N')$. On the other hand, if $r \in S$ satisfies $p_r < q_r$, then

$$\left(\frac{w}{q_s}, s\right) I_i \left(\frac{w}{(p \wedge q)_s}, s\right) R_i \left(\frac{w}{(p \wedge q)_r}, r\right) P_i \left(\frac{w}{q_r}, r\right) \tag{5}$$

and therefore $r \notin D_S(R_i, q)$. Thus, for each $r \in S \setminus S_q$ and each $N' \subseteq N \setminus N_p$, $\hat{\rho}_r(N') = 0$. We apply Edmonds' Theorem again: since $\left(\mathcal{M}_s(p \wedge q) \setminus N_p\right)_{s \in S_q}$ is not partitionable, there exists $N' \subseteq N \setminus N_p$ such that $|N'| > \sum_{s \in S_q} \rho_s(N')$. Using what we derived about $\hat{\rho}$, we get the following string of inequalities

$$|N'| \quad > \quad \sum_{s \in S_q} \rho_s(N') \geq \sum_{s \in S_q} \hat{\rho}_s(N') = \sum_{s \in S_q} \hat{\rho}_s(N') + \sum_{s \in S \setminus S_q} \hat{\rho}_s(N') = \sum_{s \in S} \hat{\rho}(N').$$

These yield $|N'| > \sum_{s \in S} \hat{\rho}(N')$ implying, again via Edmonds' Theorem, that $(\mathcal{M}_s(q))_{s \in S}$ is not partitionable. ■

Appendix B. Proofs For Section §3

For each $p \in \overline{\mathbb{R}}_+^S$, each $s \in S$, let $|p, s|$ denote the bundle $\left(\frac{w}{p_s}, s\right) \in \mathbb{R}_+ \times S$. For site assignment $\alpha \in S^N$, let $|p, \alpha|$ denote the allocation such that each $i \in N$ consumes $|p, \alpha(i)|$. Allocation $|p, \alpha|$ need not be feasible and each agent's bundle need not be an optimal choice from the p budget set, but it is clear that we may restrict our attention to allocations of this form when searching for equilibria.

B.1. **Some topological properties.** Fix the consumption space

$$Z := \{(x_0, s_0) \in \mathbb{R} \times S : 0 \leq x_0 \leq f_{s_0}(|N|)\}.$$

Endow Z with the metric ρ defined for each pair $\left\{(x_0, s_0), (y_0, s_0')\right\}$ by setting

$$\rho\left[(x_0, s_0), (y_0, s_0')\right] := \begin{cases} \frac{|x_0 - y_0|}{|x_0 - y_0| + 1} & s_0 = s_0' \\ 1 & s_0 \neq s_0'. \end{cases}$$

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Let the metric on $Z \times Z$ be given, for each pair $\left\{(z^1, z^2), (\hat{z}^1, \hat{z}^2)\right\}$, by $\max\left\{\rho(z^1, \hat{z}^1), \rho(z^2, \hat{z}^2)\right\}$. A preference relation is a closed subset $R \subset Z{\times}Z$. Since $Z{\times}Z$ is compact, the Hausdorff distance H^{Δ} is a metric for the space of preference relations $\mathcal{R}$. In fact, the topology induced by H^{Δ} is precisely the topology of closed-convergence. See Hildenbrand (1974). Endow $\mathcal{R}^N$ with the product topology.

The following lemma simply asserts the lower-hemicontinuity of demand, and so its proof is omitted.

Lemma 3. *Let $p^n \to p \in \mathbb{R}^S_{++}$ and $R^n_i \to R_i$. There exists $\bar{n} \in \mathbb{N}$ such that for each $n \geq \bar{n}$, $D_S\left(R^n_i, p^n\right) \subseteq D_S\left(R_i, p\right)$.*

We define the *limit inferior* of a sequence $x^n \in \mathbb{R}^K$ component-wise: for each k, let $\underline{x}_k := \liminf x^n_k$. Then define $\liminf x^n := \underline{x}$. Note that

$$\liminf x^n = \lim_{n\to\infty}\left[\inf_{\leq}\left\{x^{\tilde{n}} : \tilde{n} \geq n\right\}\right],$$

since the interior infimum can be found component-wise and still results in a non-decreasing, ordered sequence in the vector partial order. The *limit superior* is symmetric.

Lemma 4. *p^* is lower semi-continuous.*

Proof. Let $R^n \to R$. Let $p := \liminf p^*\left(R^n\right)$. Let $s \in S$. There is a sub-sequence $R^{\sigma(n)}$ such that $\lim p^*_s\left(R^{\sigma(n)}\right) = p_s$. Let $p^1 := \liminf p^*\left(R^{\sigma(n)}\right)$. For $t \neq s$, there is a further sub-sequence $R^{\tau(n)}$ such that $\lim p^*_t\left(R^{\tau(n)}\right) = p^1_t$. By repeating the process, we find a sub-sequence $R^{\nu(n)}$ and a price vector p^s such that $\lim p^*\left(R^{\nu(n)}\right) = p^s$ and $p^s_s = p_s$. Since S^N is finite, there is a site assignment $\alpha \in S^N$ and a further sub-sequence $R^{\tilde{\nu}(n)}$ such that for each n, $\alpha \in \mathcal{A}^*\left(R^{\tilde{\nu}(n)}\right)$. By Lemma 3, for each $i \in N$, $\alpha(i) \in D_S\left(R_i, p^s\right)$. Therefore, $|p^s, \alpha|$ is an equilibrium for R and $p^s \in \mathbb{P}(R)$. Since s was arbitrary, we note that, by the lower semi-lattice property of $\mathbb{P}(R)$ (Theorem 1) $p = \bigwedge_{s\in S} p^s \in \mathbb{P}(R)$. Finally, minimality of p^* yields

$$p^*(R) \leq p = \liminf p^*\left(R^n\right).$$

■

B.2. **Blocking via Indifference.** To facilitate discussion of blocking via indifference, it is helpful to borrow language from graph theory. Given economy $R \in \mathcal{R}^N$, prices $p \in \mathbb{R}^S$, and site-assignment $\alpha \in \mathcal{A}^*(R)$, we shall construct a directed graph with labelled arcs, denoted $\Gamma(R; p, \alpha)$. The vertex set is S and the label set is N. Since these remain constant, reference to them is suppressed, and

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so, with abuse of notation, $\Gamma(R; p, \alpha)$ refers to the set of arcs. If agent $i \in N$ is assigned $\alpha(i) = s$, and $|p, t|\ R_i\ |p, s|$, where $t \neq s$, then $(s, t, i) \in \Gamma(R; p, \alpha)$. In the visual representation of the graph, there is an arc from s to t with label i. We may write this $s \xrightarrow{i} t \in \Gamma(R; p, \alpha)$, and we may suppress the arc label where it is not important. Note that if (p, x, α) is an equilibrium, then each arc in $\Gamma(R; p, \alpha)$ represents an indifference relation. A generic path from r^0 to r^m is denoted $r^0 \rightsquigarrow r^m$. A path $r^0 \rightsquigarrow r^m$ is maximal in the graph Γ if there is no arc $\left(r', r^0\right) \in \Gamma$ and no arc $(r^m, r'') \in \Gamma$.

Theorem 2 is the union of Lemma 5, Lemma 7, and Lemma 8.

Lemma 5. *Let $R \in \overline{\mathcal{R}}^N$ and $(x, \alpha) \in F^*(R)$ be supported by prices p. Assume that s is not exhausted at equilibrium (p, x, α). Then s is blocked via indifference by a site s' that is exhausted at (p, x, α).*

Proof. Assume $s^* \in S$ is not exhausted at (p, x, α). Let $\Gamma' \subseteq \Gamma(R; p^*(R), \alpha)$ be the graph composed of all paths ending in s^*. That is, $r \xrightarrow{i} t \in \Gamma'$ if and only if there is a path $s \rightsquigarrow s^* \subseteq \Gamma(R; p^*(R), \alpha)$ such that $r \xrightarrow{i} t \in s \rightsquigarrow s^*$. Let $V \subseteq S$ be the sites adjacent to some arc in Γ'. By hypothesis, no element of V is exhausted at $|p^*(R), \alpha|$. By construction, V is a **source** of $\Gamma(R; p^*(R), \alpha)$: there is no arc $r \to t \in \Gamma(R; p^*(R), \alpha)$ with $r \in S \setminus V$ and $t \in V$. By continuity of preferences, there is $\varepsilon > 0$ and a price vector $p^\varepsilon \in \mathbb{R}^S$, given for each $s \in S$ by

$$p^\varepsilon_s := \begin{cases} p^*_s(R) - \varepsilon & s \in V \\ p^*_s(R) & s \notin V, \end{cases}$$

such that $|p^\varepsilon, \alpha|$ is feasible and V is also a source of $\Gamma(R; p^\varepsilon, \alpha)$. Note that $|p^\varepsilon, \alpha|$ is *not necessarily* an equilibrium; if it were, the proof would be finished here.

Focus attention on the sub-economy given by sites V and agents $N^\star := \alpha^{-1}(V)$. Denote this sub-economy $\mathcal{E}$. We transform $\mathcal{E}$ as follows. Let $\underline{c} \in \mathbb{N}^V$ satisfy, for each $s \in V$, $\underline{c}_s = \left|\alpha^{-1}(s)\right|$. Construct the set $V^\star$ from V by having each site s exist as $\underline{c}_s$ identical copies. Copies of s are denoted s^a, s^b, etc. Agent i's preferences $R^\star_i$ over $\mathbb{R} \times V^\star$ are defined in the natural way: for each $x \in \mathbb{R}$, $\left\{s^a, s^b\right\} \subseteq \{s\}^\star$ implies $(x, s^a)\ I^\star_i\ \left(x, s^b\right)$ and otherwise $R^\star_i$ respects R_i. We set the production function for each site copy $s^a \in \{s\}^\star$ to a constant function with value $e^\star_{s^a} := {}^{w}/_{p^\varepsilon_s}$. Finally define capacity vector $c^\star \in \mathbb{N}^{V^\star}$ by $c^\star = (1, 1, \ldots, 1)$. Let $\mathcal{E}^\star$ be the economy consisting of sites $V^\star$ with capacities $c^\star$, fixed endowments $e^\star$, and agents $N^\star$ having preferences $R^\star$. This economy admits an equilibrium $\left|p^\star, \alpha^\star\right|$ by adapting $|p, \alpha|$ in the obvious way. Then by Lemma 3 in Demange & Gale

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(1985), it admits an equilibrium $\left|q^\star, \beta^\star\right|$ such that there is a site r, having a copy r^a, such that

$$\frac{w}{q^\star_{r^a}} = e^\star_{r^a},$$

implying that $q^\star_{r^a} = p^\varepsilon_r$.[9] Moreover the lattice structure allows us to assume $q^\star \le p^\star$.

Let $\left\{t^a, t^b\right\} \subseteq \{t\}^\star$. By the construction of $R^\star$, $q^\star_{t^a} = q^\star_{t^b}$. Thus, $q_t := q^\star_{t^a}$ is well-defined. Define β for each site $s \in V$ by $\beta^{-1}(s) := \cup_{s^a \in \{s\}^\star}\{\beta^{\star -1}(s^a)\}$. We have constructed an equilibrium $|q, \beta|$ for $\mathcal{E}$ such that for each $s \in V$, $q_s \le p_s$. Moreover,

$$q_r = p^\varepsilon_r < p_r.$$

Now we map back to the original economy, with sites S and agents N. Extend q to each $t \in S \setminus V$ by setting $q_t := \infty$. Let

$$\gamma(j) := \begin{cases} \alpha(j) & j \in N \setminus N^\star \\ \beta(j) & j \in N^\star. \end{cases}$$

We need only ensure feasibility of the consumption of the V sites. By construction, at γ, at most $\underline{c}_s$ agents are consuming at $s \in V$. Each such agent is consuming at most w/p^ε_s units of commodity. The price p^ε was constructed so that this would be feasible given assignment α, and by definition, $\underline{c}_s = \left|\alpha^{-1}(s)\right|$, so since $\left|\beta^{-1}(s)\right| \le \left|\alpha^{-1}(s)\right|$,

$$\frac{f_s\left(\left|\beta^{-1}(s)\right|\right)}{\left|\beta^{-1}(s)\right|} \ge \frac{f_s(\underline{c}_s)}{\underline{c}_s} \ge \frac{w}{p^\varepsilon_s}.$$

Since each $j \notin N^\star$ consumes at $r \notin V$, $|p \wedge q, \gamma|$ is an equilibrium for R, contradicting the minimality of $p^*(R)$. ■

Given $\Gamma(R; p, \alpha)$ and a path $r^0 \rightsquigarrow r^m \subseteq \Gamma(R; p, \alpha)$, site-assignment $\alpha' \in S^N$ is the **path shift** of α **via** $r^0 \rightsquigarrow r^m$ if, for each $i \in N$,

$$\alpha'(i) := \begin{cases} r^{l+1} & \left(r^l, r^{l+1}, i\right) \in r^0 \rightsquigarrow r^m \\ \alpha(i) & \text{otherwise.} \end{cases}$$

It is easy to find the relation between $\Gamma(R; p, \alpha)$ and $\Gamma(R; p, \alpha')$ if $|p, \alpha|$ is an equilibrium: the arcs in $r^0 \rightsquigarrow r^m$ are reversed and all else is the same.

[9] View each site $s^a \in V^\star$ as an agent with utility function $v_{s^a}(\left|q^\star, \gamma^\star\right|) = e_{s^a} - w[q^\star_s]^{-1}\mathbb{1}[\exists i \in N^\star, \gamma^\star(i) = s^a]$. Let the outside option of each s^a be 0. We have a model of one to one matching where stability coincides with equilibrium. Our $V^\star$ corresponds to Q in Demange & Gale (1985), and our $N^\star$ corresponds to P. By construction, $\left|V^\star\right| \ge \left|N^\star\right|$.

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Lemma 6. *Let $\alpha \in \mathcal{A}^*(R)$ and $r \rightsquigarrow s \subseteq \Gamma(R; p^*(R), \alpha)$. Let α' be the path shift of α via $r \rightsquigarrow s$ and assume that $|p^*(R), \alpha'|$ is feasible. Then $\alpha' \in \mathcal{A}^*(R)$.*

Proof. Let $i \in N$. If $\alpha'(i) = \alpha(i)$ then clearly $\alpha(i) \in D_S(R_i; p^*(R))$. Otherwise, there is an arc $r^l \xrightarrow{i} r^{l+1} \in r \rightsquigarrow s \subseteq \Gamma(R; p^*(R), \alpha)$. Then by definition, $\left|p^*(R), r^{l+1}\right| R_i \left|p^*(R), r^l\right|$. Therefore, since $r^l = \alpha(i) \in D_S(R_i; p^*(R))$, it follows that $r^{l+1} \in D_S(R_i; p^*(R))$. ■

Lemma 7. *For each profile $R \in \overline{\mathcal{R}}^N$ such that $\mathbb{P}(R) \neq \emptyset$, there is a balanced site assignment $\alpha \in \mathcal{A}^*(R)$.*

Proof. Let $R \in \mathcal{R}^N$, $p := p^*(R)$, and $\alpha \in \mathcal{A}^*(R)$. Given p there is $b \in \mathbb{N}^S$ such that, if each site $s \in S$ has at least b_s agents consuming its resource, then balance is achieved. Note that if s is exhausted at $|p, \alpha|$, then $\left|\alpha^{-1}(s)\right| = b_s + 1$. Also note that each $b_s \leq \mathsf{c}_s(p)$.

Suppose there is $s \in S$ with $\left|\alpha^{-1}(s)\right| < b_s$. By Lemma 5, there is a path $s^0 \rightsquigarrow s \subseteq \Gamma(R; p, \alpha)$ such that s^0 is exhausted at $|p, \alpha|$. Let β be the path-shift of α via $s^0 \rightsquigarrow s$. Since $\left|\alpha^{-1}(s)\right| < b_s$, $\left|\beta^{-1}(s)\right| \leq b_s$, so $|p, \beta|$ is feasible. It follows from Lemma 6 that $\beta \in \mathcal{A}^*(R)$. Since site s^0 is exhausted under $|p, \alpha|$, $\left|\beta^{-1}\left(s^0\right)\right| = b_s$. For each $s' \in S \setminus \left\{s, s^0\right\}$, $\left|\beta^{-1}(s')\right| = \left|\alpha^{-1}(s')\right|$. Finally, $\left|\beta^{-1}(s)\right| = \left|\alpha^{-1}(s)\right| + 1$.

If $\left|\beta^{-1}(s)\right| < b_s$, we repeat the exercise. In the next iteration, there is $s^1 \rightsquigarrow s \subseteq \Gamma(R; p, \alpha)$ such that $s^1 \neq s^0$ and s^1 is exhausted at $|p, \beta|$. Proceeding thus, we generate a list of sites $\left\{s^0, s^1, \ldots, s^k\right\}$ and a site assignment $\gamma \in \mathcal{A}^*(R)$ such that for each $s^l \in \left\{s^0, s^1, \ldots, s^k\right\}$, $\left|\gamma^{-1}\left(s^l\right)\right| = b_{s^l}$. Moreover, $\left|\gamma^{-1}(s)\right| = \left|\alpha^{-1}(s)\right| + k + 1$. Finally, for each $s' \notin \{s, s^0, \ldots, s^k\}$, $\left|\gamma^{-1}(s')\right| = \left|\alpha^{-1}(s')\right|$. For k large enough, either s is no longer blocked via indifference by an exhausted site, or $\left|\gamma^{-1}(s)\right| = b_s$. The former case is ruled-out by Lemma 5. Thus the latter case is true and the lemma is proved. ■

Lemma 8. *Let $|p, \alpha|$ be a balanced equilibrium such that each site $s \in S$ is blocked via indifference by an exhausted site. Then $|p, \alpha| \in F^*(R)$.*

Proof. Assume $|p, \alpha|$ is a balanced equilibrium. Assume there are a price $p' \lneq p$ and a site assignment $\beta \in S^N$ such that $|p', \beta|$ is an equilibrium for R. Let $S^- := \{s \in S : p'_s < p_s\}$. Since $|p, \alpha|$ is a balanced allocation, if $s \in S^-$ with $\left|\alpha^{-1}(s)\right| = \mathsf{c}_s(p) - 1$, then s has endowment divisible value, and therefore $\mathsf{c}_s(p') \leq \mathsf{c}_s(p) - 1$. Thus,

$$\forall s \in S^-, \ \left|\alpha^{-1}(s)\right| \geq \mathsf{c}_s(p'). \tag{6}$$

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If $D_S(R_i, p) \cap S^- \neq \emptyset$, then clearly $D_S(R_i, p') \subseteq S^-$. Since each $i \in \alpha^{-1}(S^-)$ has $D_S(R_i, p) \cap S^- \neq \emptyset$, $\beta^{-1}(S^-) \supseteq \alpha^{-1}(S^-)$. Line 6 further implies that $\left|\beta^{-1}(S^-)\right| = \left|\alpha^{-1}(S^-)\right|$ so $\beta^{-1}(S^-) = \alpha^{-1}(S^-)$. It follows that, $\beta^{-1}(S \setminus S^-) = \alpha^{-1}(S \setminus S^-)$. If $D_S(R_i, p') \cap (S \setminus S^-) \neq \emptyset$, then clearly $D_S(R_i, p) \subseteq S \setminus S^-$. Since each $i \in \beta^{-1}(S \setminus S^-) = \alpha^{-1}(S \setminus S^-)$ has $D_S(R_i, p') \cap (S \setminus S^-) \neq \emptyset$, conclude that, for each $s \in S^-$, $s \in D_S\left(R_j, p\right)$ if and only if $\alpha(j) \in S^-$. Thus, S^- is a source in the graph $\Gamma(R; p, \alpha)$. If at $|p, \alpha|$, each site is blocked via indifference by an exhausted site, S^- must contain a site, s^*, exhausted at $|p, \alpha|$. This site has endowment divisible value at p and $\left|\alpha^{-1}(s^*)\right| = \mathsf{c}_{s^*}(p)$, so

$$\left|\alpha^{-1}(s^*)\right| = \mathsf{c}_{s^*}(p) \geq \mathsf{c}_{s^*}(p') + 1. \tag{7}$$

Then, since $\left|\beta^{-1}(S^-)\right| = \left|\alpha^{-1}(S^-)\right|$, lines 6 and 7 yield

$$\left|\beta^{-1}(S^-)\right| = \left|\alpha^{-1}(S^-)\right| \geq \mathsf{c}_{s^*}(p')) + 1 + \sum_{s \in S^- \setminus s^*} \mathsf{c}_s(p'),$$

contradicting feasibility. ∎

B.3. **Proof of Lemma 1.** We prove Lemma 1 in parts, with Lemma 2 proved in the process. We can simplify notation as follows: for each $d \in \mathbb{R}$, $R^d := \left(R_i^{s,d}, R_{-i}\right)$.

Proof of Property 2. Since $s \notin D_S(R_i, \pi(0))$, for each $\varepsilon \in \mathbb{R}$ with $|\varepsilon|$ sufficiently small, $s \notin D_S(R_i^{s,\varepsilon}, \pi(0))$. For each $\alpha \in \mathcal{A}^*(R)$, $|\pi(0), \alpha|$ is an equilibrium for R^ε. Thus $\pi(\varepsilon) \leq \pi(0)$.

Case 1. There is a sequence ε^n converging to zero such that for each $n \in \mathbb{N}$, $s \in D_S\left(R_i^{s,\varepsilon^n}, \pi(\varepsilon^n)\right)$. We showed that for each $\varepsilon > 0$ sufficiently small, $\pi(\varepsilon) \leq \pi(0)$. Combined with Lemma 4, we may write

$$\pi(0) \geq \limsup_{n\to\infty} \pi(\varepsilon^n) \geq \liminf_{n\to\infty} \pi(\varepsilon^n) \geq \pi(0).$$

Therefore $\lim_{n\to\infty} \pi(\varepsilon^n) = \pi(0)$. By Lemma 3, $s \in D_S(R_i, \pi(0))$, a contradiction.

Case 2. There is a sequence ε^n converging to zero such that for each $n \in \mathbb{N}$, $s \in D_S(R_i, \pi(\varepsilon^n))$. The argument for the previous case remains valid.

Case 3. There is an open neighborhood V containing 0 such that for each $\varepsilon \in V$, $s \notin D_S\left(R_i^{s,\varepsilon}, \pi(\varepsilon)\right)$. If $|\pi(\varepsilon), \beta|$ is an equilibrium for R^ε, since Case 1 is false, $\beta(i) \neq s$. Since $R^\varepsilon_{-i} = R_{-i}$, and since Case 2 is also false, $|\pi(\varepsilon), \beta|$ is also an equilibrium for R. Thus by minimality, $\pi(0) \leq \pi(\varepsilon)$ and we conclude that $\pi(\varepsilon) = \pi(0)$.

∎

Lemma 9. *Assume $\alpha(i) = s$. Then for each $d > 0$, each $\beta \in \mathcal{A}^*\left(R^d\right)$, $\beta(i) = s$.*

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Proof. Suppose not: there exist $d > 0$ and a site assignment $\beta \in \mathcal{A}^*\left(R^d\right)$ such that $\beta(i) = s' \neq s$. Note that $|p^*(R), \alpha|$ is an equilibrium for R^d. Therefore, $p^*\left(R^d\right) \leq p^*(R)$. Since $s' \notin D_S\left(R_i^{s,d}, p^*(R)\right)$, it follows by monotonicity of preferences that $p^*_{s'}\left(R^d\right) < p^*_{s'}(R)$. Since R_i is a negative s-translation of $R_i^{s,d}$, $\left|p^*\left(R^d\right), \beta\right|$ is an equilibrium for R, contradicting the minimality of $p^*_{s'}(R)$. ∎

Proof of Lemma 2 . By Property 2, if $s_i \notin D_S(R_i, p^*(R))$, then there exists $\bar{d}_i$ such that $p^*\left(R_i^{s_i,\bar{d}_i}, R_{-i}\right) = p^*(R)$ and $s_i \in D_S\left(R_i^{s_i,\bar{d}_i}, p^*(R)\right)$. The same holds for each $i \in N'$: $p^*\left(\left(R_j^{s_j,\bar{d}_j}\right)_{i\in N'}, R_{N\setminus N'}\right) = p^*(R)$. Thus we assume without loss of generality that $R = \left(\left(R_j^{s_j,\bar{d}_j}\right)_{i\in N'}, R_{N\setminus N'}\right)$.

Assume $\alpha \in \mathcal{A}^*(R)$ is a balanced site-assignment. Let $\bar{R} := \left(\left(R_j^{s_j,d}\right)_{j\in N'}, R_{N\setminus N'}\right)$, $\hat{S} := \{t \in S : p^*_t\left(\bar{R}\right) < p^*_t(R)\}$, and $\hat{N} := \{i \in N : F^*(\bar{R})\ P_i\ F^*(R)\}$. It is clear that if $t \in \hat{S}$ then $\alpha^{-1}(t) \subseteq \hat{N}$. Therefore

$$\alpha^{-1}\left(\hat{S}\right) \subseteq \hat{N}. \tag{8}$$

If $s' \in \hat{S}$ has endowment divisible value at $p^*(R)$, then $\tilde{c}_{s'}\left(p^*(\bar{R})\right) \leq \tilde{c}_{s'}(p^*(R)) - 1$. Since α is balanced, $\min\{c_t, \tilde{c}_t(p^*(R)) - 1\} \leq \left|\alpha^{-1}(t)\right|$. Thus if $\min\{c_t, \tilde{c}_t(p^*(R)) - 1\} = \tilde{c}_{s'}(p^*(R)) - 1$,

$$\mathsf{c}_{s'}(p^*(\bar{R})) \leq \tilde{c}_{s'}\left(p^*(\bar{R})\right) \leq \tilde{c}_{s'}(p^*(R)) - 1 \leq \left|\alpha^{-1}(t)\right|.$$

Otherwise, $\min\{c_t, \tilde{c}_t(p^*(R)) - 1\} = c_t$ and we have

$$\mathsf{c}_{s'}(p^*(\overline{R})) \leq c_t \leq \left|\alpha^{-1}(t)\right|.$$

Therefore $\mathsf{c}_{s'}(p^*(\overline{R})) \leq \left|\alpha^{-1}(t)\right|$. If $t \in \hat{S}$ does not have endowment divisible value, $\mathsf{c}_{s'}(p^*(\overline{R})) \leq \mathsf{c}_{s'}(p^*(R)) = \left|\alpha^{-1}(t)\right|$. In sum,

$$\forall t \in \hat{S},\ \left|\alpha^{-1}(t)\right| \geq c_t\left(p^*(\bar{R})\right). \tag{9}$$

To arrive at a contradiction, assume that $\alpha(N') \subseteq \hat{S}$. By construction, for $k \in N'$, $D_S\left(\bar{R}_k, p(\bar{R})\right) \subseteq \left(s_j\right)_{j\in N'} \subseteq \hat{S}$. For $k \in \hat{N}\setminus N'$, since $\bar{R}_k = R_k$ and preferences are increasing, $F^*\left(\bar{R}\right) P_k\ F^*(R)$ implies $D_S\left(R_k, p^*\left(\bar{R}\right)\right) \subseteq \hat{S}$. In sum, $N' \subseteq \hat{N}$, and for each $k \in \hat{N}$, $D_S\left(\bar{R}_k, p^*\left(\bar{R}\right)\right) \subseteq \hat{S}$. Let $\beta \in \mathcal{A}^*\left(\bar{R}\right)$. Then $\hat{N} \subseteq \beta^{-1}(\hat{S})$. By 9,

$$\left|\hat{N}\right| \leq \left|\beta^{-1}\left(\hat{S}\right)\right| \leq \sum_{r\in\hat{S}} \mathsf{c}_r(p^*(\overline{R})) \leq \sum_{r\in\hat{S}} \left|\alpha^{-1}(r)\right| = \left|\alpha^{-1}\left(\hat{S}\right)\right|.$$

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Combined with line 8, we deduce that $\hat{N} = \alpha^{-1}\left(\hat{S}\right)$.

By Lemma 5, for each $r \in \left(s_j\right)_{j \in N'}$ there is a path $t \xrightarrow{1} t^2 \xrightarrow{2} \cdots \xrightarrow{n} r \subseteq \Gamma(R; p^*(R), \alpha)$ such that t is exhausted at (x, α). Note that if $D_S(R_k, p^*(R)) \cap \hat{S} \neq \emptyset$, then $D_s\left(R_k, p^*\left(\bar{R}\right)\right) \subseteq \hat{S}$ and $F^*\left(\overline{R}\right) P_k\ F^*(R)$. Therefore, for each $k \notin \hat{N}$, $D_S(R_k, p^*(R)) \cap \hat{S} = \emptyset$. It follows that, since $r \in \hat{S}$, $i^n \in \hat{N}$. Then $\alpha(i^n) \in \hat{S}$, and it follows by the same argument that $i^{n-1} \in \hat{N}$, and so on. Conclude that $\left\{i^1, \ldots, i^n\right\} \subseteq \hat{N}$ and therefore that $t \in \hat{S}$. Since t is exhausted at (x, α), it has endowment divisible value at $p^*(R)$ and $\left|\alpha^{-1}(t)\right| = \mathsf{c}_t(p^*(R))$. Moreover, $\mathsf{c}_t(p^*(\overline{R})) = \mathsf{c}_t(p^*(R)) - 1$. Thus, $\mathsf{c}_t(p^*(\overline{R})) = \left|\alpha^{-1}(t)\right| - 1$. Therefore, since $\hat{N} = \alpha^{-1}\left(\hat{S}\right)$, line (9) then implies

$$\left|\beta^{-1}(\hat{S})\right| \geq \left|\hat{N}\right| = \left|\alpha^{-1}\left(\hat{S}\right)\right| > \sum_{\hat{s} \in \hat{S}} c_{\hat{s}}\left(p^*(\bar{R})\right),$$

a contradiction. ■

With Lemma 2 and Property 1shown, we may now use the fact that F^* is *weakly group-strategy-proof*.

Proof of Property 1. By Property 2, we may confine attention to the case when $s \in D_S(R_i, \pi(0))$. Suppose that for $d > 0$, $\pi_s(d) < \pi_s(0)$. If $s \notin D_S\left(R_i^d, \pi(d)\right)$, then we apply Property 2 to conclude that $\pi_s(0) = \pi_s(d) < \pi_s(0)$. Therefore, $s \in D_S\left(R_i^d, \pi(d)\right)$. By definition of R^d, for each pair of bundles (x, s) and (y, t) with $t \neq s$, if $(x, s)\ I_i^d\ (y, t)$ then $(y, t)\ P_i\ (x, s)$. Therefore, for each $\alpha \in \mathcal{A}^*\left(R^d\right)$,

$$|\pi(d), \alpha(i)|\ R_i \left(\frac{w}{\pi_s(d)}, s\right) P_i\ F^*(R),$$

contradicting *strategy-proofness*. ■

Appendix C. Proof of Theorem 5

Proposition 2. *F^* satisfies w-Con.*

Proof. Briefly consider an alternate definition of equilibrium. Triple (p, x, α) is a **constrained price equilibrium** if it is an equilibrium and, for each site $s \in S$ with $\alpha^{-1}(s) = \emptyset$, $p_s = {}^{w}\!/_{f_s(1)}$. Given an equilibrium, (q, y, β), define for each $s \in S$, $p_s := \max\{q_s, {}^{w}\!/_{f_s(1)}\}$. It is easy to verify that (p, y, β) is a constrained price equilibrium. It follows that $p^{**}(R)$ defined from $p^*(R)$ by this mapping is the unique minimal constrained equilibrium price for economy R. Let F^{**} be the rule such that each $F^{**}(R)$ is the set of equilibria supported by $p^{**}(R)$. If there is $\alpha \in \mathcal{A}^*(R)$ and $s \in S$

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with $\alpha^{-1}(s) \neq \emptyset$, then feasibility implies $p^*_s(R) \geq {}^{w}/_{f_s(1)}$, which further implies $p^{**}_s(R) = p^*_s(R)$. Thus, $F^{**} = F^*$.

Let $\varphi \in F^*$. Let $R^n \in \mathcal{R}^N$ be a sequence converging to $R \in \mathcal{R}^N$. Let (x, α) be an allocation such that for each $n \in \mathbb{N}$, $\varphi(R^n) = (x, \alpha)$. Let $\hat{S} := \alpha(N)$. For each $s \notin \hat{S}$, $p^{**}_s(R) = we^{-1}_s$. Clearly, for each $s \in \hat{S}$, $p^{**}_s(R^n)$ is a constant sequence. Define $\bar{p} := p^{**}(R^n)$. By Lemma 3, there is $\bar{n} \in \mathbb{N}$ such that for each $n > \bar{n}$, and each $i \in N$, $D_S(R_i, \bar{p}) \supseteq D_S\left(R^n_i, \bar{p}\right)$. This implies moreover that $\bar{p} \in \mathbb{P}(R)$ and therefore, $p^{**}(R) \leq \bar{p}$.

Let $n \geq \bar{n}$. For each $i \in N$, construct preference relation $\hat{R}_i$ such that $D_S\left(\hat{R}_i, \bar{p}\right) = D_S(R_i, \bar{p})$ by performing successive, positive site-translations. By Property 2 , $p^{**}\left(\hat{R}_1, R^n_{-1}\right) = \bar{p}$, $p^{**}\left(\hat{R}_1, \hat{R}_2, R^n_{N\setminus\{1,2\}}\right) = \bar{p}$ and so on. Conclude that $p^{**}\left(\hat{R}\right) = \bar{p}$. Next, for each i, construct $\tilde{R}_i$ from $\hat{R}_i$ by site translations such that $(x, s)\ \tilde{I}_i\ D\left(\tilde{R}_i, \bar{p}\right)$ if and only if $(x, s)\ I_i\ D(R_1, \bar{p})$. That is, the optimizing indifference set of $\tilde{R}_i$ for prices $\bar{p}$ is identical to the optimizing indifference set of R_i for prices $\bar{p}$. Since we have already set $D_S\left(\hat{R}_i, \bar{p}\right) = D_S(R_i, \bar{p})$, this operation involves sites $s \notin D_S\left(\hat{R}_i, \bar{p}\right)$. Therefore, by Property 2 again conclude that $p^{**}\left(\tilde{R}\right) = \bar{p}$. By *strategy-proofness*, $\varphi\left(R_1, \tilde{R}_{-1}\right)\ R_1\ \varphi\left(\tilde{R}\right)$. If $\varphi\left(R_1, \tilde{R}_{-1}\right)\ P_1\ \varphi\left(\tilde{R}\right)$, then by construction, $\varphi\left(R_1, \tilde{R}_{-1}\right)\ \tilde{P}_1\ \varphi\left(\tilde{R}\right)$, contradicting *strategy-proofness*. Therefore, $\varphi\left(R_1, \tilde{R}_{-1}\right)\ I_1\ \varphi\left(\tilde{R}\right)$. It follows that, since the preferences of other agents remain constant, $\varphi\left(R_1, \tilde{R}_{-1}\right)$ is an equilibrium for $\tilde{R}$ and $\bar{p} = p^{**}\left(\tilde{R}\right) \leq p^{**}\left(R_1, \tilde{R}_{-1}\right)$. We conclude then that $p^{**}\left(R_1, \tilde{R}_{-1}\right) = \bar{p}$. Proceed inductively to conclude that $p^{**}(R) = p^{**}\left(\tilde{R}\right) = \bar{p}$. ■

C.1. **The domain of unique assignment cardinality.** For each $k \in \mathbb{Z}_+$, define

$$\mathcal{D}^k := \left\{R \in \mathcal{R}^N : \sum \mathsf{c}_s(p^*(R)) \leq |N| + k\right\}.$$

If $R \in \mathcal{D}^0$, then $\sum_{s\in S} \mathsf{c}_s(p^*(R)) = |N|$. It follows that for each $\alpha \in \mathcal{A}^*(R)$ and each $s \in S$, $\left|\alpha^{-1}(s)\right| = \mathsf{c}_s(p^*(R))$.

In service of proving the main theorem, we first prove

Theorem 8. *Let φ be a single-valued rule. Assume φ is strategy-proof, welfare anonymous, constant sequence continuous, and strongly undominated in these properties. Then for each $R \in \mathcal{D}^0$, $\varphi(R) \in F^*(R)$.*

A single-valued rule φ is **individually invariant to unilateral monotonic transformation**, or **unilaterally invariant** for short, if for each $R \in \mathcal{R}^N$, and each $i \in N$, if $(x_i, s) = \varphi_i(R)$, then for each $R'_i \in \mathcal{T}(R_i, (x_i, s))$, $(x_i, s) = \varphi_i\left(R'_i, R_{-i}\right)$. There is a well-known result in the literature, called

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the "Invariance Lemma" by Thomson (2014), that implies each *strategy-proof* rule is *unilaterally invariant*. *Unilateral invariance*, while being implied by *strategy-proofness*, is in fact closely related to *strategy-proofness*. See Klaus & Bochet (2013) for a thorough study.

Let $R \in \mathcal{D}^0$ and $p := p^*(R)$. Let $\overline{c} := \mathsf{c}_s(p^*(R))$. Let R_0^p be a preference relation such that B, the budget set given by prices p, is an indifference set of R_0^p. That is, given prices p, an agent with preferences R_0^p is indifferent as to which site's commodity he wants to consume. Since $\mathcal{R}$ is sufficiently rich, $R_0^p \in \mathcal{R}$. Let $R^p \in \mathcal{R}^N$ be the profile such that for each $i \in N$, $R_i^p = R_0^p$.

By repeated applications of Property 2, conclude that $p^*(R^p) = p$ and therefore that $\sum_{s\in S} \overline{c}_s = \sum_{s\in S} \mathsf{c}_s(p^*(R^p))$. Since $\sum_{s\in S} \overline{c}_s = |N|$, $R^p \in \mathcal{D}^0$.

For $\varphi(R^p)$, *welfare anonymity* implies that all agents consume on a common indifference set. *Strong-undomination* implies that each agent must find their allocation at least as good as $F^*(R^p)$. Thus by feasibility, we deduce that there is $\beta \in S^N$ such that, for each $i \in N$ $\varphi(R^p) = |p, \beta|$, and therefore, $\varphi(R^p) \in F^*(R^p)$. Note that, typically, $F^*(R^p) \supsetneq F^*(R)$; this fact accounts for most of the work in what follows.

Lemma. *Let $i \in N$ and $R' := \left(R_{-i}, R_i^p\right)$. For each $j \in N$, $\varphi_j(R') \in D(R'_j, p)$.*

Proof. Let $(x, \alpha) \in F^*(R)$ and let $i^* \in N$ be such that $s^* := \alpha(i^*)$ is exhausted at (x, α). For each $\varepsilon > 0$ and each $j \neq i^*$, let $R_j^\varepsilon \in \mathcal{R}$ satisfy $H^\Delta\left(R_j, R_j^\varepsilon\right) < \varepsilon$ and $R_j^\varepsilon \in \mathcal{T}\left(R_j, (x_j, \alpha(j))\right)$. We first show by induction that the lemma is true for profile $\left((R_j^\varepsilon)_{j\neq i^*}, R_{i^*}^p\right)$.

Inductive Base: We showed above that there is $\beta \in S^N$ with $\varphi(R^p) = |p, \beta|$. Since $R^p \in \mathcal{D}^0$, and $p^*(R^p) = p^*(R)$, for each $s \in S$, $\left|\beta^{-1}(s)\right| = \left|\alpha^{-1}(s)\right|$. It follows that for each $i \in N$, there is $j \in \beta^{-1}(\alpha(i))$, which further implies that $\varphi_j(R^p) = (x_i, \alpha(i))$. Let $R''_j := R_i^\varepsilon$. By *unilateral invariance*, $\varphi_j\left(R''_j, R_{-j}^p\right) = (x_i, \alpha(i))$. By *welfare anonymity*, $\varphi_i\left(R_i^\varepsilon, R_{-i}^p\right)\ I_i^\varepsilon\ \left(z_j, \gamma(j)\right)$ and therefore $\varphi_i\left(R_i^\varepsilon, R_{-i}^p\right)\ R_i^\varepsilon\ B$. By *strategy-proofness*, $(x_i, \alpha(i)) = \varphi_i(R^p)\ R_i^p\ \varphi_i\left(R_i^\varepsilon, R_{-i}^p\right)$. Since $R_i^\varepsilon \in \mathcal{T}(R_i, (x_i, \alpha(i))) \subseteq \mathcal{T}(R^p, (x_i, \alpha(i)))$, conclude that $\varphi_i\left(R_i^\varepsilon, R_{-i}^p\right) = (x_i, \alpha(i))$. Let $(z, \gamma) := \varphi\left(R_i^\varepsilon, R_{-i}^p\right)$.

For each $j \in N \setminus i$, $F^*\left(R_i^\varepsilon, R_{-i}^p\right)\ I_j^p\ B$, therefore by *strong-undomination*, there exists $k \in N \setminus i$ such that $\varphi_k(R_i^\varepsilon, R_{-i}^p)\ R_k^p\ B$. By *welfare anonymity*, for *each* $j \in N \setminus i$, $\left(z_j, \gamma(j)\right)\ I_j^p\ (z_k, \gamma(k))$. In sum, for each $j \in N \setminus i$, $\varphi_j(R_i^\varepsilon, R_{-i}^p)\ R_j^p\ B$. Since preferences are increasing, $z_k \geq wp_{\gamma(k)}^{-1}$. It follows by feasibility that $\left|\gamma^{-1}(\gamma(k))\right| \leq c_{\gamma(k)}(p) = \left|\alpha^{-1}(\gamma(k))\right|$. Since k is arbitrary, for each $s \in S$, $\left|\gamma^{-1}(s)\right| \leq \left|\alpha^{-1}(s)\right|$. Now since, for each $s \in S$, $\left|\alpha^{-1}(s)\right| = \overline{c}_s$, and since $\sum \overline{c}_s = |N|$, it follows that $\left|\gamma^{-1}(s)\right| = \left|\alpha^{-1}(s)\right|$. Since $i \neq i^*$, this further implies there is $k^* \in N \setminus i$ satisfying $\gamma(k^*) = s^*$ and

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$z_{k^*} \geq x_{i^*}$.[10] Since for each $k \in \gamma^{-1}(s^*)$, $z_k \geq x_{i^*}$, by feasibility, $z_{k^*} = x_{i^*}$ (recall that $s^* = \alpha(i^*)$ is exhausted at (x, α)). Since k^*'s preferences are R_0^p, *welfare anonymity* implies that for each $k \in N \setminus i$, $\varphi_k\left(R_i^\varepsilon, R_{-i}^p\right) \in B$.

Induction Step: Fix $n \in \mathbb{N}$. The induction hypothesis is as follows: Let $R' := \left(R_{N'}^\varepsilon, R_{N\setminus N'}^p\right) \in \mathcal{R}^N$. Assume that $|N'| \leq n$ and $i^* \notin N'$. Then, for each $j \in N$, $\varphi_j(R') \in D\left(R_j', p\right)$.

Let R' satisfy the induction hypothesis and let $(y, \beta) := \varphi(R')$.

Claim 6. For each $i \in N \setminus N'$, there is $j \in N \setminus N'$ such that $\beta(j) = \alpha(i)$.

Proof of Claim. Let $\alpha(i) = s$. The claim is thus

$$i \in N \setminus N' \implies (\exists j \in N \setminus N' \; s.t. \, \beta(j) = s).$$

Since $\varphi(R')$ assigns only bundles in B, for each $s' \in S$, $\left|\beta^{-1}(s')\right| \leq \bar{c}_{s'}$. Since $\sum \bar{c}_{s'} = |N|$, feasibility implies that in fact $\left|\beta^{-1}(s')\right| = \bar{c}_{s'}$. For each $k \in N'$, since $\{(x_k, \alpha(k))\} = D\left(R_k^\varepsilon, p\right)$, the induction hypothesis that $(y_k, \beta(k)) \in D\left(R_k^\varepsilon, p\right)$ yields $(y_k, \beta(k)) = (x_k, \alpha(k))$. Thus, $\left(\beta^{-1}(s) \cap N'\right) \subseteq \alpha^{-1}(s)$.

The contrapositive hypothesis is $\beta^{-1}(s) \subset N'$. Then $\beta^{-1}(s) \subseteq \alpha^{-1}(s)$, and we deduce

$$\bar{c}_s = \left|\beta^{-1}(s)\right| = \left|\alpha^{-1}(s) \cap \beta^{-1}(s)\right| \leq \left|\alpha^{-1}(s) \cap N'\right|.$$

Since $\bar{c}_s = \left|\alpha^{-1}(s)\right|$ by definition, this further implies $\alpha^{-1}(s) \subseteq N'$ and $i \in N'$, which is the contrapositive conclusion. □

Let $i \in N \setminus (N' \cup i^*)$. By Claim 6, there is $j \in N \setminus N'$ such that $\beta(j) = \alpha(i)$. Let $R_j'' := R_i^\varepsilon$, and denote $R'' := \left(R_j'', R_{-j}'\right)$. *Unilateral invariance* implies that $\varphi_j(R'') = (x_i, \alpha(i))$. By *welfare anonymity*, $\varphi_i\left(R_i^\varepsilon, R_{-i}'\right) \; I_i^\varepsilon \; \varphi_j\left(R_j'', R_{-j}'\right)$ and therefore $\varphi_i\left(R_i^\varepsilon, R_{-i}'\right) \; R_i^\varepsilon \; B$. By *strategy-proofness*, $\varphi_i(R') \; R_i^p \; \varphi_i\left(R_i^\varepsilon, R_{-i}'\right)$, and recall $\varphi_i(R') \; I_i^p \; B$ by the induction premise. Since $R_i^\varepsilon \in \mathcal{T}(R_i, (x_i, \alpha(i))) \subseteq \mathcal{T}(R^p, (x_i, \alpha(i)))$, conclude that $\varphi_i\left(R_i^\varepsilon, R_{-i}'\right) = (x_i, \alpha(i))$. Let $(z, \gamma) := \varphi\left(R_i^\varepsilon, R_{-i}'\right)$.

Claim 7. For each $j \in N \setminus i$,

$$\left(z_j, \gamma(j)\right) R_j' \, D\left(R_j', p\right)$$

Proof of claim. Note that the previous paragraph establishes this claim for i. The proof for the remaining agents is by contradiction. Assume there are $j \in N$ and $\left(y_j, r\right) \in D\left(R_j', p\right)$ such that

$$\left(y_j, r\right) P_j' \left(z_j, \gamma(j)\right).$$

[10] If $\left|\gamma^{-1}(\alpha(i^*))\right| = 1$, then since $\left|\gamma^{-1}(\alpha(i^*))\right| = \left|\alpha^{-1}(\alpha(i^*))\right|$, we have $\alpha^{-1}(\alpha(i^*)) = \{i^*\}$. Since $i \neq i^*$, $\alpha(i) \neq \alpha(i^*)$, and therefore $\gamma(j) \neq \alpha(i^*)$.

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Assume first that $j \in N \setminus N'$. Thus, j's preferences are R_0^p. Since for each $k \in N$ and each $\psi \in F^*$, $\psi(R_i^\varepsilon, R'_{-i})\ I'_k\ D\left(R'_k, p\right)$, *strong undomination* implies there is $k' \in N \setminus i$ for whom

$$(z_{k'}, \gamma(k'))\ P'_{k'}\ D(R'_{k'}, p),$$

which further implies, since preferences are increasing, that $z_{k'} > wp^{-1}_{\gamma(k')}$. *Welfare anonymity* implies that the preferences of j and k' differ and therefore that $k' \in N'$. Profile $R'' := \left(R^p_{k'}, R^\varepsilon_i, R'_{-i-k'}\right)$ satisfies the induction hypothesis and therefore $\varphi_{k'}(R'') \in D\left(R''_{k'}, p\right) \subset B$. Then if R'' is the true profile, k' will manipulate by reporting $R^\varepsilon_{k'}$, contradicting *strategy-proofness*. Conclude that $j \notin N \setminus N'$.

Thus $j \in N'$; j's preferences are $R'_j = R^\varepsilon_j$. Profile $R'' := \left(R^p_j, R^\varepsilon_i, R'_{-i-j}\right)$ satisfies the induction hypothesis and therefore $\varphi(R'') \in B^N$. We also apply Claim 6: there is $k \in N$ with preferences R^p_0 such that $\varphi_k\left(R^p_j, R^\varepsilon_i, R'_{-i-j}\right) = \left(x_j, \alpha(j)\right)$. Let $R'''_k := R^\varepsilon_j$ and denote $R''' := \left(R'''_k, R''_{-k}\right)$. By *unilateral invariance*, $\varphi_k(R''') = \left(x_j, \alpha(j)\right)$. However, since $(x_j, \alpha(j)) \in D\left(R^\varepsilon_j, p\right)$,

$$\left(y_j, r\right)\ I_j\ (x_j, \alpha(j))\ P_j\ (z_j, \gamma(j)),$$

contradicting *welfare anonymity*. In sum, we have deduced that $j \notin N$. □

Claim 8. For each $j \in N'$, $(z_j, \gamma(j)) \in D\left(R'_j, p\right)$.

Proof of claim. Let $(y_j, \gamma(j)) \in B$ and suppose $z_j > y_j$. Then by the induction hypothesis, at profile $\left(R^\varepsilon_i, R^p_j, \hat{R}_{-i-j}\right)$, agent j successfully manipulates the rule by reporting R^ε_j. Thus, $y_j \geq z_j$, and by Claim 7, $y_j = z_j$. □

Claim 9. For each $j \in N \setminus N'$, $(z_j, \gamma(j)) \in D\left(R'_j, p\right)$.

Proof of claim. This argument follows mostly from feasibility. By their preferences and by *welfare anonymity*, all the agents in question consume on a set that is parallel to B. We shall show that there remains an agent $j \in N \setminus N'$ such that $\gamma(j)$ is precisely the site s^* where the special agent i^* consumes under $|p, \alpha|$. Recalling that this site is exhausted, it must be that $z_j = wp^{-1}_{\gamma(j)}$.

Since preferences are increasing, by Claim 7, $z_j \geq wp^{-1}_{\gamma(j)}$. Since we have shown in 8 that $z_j = wp^{-1}_{\gamma(j)}$ for each $k \in N' \cap \gamma^{-1}(\gamma(j))$, feasibility implies $\left|\gamma^{-1}(\gamma(j))\right| \leq \bar{c}_{\gamma(j)}$. Since j is arbitrary, for each $s \in S$, $\left|\gamma^{-1}(s)\right| \leq \bar{c}_s$. Now since, for each $s \in S$, $\left|\alpha^{-1}(s)\right| = \bar{c}_s$, and since $\sum \bar{c}_s = |N|$, it follows that $\left|\gamma^{-1}(s)\right| = \left|\alpha^{-1}(s)\right|$.

To arrive at a contradiction, suppose that $\gamma^{-1}(s^*) \subseteq N' \cup i$. Let $j \in \gamma^{-1}(s^*)$, and so j's preferences are R^ε_j. We established that $\varphi_j\left(R^\varepsilon_i, R'_{-i}\right) \in D\left(R^\varepsilon_j, p\right) = \left\{(x_j, \alpha(j))\right\}$. Therefore, $\varphi_j\left(R^\varepsilon_i, R'_{-i}\right) =$

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$\left(x_j, \alpha(j)\right)$. Thus, $\alpha(j) = s^*$. Since j is an arbitrary member of $\gamma^{-1}(s^*)$, $\gamma^{-1}(s^*) \subseteq \alpha^{-1}(s^*)$. Since $\left|\gamma^{-1}(s)\right| = \left|\alpha^{-1}(s)\right|$, $\gamma^{-1}(s^*) = \alpha^{-1}(s^*)$, and so $\alpha^{-1}(s^*) \subseteq (N' \cup i)$. But this implies $i^* \in N' \cup i$, a contradiction to the induction hypothesis.

Therefore, there is an agent $j \in N \setminus N'$ such that $\gamma(j) = s^*$. As above, $z_k \geq wp_{s^*}^{-1}$ for each $k \in \gamma^{-1}(s^*) \cap (N \setminus N')$ and $z_k = wp_{s^*}^{-1}$ for each $k \in \gamma^{-1}(s^*) \cap N'$. Since s^* is exhausted at $|p, \alpha|$, it has endowment divisible value under p, so $\overline{c}_{s^*} wp_{s^*}^{-1} = f_{s^*}(\overline{c}_{s^*})$. Since $\left|\gamma^{-1}(s^*)\right| = \overline{c}_{s^*}$, we have that for each $k \in \gamma^{-1}(s^*)$, $z_k = wp_{s^*}^{-1}$. Since j has preferences R_0^p, *welfare anonymity* implies that, for each $k \in N \setminus N'$, $\varphi_k\left(R_i^\varepsilon, R'_{-i}\right) \in D(R_0^p, p) = B$. □

Our induction argument has proven an approximate version of the lemma (because agents have R_i^ε and not R_i) for the special case $R' := \left(R_{i^*}^p, R_{-i}\right)$. Now let $i \in N$ be arbitrary. If $\alpha(i)$ is not exhausted at $|p, \alpha|$, then there exists a path $r \rightsquigarrow \alpha(i) \subset \Gamma(R; p, \alpha)$ such that r is exhausted at $|p, \alpha|$. Let α' be the path shift of α via $r \rightsquigarrow \alpha(i)$ and define α^* so that, for each $j \in N$,

$$\alpha^*(j) := \begin{cases} r & j = i \\ \alpha'(j) & \text{otherwise.} \end{cases}$$

Note that $p^*\left(R_i^p, R_{-i}\right) = p^*(R)$ and $\alpha^* \in \mathcal{A}^*\left(R_i^p, R_{-i}\right)$. Therefore, our argument holds for profile $\left(R_i^p, R_{-i}\right)$ by setting $i^* = i$ and using site assignment α^*. Conclude that for each $j \neq i$, $\varphi_j\left(R_i^p, R_{-i}^\varepsilon\right) \in D\left(R_j, p\right)$ and $\varphi_i\left(R_i^p, R_{-i}^\varepsilon\right) \in D\left(R_i^p, p\right)$, giving the approximate version of the lemma in the general case. It remains to let R_{-i}^ε converge to R_{-i}, which we do in the next paragraph.

We have shown that for each $i \in N$, each site assignment $\alpha \in S^N$ satisfying either $\alpha \in \mathcal{A}^*(R)$ or α is constructed as α^*, each $\varepsilon > 0$, and each $j \neq i$, $\varphi_j^{i,\varepsilon} := \varphi_j\left(\left(R_j^\varepsilon\right)_{j \neq i}, R_i^p\right) \in D\left(R_j^\varepsilon, p\right)$, where demand is single-valued by construction of R_j^ε. Moreover, $\varphi_i^{i,\varepsilon} := \varphi_i\left(\left(R_j^\varepsilon\right)_{j \neq i}, R_i^p\right) \in D\left(R_i^p, p\right)$. Therefore, there exists $\beta \in S^N$ such that $\varphi^{i,\varepsilon} = |p, \beta|$ and since $\sum_{s \in S} \overline{c}_s = |N|$, this leaves $\varphi_i^{i,\varepsilon} = |p, \beta(i)|$. The single valuedness of demand implies that in fact for *each* $\varepsilon > 0$, $\varphi^{i,\varepsilon} = |p, \beta|$. Since $\varepsilon > 0$ was arbitrary, *constant sequence continuity* implies that $\varphi_i\left(R_{-i}, R_i^p\right) \in D\left(R_i^p, p\right)$ and, for each $j \neq i$, $\varphi_j\left(R_{-i}, R_i^p\right) \in D\left(R_j, p\right)$. ■

The lemma proven, it remains only to show that at $\varphi(R)$, all agents are maximizing their R preferences on B. If there is an agent i with

$$\varphi_i(R)\ P_i\ D(R_i, p),$$

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then since preferences are increasing, $\varphi_i(R)$ is above B. When $\left(R_i^p, R_{-i}\right)$ is the true profile, i manipulates by reporting R_i, a contradiction. Thus we have that for each agent i,

$$D(R_i, p)\ R_i\ \varphi_i(R).$$

But then by *strong undomination* we have for each agent i that $\varphi_i(R)\ R_i\ D(R_i, p)$, and the proof of Theorem 8 is complete. ■

C.2. **Proof of Theorem 5.** An edge-labeled graph Γ is **simple** if for each $(r, s) \in S \times S$, there is at most one $i \in N$ such that $(r, s, i) \in \Gamma$. A **tree** is an acyclic, connected graph Γ such that, for each $s \in S$, there is at most one $r \in S$ with $r \to s$.[11] Thus, each tree has a unique vertex, called the **root**, with no incoming edges. A graph is a **forest** if it is the disjoint union of trees. A graph $\Gamma(R; p^*(R), \alpha)$, generated by an equilibrium assignment $|p^*(R), \alpha|$, is a **minimal simple forest** if it is a simple forest with the following property: for each tree $T \subseteq \Gamma(R; p^*(R), \alpha)$, site $r \in T$ is exhausted at $|p^*(R), \alpha|$ if and only if it is the root. For each $k \in \mathbb{Z}_+$, let $\mathcal{D}^{k-1/2} \subseteq \mathcal{D}^k$ be the set of all economies $R \in \mathcal{D}^k$ with $\alpha \in \mathcal{A}(R)$ such that $\Gamma(R; p^*(R), \alpha)$ is a minimal simple forest. This forces us to live with the awkward notation $\mathcal{D}^{-1/2}$ for the elements of $\mathcal{D}^0$ with the simple-forest property. This cost is recouped in the proof of Theorem 5 below, which is just an exercise in cumbersome notation once Lemma 10, Lemma 11, and Lemma 12 are shown.

Lemma 10. *Assume $k > 0$ and let $R \in \mathcal{D}^{k-1/2}$ and $p := p^*(R)$. For each $i \in N$, there is a site $t \in S$ and a sequence $\left\{R_i^n\right\} \subseteq \mathcal{T}\left(R_i^p, |p, t|\right)$ such that $R_i^n \to R^p$ and, for each $n \in \mathbb{N}$, $(R_i^n, R_{-i}) \in \mathcal{D}^{k-1}$.*

Proof. Let $i \in N$, $\alpha \in \mathcal{A}(R)$, and $s := \alpha(i)$. Assume $\Gamma(R; p, \alpha)$ is a minimal simple forest. Let $T \subseteq \Gamma(R; p, \alpha)$ be the tree to which s belongs and r^0 its root. There is $t \in S$, such that $\left|\alpha^{-1}(t)\right| \leq \mathsf{c}_t(p) - 1$ (otherwise $R \in \mathcal{D}^0$).

There is a unique path $r^0 \rightsquigarrow s := \left[r^0 \to r^1 \to \cdots \to (r^m := s)\right]$ in $\Gamma(R; \alpha)$ from r^0 to s. Assume first that $t \in \left(r^l\right)_{l=0}^m$, so there is $m' \in \{0, \ldots, m\}$ such that $t = r^{m'}$. Let $r^0 \rightsquigarrow r^{m'} \subseteq r^0 \rightsquigarrow s$ be the subpath from r^0 to $r^{m'}$. Let α' be the path shift of α via $r^0 \rightsquigarrow r^{m'}$. Since $\left|\alpha^{-1}(r^{m'})\right| \leq c_{r^{m'}}(p) - 1$, α' is feasible. Therefore, by Lemma 6, $\alpha' \in \mathcal{A}(R)$.

Tree $T \subseteq \Gamma(R; p, \alpha)$ is transformed into tree $T' \subseteq \Gamma(R; p, \alpha')$ by simply reversing the path $r^0 \rightsquigarrow r^{m'}$. Since $\alpha' \in \mathcal{A}(R)$, there must be some exhausted site in T'. The number of consumers

[11]This is a departure from the standard terminology. The word *tree* is usually reserved for undirected graphs, whereas the structure we describe is an *arborescence.* Since we are concerned only with directed graphs, we use the simpler term.

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has only changed for r^0 and $r^{m'}$, with r^0 losing a consumer and r^m gaining one. Thus, $r^{m'}$ is now the unique exhausted site in T', and $\Gamma(R; p, \alpha')$ remains a minimal simple tree.

If there is $t' \in \left(r^l\right)_{l=m'}^m$ such that $\left|\alpha^{-1}(t')\right| \le c_{t'}(p) - 1$, then we repeat the process in the previous paragraph to construct a site-assignment $\alpha'' \in \mathcal{A}(R)$ and find that s now belongs to tree T'' with t' as root. In sum, we may assume without loss of generality that we have chosen $\alpha \in \mathcal{A}(R)$ such that the site t is not in $r^0 \rightsquigarrow s$.

Fix $\varepsilon > 0$ and let $R_i^\varepsilon \in \mathcal{T}\left(R_i^p, |p, t|\right)$ satisfy $H^\Delta\left(R_i^\varepsilon, R_i^p\right) < \varepsilon$ (recall that since $\mathcal{R}$ is a rich domain, both $R_i^p \in \mathcal{R}$ and $R_i^\varepsilon \in \mathcal{R}$). Let $\beta \in S^N$ be given, for each $j \in N$, by $\beta(j) = t$ if $j = i$, and $\beta(j) = \alpha(j)$ otherwise. Since $\left|\alpha^{-1}(t)\right| \le c_t(p) - 1$, $|p, \beta|$ is an equilibrium for $R' := \left(R_i^\varepsilon, R_{-i}\right)$, so $p \in \mathbb{P}(R')$.

Since $\alpha(i) = s$, and since $\Gamma(R; p, \alpha)$ is a minimal simple tree, i labels no arcs in the path $r^0 \rightsquigarrow s \subseteq \Gamma(R; p, \alpha)$. Therefore, it is still the case that $r^0 \rightsquigarrow s \subseteq \Gamma(R'; p, \beta)$. Let γ be the path shift of β via $r^0 \rightsquigarrow s$. Since i has left s, $|p, \gamma|$ is feasible. Then, by Lemma 6, $|p, \gamma|$ is an equilibrium for R'. In $\Gamma(R'; p, \gamma)$, the path $r^0 \rightsquigarrow s$ has been reversed: $s \rightsquigarrow r^0 \subset \Gamma(R; p, \gamma)$.

We construct $p' \in \mathbb{R}^S$ such that for each $r^l \in s \rightsquigarrow r^0$, there is $\eta_l > 0$ with $p'_s = p_s - \eta_l$, and for each $t' \notin s \rightsquigarrow r^0$, $p'_{t'} = p^*_{t'}(R)$. Our construction is inductive, so for each $l \in \{0, \ldots, m\}$ and each $t' \in S$, let

$$p_{t'}^l := \begin{cases} p_{t'} - \eta_n & \exists n \ge l,\ t' = r^n \\ p_{t'} & \text{otherwise.} \end{cases}$$

We shall first note that these changes are feasible, for small η's, by considering the two cases. Suppose first that $r^0 = s$, then the path $r^0 \rightsquigarrow s \subset \Gamma(R; p, \alpha)$ was trivial, and $\gamma^{-1}(s) = \beta^{-1}(s) = \alpha^{-1}(s) \setminus i$. Moreover, s was exhausted at $|p, \alpha|$ yielding

$$\frac{w}{p_s}\left|\gamma^{-1}(s)\right| = \frac{w}{p_s}\left(\left|\alpha^{-1}(s)\right| - 1\right) = \frac{w}{p_s}\left(\mathsf{c}_s(p) - 1\right) < \frac{w}{p_s}\mathsf{c}_s(p) = f_s\left(\mathsf{c}_s(p)\right).$$

Thus we may lower the price of s and maintain feasibility. If $r^0 \ne s$, then i is replaced by another agent at s in going from α to β to γ. Thus, $\left|\gamma^{-1}(s)\right| = \left|\alpha^{-1}(s)\right|$. Since $\Gamma(R; p, \alpha)$ is a minimal simple tree, s was not exhausted at $|p, \alpha|$, so

$$\frac{w}{p_s}\left|\gamma^{-1}(s)\right| = \frac{w}{p_s}\left|\alpha^{-1}(s)\right| < f_s\left(\mathsf{c}_s(p)\right).$$

Thus, in the following arguments, setting each η_n sufficiently small retains feasibility.

Since there are no arcs in $\Gamma(R'; p^*(R), \gamma)$ ending at s (s is now the root of a tree), we may set η_m sufficiently small that if $s \in D\left(R'_j; p^m\right)$ then $\gamma(j) = s$. Since the price of s has decreased, for each

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$j \in \gamma^{-1}(s)$, $r^{m-1} \notin D\left(R'_j; p^m\right)$. Furthermore, since $(s = r^m) \to r^{m-1} \in \Gamma(R'; p, \gamma)$, and $\Gamma(R'; p, \gamma)$ is a simple graph, for each $j' \notin \gamma^{-1}(s) \cup \gamma^{-1}(r^{m-1})$, $r^{m-1} \notin D\left(R'_{j'}; p^*(R)\right)$. Then it remains the case that $r^{m-1} \notin D\left(R'_{j'}; p^m\right)$. Thus we may set η_{m-1} sufficiently small that if $r^{m-1} \in D_S\left(R'_k; p^{m-1}\right)$ then $r^{m-1} = \gamma(k)$. Continue inductively to define each η_l, for $l \in \{0, 1, \ldots, m\}$ and therefore define p'.

We now show that $p' \in \mathbb{P}(R')$. Let $j \in \gamma^{-1}\left(\{r^0, r^1, \ldots, s\}\right)$. By construction, $D_S\left(R'_j; p'\right) = \{\gamma(j)\}$. Now let $j \in N \setminus \gamma^{-1}\left(\{r^0, r^1, \ldots, s\}\right)$. Then since there are no arcs $r \xrightarrow{j} r' \in \Gamma(R'; p, \beta)$ with $r' \in \{r^0, r^1, \ldots, s\}$, $D_S\left(R_j; p\right) \cap \{r^0, r^1, \ldots, s\} = \emptyset$. Since each η_l is small, $D_S\left(R_j; p'\right) \cap \{r^0, r^1, \ldots, s\} = \emptyset$. Since the prices of the sites $S \setminus \{r^0, r^1, \ldots, s\}$ remain unchanged under p, $D\left(R'_j; p'\right) = D\left(R'_j; p\right)$. We conclude that $|p', \gamma|$ is an equilibrium for R' and therefore that $p' \in \mathbb{P}(R')$.

Since r^0 is exhausted at $|p, \alpha|$, $\left|\alpha^{-1}(r^0)\right| = \tilde{c}_{r^0}(p)$, and therefore $\mathsf{c}_{r^0}(p) = \tilde{c}_{r^0}(p)$. Since, $\left|\gamma^{-1}(r^0)\right| = \left|\alpha^{-1}(r^0)\right| - 1$, we have $\left|\gamma^{-1}(r^0)\right| \leq \tilde{c}_{r^0}(p) - 1$. Recalling that exhaustion implies

$$\frac{w}{p_{r^0}} = \frac{f\left(\tilde{c}_{r^0}(p)\right)}{\tilde{c}_{r^0}(p)},$$

congestion implies that, for η_0 sufficiently small, $\tilde{c}_{r^0}(p') = \tilde{c}_{r^0}(p) - 1$. Therefore, $\mathsf{c}_{r^0}(p') = \tilde{c}_{r^0}(p')$ and $\mathsf{c}_{r^0}(p') \leq \mathsf{c}_{r^0}(p) - 1$. We may thus write

$$\begin{aligned}
\sum \mathsf{c}_r(p') &= \mathsf{c}_{r^0}(p') + \sum_{r \in \{r^1, \ldots, r^m\}} \mathsf{c}_r(p') + \sum_{r \in S \setminus \{r^0, r^1, \ldots, r^m\}} \mathsf{c}_r(p') \\
&\leq \mathsf{c}_{r^0}(p) - 1 + \sum_{r \in \{r^1, \ldots, r^m\}} \mathsf{c}_r(p') + \sum_{r \in S \setminus \{r^0, r^1, \ldots, r^m\}} \mathsf{c}_r(p') \\
&\leq \mathsf{c}_{r^0}(p) - 1 + \sum_{r \in \{r^1, \ldots, r^m\}} \mathsf{c}_r(p) + \sum_{r \in S \setminus \{r^0, r^1, \ldots, r^m\}} \mathsf{c}_r(p') \\
&= \mathsf{c}_{r^0}(p) - 1 + \sum_{r \in \{r^1, \ldots, r^m\}} \mathsf{c}_r(p) + \sum_{r \in S \setminus \{r^0, r^1, \ldots, r^m\}} \mathsf{c}_r(p) \\
&= -1 + \sum c_r(p).
\end{aligned}$$

∎

Lemma 11. *Assume φ is strategy-proof, welfare anonymous, constant sequence continuous, and strongly undominated in these properties. Let $k \in \mathbb{Z}_+$, and $R \in \mathcal{D}^{k+1/2}$. Assume also that, for each $R' \in \mathcal{D}^k$, $\varphi(R') \in F^*(R')$. Then $\varphi(R) \in F^*(R)$.*

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Proof. Let $i \in N$. Since $R_i^p \in \mathcal{R}$, $p^*(R_i^p, R_{-i}) = p$, and, since F^* is *strategy-proof*, for each $(y,t) \in F^*(\mathcal{R}, R_{-i})$, $F^*\left(R_i^p, R_{-i}\right) \; R_i^p \; (y,t)$. It follows that each $(y,t) \in F^*(\mathcal{R}, R_{-i})$ is affordable at prices p.

By Lemma 10, there is $t \in S$ and a sequence $\left\{R_i^n\right\} \subseteq \mathcal{T}\left(R_i^p, |p,t|\right)$ such that $R_i^n \to R_i^p$ and, for each $n \in \mathbb{N}$, $\left(R_i^n, R_{-i}\right) \in \mathcal{D}^k$. By hypothesis, for each $n \in \mathbb{N}$, $\varphi\left(R_i^n, R_{-i}\right) \in F^*(R_i^n, R_{-i})$. The sequence of lower-contour sets $\left\{R_i^n \downarrow F^*\left(R_i^n, R_{-i}\right)\right\}^{n\in\mathbb{N}}$ converges to the budget set $B := \{(x_i, s) \in \mathbb{R} \times S \,:\, p_s x_i \leq w\}$. By *strategy-proofness*, for each $n \in \mathbb{N}$, $\varphi_i(R) \in R_i^n \downarrow \varphi_i\left(R_i^n, R_{-i}\right) = R_i^n \downarrow F_i^*\left(R_i^n, R_{-i}\right)$. Conclude that $\varphi_i(R) \in B$.

Since $i \in N$ was arbitrary, in fact each bundle assigned at $\varphi(R)$ is in B. *Strong undomination* then yields $\varphi(R) \in F^*(R)$. ■

Lemma 12. *Fix $k \in \mathbb{Z}_+$. Assume that, for each $R' \in \mathcal{D}^{k-1/2}$, $\varphi(R') \in F^*(R')$. Let $R \in \mathcal{D}^k$. There is a sequence $\{R^n\} \subseteq \mathcal{D}^{k-1/2}$ such that $R^n \to R$; if φ is constant sequence continuous, then $\varphi(R) \in F^*(R)$.*

Proof. Let $R \in \mathcal{D}^k \setminus \mathcal{D}^{k-1/2}$ and $\alpha \in \mathcal{A}^*(R)$. Assume α is balanced. We shall find a minimal simple tree $\Gamma' \subseteq \Gamma(R; p^*(R), \alpha)$ by selecting paths. Let $s \in S$. By Lemma 5, there is a path $r \rightsquigarrow s$ in $\Gamma(R; p^*(R), \alpha)$ such that r is exhausted at $|p^*(R), \alpha|$. Moreover, we may choose $r \rightsquigarrow s$ such that r is the *only* exhausted site in the path. We extend this path as follows: if there is $i \in N$ and $t \notin r \rightsquigarrow s$ such that i) t is *not* exhausted at $|p^*(R), \alpha|$ and ii) $(s,t,i) \in \Gamma(R; p^*(R), \alpha)$, then append the arc (s,t,i) to the path. Continue on in this way until it is no longer possible and add the resulting path to Γ'. Repeat the process until each $s' \in S$ is connected to one and only one exhausted site via our chosen paths (some paths may consist of a single, exhausted site and no arcs). Let Γ' be the graph resulting from the union of these paths. Clearly, Γ' is a minimal simple forest such that for each unexhausted $s \in S$, there is a path $r^0 \rightsquigarrow s \subseteq \Gamma'$ with r^0 exhausted.

For each $\varepsilon > 0$, construct economy $R^\varepsilon \in \mathcal{R}^N$ as follows. For each $i \in N$, identify the set $\Gamma_i' := \{\tilde{s} \in S \,:\, \exists t \in S, \text{ s.t. } (\tilde{s}, t, i) \in \Gamma' \text{ or } (t, \tilde{s}, i) \in \Gamma'\}$. Let $A_i := \left\{|p, s| \in \mathbb{R} \times S \,:\, s \in \Gamma_i'\right\}$. Now let $R_i^\varepsilon \in \mathcal{T}(R_i, A_i)$ be such that $H^\Delta\left(R_i^\varepsilon, R_i\right) < \varepsilon$ (again, richness ensures $R_i^\varepsilon \in \mathcal{R}$). By construction, $\Gamma(R^\varepsilon; p^*(R)\alpha) = \Gamma'$. Moreover, since α is balanced, Theorem 2 implies $|p^*(R), \alpha| \in F^*(R^\varepsilon)$. Therefore, $R^\varepsilon \in \mathcal{D}^{k-1/2}$.

It is also easy to verify that for each R^ε, $|F^*(R^\varepsilon)| = 1$, and, moreover, for each pair ε, ε', $F^*(R^\varepsilon) = F^*(R^{\varepsilon'})$. Therefore, $\varphi(R^\varepsilon) = \varphi\left(R^{\varepsilon'}\right)$. We generate the desired sequence R^n via a decreasing sequence $\varepsilon^n \in \mathbb{R}$ such that $\varepsilon^n \to 0$. *Constant sequence continuity* then implies that $\varphi(R)$ is welfare equivalent to $F^*(R)$.

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Now suppose there is $i \in N$ consuming above the $F^*(R)$ budget set: $\varphi_i(R) = (x_i, s)$ and $x > wp^*_s(R)^{-1}$. Instead have agent i declare a sequence of preferences $R_i^{\varepsilon n} \in \mathcal{T}\left(R_i^{p^*(R)}, A_i\right)$. The above arguments all go through: $\varphi\left(R_i^{p^*(R)}, R_{-i}\right)$ is welfare equivalent to $F^*(R_i^{p^*(R)}, R_{-i})$, and, moreover, $p^*(R_i^{p^*(R)}, R_{-i}) = p^*(R)$. Thus at economy $(R_i^{p^*(R)}, R_{-i})$, i prefers to report R_i, contradicting *strategy-proofness.* ■

Proof of Theorem 5. Let $R \in \mathcal{R}^N$. There is $k \in \mathbb{Z}_+$ such that $R \in \mathcal{D}^k$. Let $R^k := R$. Let $\sigma\left(R^k\right) \subseteq \mathcal{D}^{k-1/2}$ be the sequence in Lemma 12. For each $R^{k-1/2} \in \sigma\left(R^k\right)$, there is a sequence $\sigma\left(R^{k-1/2}\right) \subseteq \mathcal{D}^{k-1}$ as defined in Lemma 10. Collect these sequences in $\Sigma(k-1)$. For each $\sigma\left(R^{k-1/2}\right) \in \Sigma(k-1)$, and each $R^{k-1} \in \sigma\left(R^{k-1/2}\right)$, let $\sigma\left(R^{k-1}\right) \subseteq \mathcal{D}^{k-1-1/2}$ be the sequence in Lemma 12. Collect these sequences in $\Sigma(k-1-1/2)$. Continuing by recursion, we define the families $\{\Sigma(k-q) : q \in \{1, \ldots, k\}\}$ and $\{\Sigma(k-q-1/2) : q \in \{1, \ldots, k\}\}$.

Fix $k'-1 \in \mathbb{Z}_+$ and $R \in \mathcal{D}^{k'}$. Assume that for each $\sigma\left(R^{k'-1/2}\right) \in \Sigma(k'-1)$, and each $R \in \sigma\left(R^{k'-1/2}\right)$, $\varphi(R) \in F^*(R)$ (note that the base case, $k'-1 = 0$, is proven by Theorem 8). Let $\sigma\left(R^{k'}\right) \in \Sigma(k'-1/2)$. Each $R^{k'-1/2} \in \sigma\left(R^{k'}\right)$ has an associated sequence $\sigma\left(R^{k'-1/2}\right) \in \Sigma(k'-1)$, which satisfies the induction hypothesis. Since $\sigma\left(R^{k'-1/2}\right)$ is the sequence for $R^{k'-1/2}$ constructed in Lemma 10, Lemma 11 then implies that $\varphi\left(R^{k'-1/2}\right) \in F^*\left(R^{k'-1/2}\right)$.

The previous implies that each $\tilde{R}^{k'-1/2} \in \sigma\left(R^{k'}\right)$ satisfies $\varphi\left(\tilde{R}^{k'-1/2}\right) \in F^*\left(\tilde{R}^{k'-1/2}\right)$. Since $\sigma\left(R^{k'}\right)$ is the sequence constructed in Lemma 12, the Lemma implies $\varphi\left(R^{k'}\right) \in F^*(R^{k'})$. ■

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