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Anomalous magnetic moment of the muon, a hybrid approach

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Abstract

A new QCD sum rule determination of the leading order hadronic vacuum polarization contribution to the anomalous magnetic moment of the muon, $a_\mu^{\rm hvp}$, is proposed. This approach combines data on e^+e^- annihilation into hadrons, perturbative QCD and lattice QCD results for the first derivative of the electromagnetic current correlator at zero momentum transfer, $\Pi'_{\rm EM}(0)$. The idea is based on the observation that, in the relevant kinematic domain, the integration kernel $K(s)$, entering the formula relating $a_\mu^{\rm hvp}$ to e^+e^- annihilation data, behaves like $1/s$ times a very smooth function of s, the squared energy. We find an expression for a_μ in terms of $\Pi'_{\rm EM}(0)$, which can be calculated in lattice QCD. Using recent lattice results we find a good approximation for $a_\mu^{\rm hvp}$, but the precision is not yet sufficient to resolve the discrepancy between the $R(s)$ data-based results and the experimentally measured value.

1 Introduction

The discrepancy between the theoretical prediction of the muon magnetic moment anomaly, a_μ, and its experimentally measured value constitutes one of the few remaining problems for the Standard Model. The main uncertainty arises from hadronic contributions. The leading order hadronic vacuum polarization contribution can be expressed as an integral over the total hadronic e^+e^- annihilation cross section $R(s)$ multiplied by a kernel $K(s)/s$ where s is the square of the center-of-mass energy. The range of integration extends from threshold, $s_{\rm thr}$, to infinity. The kernel $K(s)$ is strongly peaked at small s so that the integral is dominated by the e^+e^- cross section in the $\pi\pi$ channel. The available data have recently improved significantly, leading to the value $a_\mu^{\rm had} = (693.1 \pm 3.4) \times 10^{-10}$ [1] using the phenomenological dispersion relation analysis. This result leads to a deviation of the Standard Model prediction from the direct measurement of a_μ [2] by $a_\mu^{\rm exp} - a_\mu^{\rm SM} = (26.8 \pm 7.6) \times 10^{-10}$, corresponding to a 3.5σ discrepancy. Similar findings are reported by an alternative evaluation of the dispersion integral [3].

In addition to phenomenological determinations of $a_\mu^{\rm hvp}$ relying on experimental data, it is important to exploit fully, or partly, some theoretical estimates of the anomaly. They all start with a determination of the hadronic part of the vacuum polarization, i.e. the correlator of two electromagnetic currents, $\Pi_{\rm EM}(s)$. Most of the theoretical calculations are based on QCD sum rules (QCDSR) [3–6] or on lattice QCD (LQCD). The calculation of the correlator $\Pi_{\rm EM}(Q^2)$ in the Euclidean domain is a major activity within the LQCD community. Its main purpose is the determination of a_μ by evaluating a convolution integral involving the subtracted correlator $\Pi_{\rm EM}(Q^2) - \Pi_{\rm EM}(0)$ in the Euclidean domain, i.e., for $Q^2 < 0$ [7,8]. A large part of that calculation is concerned with the determination of the additive renormalization, $\Pi_{\rm EM}(0)$, as well as the slope $\Pi'_{\rm EM}(0)$. The latter gives the leading contribution in the representation of $\Pi_{\rm EM}(Q^2) - \Pi_{\rm EM}(0)$ in terms of Padé approximants [9–11].

Some time ago it was shown in Refs. [12, 13] that a very precise approximation of the kernel $K(s)$ by a meromorphic function allows us to reduce the required theoretical information to a few derivatives of $\Pi'_{\rm EM}(s)$ at the origin. At the time of this proposal, numerical predictions relied on model-dependent estimates of these derivatives. Since then, though, precise results from LQCD have become available [14–22]. LQCD results for the first derivative of $\Pi_{\rm EM}(s)$ at $s = 0$ have been used in [13], updating [12]. A recent systematic approach, beyond a simple Taylor expansion of $\Pi_{\rm EM}(s)$, makes use of a Mellin-Barnes representation [23]. This allows one to express $a_\mu^{\rm hvp}$ as a series over moments and log-enhanced moments. It has been argued that this provides a tool to obtain precise results from LQCD for Euclidean momenta. Convergence properties and results based on additional model assumptions have been studied recently in [24] and [25].

In this note we discuss a new hybrid method which combines QCDSR and LQCD, and requires the use of data on $R(s)$. However, the latter contributes only as a small correction so that data errors become quite irrelevant. There is also a small contribution from the asymptotic region where one can safely use perturbative QCD (PQCD). The most important contribution, by far, arises from the derivative of the electromagnetic correlator $\Pi'_{\rm EM}(s=0)$ determined by recent LQCD calculations.

2 Hybrid QCD sum rule

The standard expression of the hadronic contribution to the muon anomaly is given by

$$a_\mu = \frac{\alpha_{\rm EM}^2}{3\,\pi^2}\int_{s_{\rm thr}}^\infty \frac{ds}{s} K(s) R(s)\,, \tag{1}$$

where $\alpha_{\rm EM}$ is the electromagnetic fine structure constant and $R(s)$ is the bare cross section for $e^+e^- \to$ hadrons, normalized to the cross section for μ-pair production including final state radiation. The integral starts at a threshold $s_{\rm thr} > 0$, which is often identified with the two-pion threshold[1], $s_{\rm thr} = 4m_\pi^2$. In terms of the correlator $\Pi_{\rm EM}(s)$ of two electromagnetic currents, we have $R(s) = 3\sum_f Q_f^2\,[6\,\pi\,{\rm Im}\,\Pi_{\rm EM}(s)]$ and $\Pi_{\rm EM}(s)$ is defined as

$$\Pi^{\mu\nu}_{\rm EM}(q^2) = i\int d^4x\, e^{iqx}\langle 0|T\left(j^\mu_{\rm EM}(x)\, j^\nu_{\rm EM}(0)\right)|0\rangle = (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi_{\rm EM}(q^2)\,, \tag{2}$$

where $j^\mu_{\rm EM}(x) = \sum_c\sum_f Q_f \bar{q}_{f,c}(x)\gamma^\mu q_{f,c}(x)$ is the electromagnetic current and Q_f the quark charges. The integration kernel $K(s)$ is given by

$$K(s) = \int_0^1 dx\, \frac{x^2(1-x)}{x^2 + \frac{s}{m_\mu^2}(1-x)}\,, \tag{3}$$

where $m_\mu = 0.105658$ GeV is the muon mass. An analytic form can be found, e.g. in Ref. [26] and will be used for our numerical evaluations. The function $sK(s)$ is slowly varying in the range of integration. It increases monotonically from 2.36×10^{-3} GeV^2 at $s = 4m_\pi^2$ to $m_\mu^2/3 \simeq 3.72\times 10^{-3}$ at $s=\infty$. This means that the kernel $K(s)/s$ behaves roughly like $1/s^2$ over the range of integration. Thus the kernel gives a very large weight to the low-energy region, in particular to the ρ-meson contribution.

At first, we consider in Eq. (1) only the light-quark contribution. Hence, the integral extends from $s_{\rm thr} = 4m_\pi^2 \simeq 0.078$ GeV^2 to $s = s_0$ where s_0 should be in the scaling region, but below the heavy quark threshold, i.e. in the range 3 GeV$^2 \leq s_0 \leq 9$ GeV^2. For definiteness, we choose $s_0 = 4$ GeV^2. The remaining integral from s_0 to ∞ can be safely computed in PQCD; it is anyway small due to the rapid fall-off of $K(s)$.

In order to use finite energy QCD sum rules (FESR) one needs to approximate $K(s)$ by a meromorphic function $K_1(s)$. As $K(s)/s$ behaves approximately like $1/s^2$, we can choose,

[1] The correct threshold is $s_{\rm thr} = m_{\pi^0}^2$ for the $\pi^0\gamma$ final state. Its contribution to $a_\mu^{\rm hvp}$ is very small, but it is included in our data integral (see below).

for example,

$$\frac{K_1(s)}{s} = \frac{c_{-2}}{s^2} + c_0 + c_1 s \quad \text{for} \quad s_{\rm thr} \leq s \leq s_0 \,. \tag{4}$$

Below we will derive a sum rule for $a_\mu^{(u,d,s)}$ which is independent of the choice of the coefficients c_i. To be specific, we determine the constants c_{-2}, c_0, and c_1 by the conditions

$$\int_{s_{\rm thr}}^{s_0} \frac{K(s)}{s} s^n ds = \int_{s_{\rm thr}}^{s_0} \frac{K_1(s)}{s} s^n ds \tag{5}$$

for $n = 0, 1, 2$. This leads to

$$c_{-2} = 2.762 \times 10^{-3}\ {\rm GeV}^2, \quad c_0 = 4.136 \times 10^{-4}\ {\rm GeV}^{-2}, \quad c_1 = -9.914 \times 10^{-5}\ {\rm GeV}^{-4}\,. \tag{6}$$

We will use this approximation in subsequent sections to illustrate the relative size of different contributions to a_μ. However, we will also consider other possible choices for the approximate kernel.

For three flavors the integral in Eq. (1) can be separated into a low- and a high-energy part. In the low-energy part we add and subtract the approximation $K_1(s)$ to obtain

$$\begin{aligned} a_\mu^{(u,d,s)} &= \frac{\alpha_{\rm EM}^2}{3\,\pi^2} \int_{4m_\pi^2}^{s_0} \frac{ds}{s}\ [K(s) - K_1(s) + K_1(s)]\ 12\pi\ {\rm Im}\,\Pi_{\rm EM}(s) \\ &+ \frac{\alpha_{\rm EM}^2}{3\,\pi^2} \int_{s_0}^{\infty} \frac{ds}{s}\ K(s) R(s) \end{aligned}$$

Subsequently, we re-write the low-energy part in terms of $K_1(s)$:

$$\begin{aligned} &\int_{4m_\pi^2}^{s_0} \frac{ds}{s} K_1(s)\ {\rm Im}\,\Pi_{\rm EM}(s) \\ &= -\frac{1}{2i} \oint_{|s|=s_0} \frac{ds}{s} K_1(s) \Pi_{\rm EM}(s) + \ \pi {\rm Res}\left[\frac{1}{s} K_1(s) \Pi_{\rm EM}(s)\right]_{s=0} \\ &= -\frac{1}{2i} \oint_{|s|=s_0} \frac{ds}{s}\ K_1(s) \Pi_{\rm EM}(s) + \pi c_{-2} \Pi'_{\rm EM}(0)\,, \end{aligned} \tag{7}$$

where the integral around the circle of radius s_0 can be computed using PQCD.

We have thus four contributions to the anomalous magnetic moment of the muon:

$$a_\mu^{(u,d,s)} = a_\mu^{(1)} + a_\mu^{(2)} + a_\mu^{(3)} + a_\mu^{(4)} \tag{8}$$

where

$$a_\mu^{(1)} = \frac{\alpha_{\rm EM}^2}{3\,\pi^2}\, 6\pi i \oint_{|s|=s_0} \frac{ds}{s}\, K_1(s)\Pi_{\rm EM}(s)\,, \tag{9}$$

$$a_\mu^{(2)} = \frac{\alpha_{\rm EM}^2}{3\,\pi^2}\, 12\pi^2 c_{-2}\Pi'_{\rm EM}(0)\,, \tag{10}$$

$$a_\mu^{(3)} = \frac{\alpha_{\rm EM}^2}{3\,\pi^2} \int_{4m_\pi^2}^{s_0} \frac{ds}{s}\, \left[K(s) - K_1(s)\right] R(s)\,, \tag{11}$$

$$a_\mu^{(4)} = \frac{\alpha_{\rm EM}^2}{3\,\pi^2} \int_{s_0}^{\infty} \frac{ds}{s}\, K(s)\; R(s)\,. \tag{12}$$

Only the third part, $a_\mu^{(3)}$ will be calculated using experimental data, with the others obtained entirely from theory. It turns out that the data-dependent contribution is only a small correction. Experimental uncertainties are consequently considerably suppressed. It is important to emphasize that the sum rule Eq. (8) is exact. The precision of the approximation $K_1(s)$ is not relevant for the total value of $a_\mu^{(u,d,s)}$. This is because what is included is the difference $K(s)-K_1(s)$. A good approximation to the kernel will, however, reduce the impact of data uncertainties entering $a_\mu^{(3)}$.

The first derivative of $\Pi_{\rm EM}$ at $s = 0$ enters in Eq. (10) because the approximate kernel $K_1(s)$ contains a term proportional to $1/s$. In principle it is possible to include higher inverse powers of s, as e.g. in Ref. [12], where a very good approximation of the kernel $K(s)$ was found using powers s^{-n} up to $n = 3$. There, the data integral, $a_\mu^{(3)}$ turned out to be completely negligible, at the level below one permille. Instead, higher derivatives of $\Pi_{\rm EM}$ would contribute:

$$\frac{K_1(s)}{s} = \sum_{n\geq 2} \frac{c_{-n}}{s^n} + c_0 + c_1 s \quad\longrightarrow\quad a_\mu^{(2)} = \frac{\alpha_{\rm EM}^2}{3\,\pi^2}\, 12\pi^2 \sum_{n\geq 2} \frac{c_{-n}}{(n-1)!} \left(\frac{d}{ds}\right)^{n-1} \Pi_{\rm EM}(s)\Bigg|_{s=0}\,. \tag{13}$$

With such a modification of our approach one would reduce the contributions which can be calculated in PQCD and from data in favor of non-perturbative input from LQCD. Sufficiently precise results for the higher derivatives from LQCD are, however, not yet available and we will not study this possibility further. Next, one has to add the heavy quark contributions. These will be obtained from PQCD as described below. In subsequent sections we will discuss each term in Eq. (8).

3 Calculation of the low-energy PQCD part: $a_\mu^{(1)}$

The integral in Eq. (9) involving $\Pi_{\rm EM}$ contains a dominant purely perturbative term, plus corrections due to condensates and duality violations. In PQCD, the current-current correlator is usually defined for the vector current of a single quark type, so that for three light flavors, using $3\sum Q_f^2 = 2$, one has $\Pi_{\rm EM} = (2/3)\Pi^{\rm PQCD}$. We calculate the integral containing $\Pi^{\rm PQCD}$ using fixed-order perturbation theory in the light-quark sector at five-loop level. To achieve this we make use of moments defined by

$$M_N(s_0) = 4\pi^2 \int_0^{s_0} \frac{ds}{s_0}\left[\frac{s}{s_0}\right]^N \frac{1}{\pi}\,{\rm Im}\,\Pi^{\rm PQCD}(s) = \frac{-1}{2\pi i}\oint_{|s|=s_0}\frac{ds}{s_0}\left[\frac{s}{s_0}\right]^N 4\pi^2\Pi^{\rm PQCD}(s)\,, \quad (14)$$

with [27]

$$\begin{aligned} 4\pi^2\Pi^{\rm PQCD}(s) &= -L - La - a^2\left(Lk_2 - \frac{1}{2}L^2\beta_0\right) \quad (15)\\ &\quad -a^3\left(Lk_3 + \frac{1}{3}L^3\beta_0^2 + L^2\left(-\frac{1}{2}\beta_1 - \beta_0 k_2\right)\right)\\ &\quad -a^4\left(Lk_4 - \frac{1}{4}L^4\beta_0^3 + \frac{1}{6}L^3\beta_0\left(5\beta_1 + 6\beta_0 k_2\right)\right.\\ &\quad \left. +L^2\left(-\frac{1}{2}\beta_2 - \frac{3}{2}\beta_0 k_3 - \beta_1 k_2\right)\right) \end{aligned}$$

where $L = \ln(-s/\mu^2)$ and $a = \alpha_s(\mu^2)/\pi$ is the running strong coupling at a renormalization scale μ^2. The constants are $k_1 = 1$, $k_2 = 1.6398$, $k_3 = 6.3711$, $k_4 = 49.076$, $\beta_0 = \frac{9}{4}$, $\beta_1 = 4$, $\beta_2 = 10.060$, $\beta_3 = 47.228$. The result for the low-energy PQCD contribution to the anomaly is

$$a_\mu^{(1)} = \frac{\alpha_{\rm EM}^2}{3\,\pi^2}\,2s_0\left[\frac{c_{-2}}{s_0^2}M_{-2}(s_0) + c_0 M_0(s_0) + c_1 s_0 M_1(s_0)\right]\,. \quad (16)$$

Explicit expressions for the moments can be found in Ref. [28] (see also [29]). We use $\alpha_s(M_Z) = 0.1181\pm0.0011$ [2], corresponding to $\alpha_s(m_\tau^2) = 0.321\pm0.009$. With $\mu^2 = s_0 = 4$ GeV^2 we obtain

$$M_{-2} = -1.0774\pm0.0010, \quad M_0 = 1.1433\pm0.0052, \quad M_1 = 0.5559\pm0.0013\,.$$

With the values for c_{-2}, c_0 and c_1 from Eq. (6)) the low-energy PQCD part of $a_\mu^{(1)}$ becomes

$$a_\mu^{(1)} = (9.56\pm0.21)\times10^{-10}\,. \quad (17)$$

Alternatively, using contour-improved perturbation theory [30, 31], where the running (i.e. s-dependent) strong coupling constant $\alpha_s(\mu^2 = s)$ is used inside the integrals for the moments, we obtain $M_{-2} = -1.0794$, $M_0 = 1.1372$, $M_1 = 0.5509$ and $a_\mu^{(1)} = 9.44\times10^{-10}$.

This contribution to a_μ is small and its uncertainty due to scale variations or the uncertainty from α_s is negligible for the total. Finally, we consider additional uncertainties due to higher-dimensional operators, e.g. the gluon condensate, and due to duality violations. The contribution of the gluon condensate is given by

$$\Pi_V^{\rm GG}(s) = \frac{1}{s^2}\sum_{f=u,d,s} Q_f^2\frac{1}{12}\left(1+\frac{7}{6}a\right)\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle\,, \quad (18)$$

leading to

$$a_\mu^{(1)\mathrm{GG}} = \frac{\alpha_{\mathrm{EM}}^2}{3\,\pi^2}\frac{-1}{2\pi i}\oint_{|s|=s_0} ds\, c_1 8\pi^2 \Pi_V^{\mathrm{GG}}(s)\,. \tag{19}$$

The numerical value of the gluon condensate is known with large uncertainties. For illustration we use a conservative estimate based on Ref. [32], $\left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle = 0.015 \pm 0.015$ GeV^4, which covers most of the phenomenological determinations. This leads to

$$a_\mu^{(1)\,\mathrm{GG}} = (0.12 \pm 0.12) \times 10^{-10} \tag{20}$$

which is two orders of magnitude smaller than $a_\mu^{(1)}$ from PQCD. If $K_1(s)$ does not contain a linear term, i.e. if $c_1 = 0$ in Eq. 4, there is still a contribution from the higher-order term in Eq. (18) due to the logarithmic s-dependence of α_s. This contribution to $a_\mu^{(1)}$ will be even smaller since it is suppressed by an extra power of α_s. We have also checked that strange-quark mass effects are tiny ($a_\mu^{(1)\,\mathrm{strange}} = 0.05 \times 10^{-10}$ for $m_s = 0.1$ GeV) and hence can safely be neglected. Finally, also duality violations are expected to be very small due to their expected exponential fall-off with increasing energy. Indeed, using the model described in Ref. [33] we can estimate their contribution as $a_\mu^{(1)\mathrm{DV}} \simeq 0.06 \times 10^{-10}$.

4 Contribution of the pole residue part: $a_\mu^{(2)}$

In the present hybrid approach, the main contribution to a_μ is due to the residue term involving the first derivative of the electromagnetic correlator at the origin

$$a_\mu^{(2)} = \frac{\alpha_{\mathrm{EM}}^2}{3\,\pi^2}\, 12\pi^2 c_{-2} \Pi'_{\mathrm{EM}}(0)\,. \tag{21}$$

This quantity, dominated by non-perturbative physics, can be determined using recent results obtained in LQCD [20–22]. These calculations are afflicted with a number of systematic uncertainties (such as discretization artefacts, finite-volume effects or isospin-breaking contributions) which must be brought under sufficient control in order to achieve a competitive determination of a_μ^{hvp}. The slope at the origin, $\Pi'_{\mathrm{EM}}(0)$, can be obtained by performing a fit to the numerical data for $\Pi_{\mathrm{EM}}(Q^2)$. Alternatively, its value is accessible by computing the second time moment of the spatially summed vector correlator [19].

In Table 1 we show a compilation of results for $\Pi'_{\mathrm{EM}}(0)$ from LQCD. While the three calculations produce results in the same ballpark, they differ in several technical aspects. In Refs. [20] and [21] staggered quarks are employed to discretize the QCD action, and the slope is determined by evaluating the second time moment of the vector correlator. By contrast, Mainz/CLS [22] use Wilson quarks and perform a low-order Padé fit to determine the slope. Still, the results listed in Table 1 differ significantly, given the quoted errors. The most likely explanation is that systematic uncertainties are not sufficiently controlled in some (or all) of these calculations. Theoretical predictions for the hadronic vacuum

$\Pi'_{\rm EM}(0)$	Collab.	$a\,[\rm fm]$
0.0883(59)	Mainz/CLS [22]	C.L.
0.0959(30)	BMW [21]	C.L.
0.0889(16)	HPQCD [20]	0.15
0.0892(14)	HPQCD [20]	0.12

Table 1: Recent results for $\Pi'_{\rm EM}(0)$ computed in LQCD at the physical pion mass. Results labelled by "C.L." have been extrapolated to the continuum limit of vanishing lattice spacing, $a = 0$. In this case the uncertainty on $\Pi'_{\rm EM}(0)$ includes a contribution from the continuum limit extrapolation. The result of Ref. [21] includes a negative contribution from quark-disconnected contractions, and the (small) contribution from charm quarks contained in that result has been subtracted.

polarization contribution to the muon $g-2$ obtained on the basis of these results require a critical assessment.

Using the result from Ref. [22] to estimate the slope parameter we find $\Pi'_{\rm EM}(0) = (0.0883 \pm 0.0059)\ {\rm GeV}^{-2}$, which implies that the residue part of the anomaly is evaluated as $a_\mu^{(2)} = (519.7 \pm 34.6) \times 10^{-10}$. Thus, the residue term constitutes the largest contribution to $a_\mu^{(2)}$ by far. However, it also makes the largest contribution to the error, and this remains true if $\Pi'_{\rm EM}(0)$ is replaced by the estimates from Refs. [20] or [21].

5 Data integral contribution: $a_\mu^{(3)}$

In this hybrid approach the integral over the data is

$$a_\mu^{(3)} = \frac{\alpha_{\rm EM}^2}{3\,\pi^2} \int_{4m_\pi^2}^{s_0} \frac{ds}{s}\,\left[K(s) - K_1(s)\right]\,R(s)\,. \tag{22}$$

For $R(s)$ we use a previous compilation of data from [34], which was already used in [35]. The result for $s_0 = 4\ {\rm GeV}^2$ is

$$a_\mu^{(3)} = (55.5 \pm 0.6) \times 10^{-10}\,. \tag{23}$$

Hence, in this hybrid approach, the data contribution to the anomaly is small, i.e. around 8 % of the total hadronic contribution. Data errors are reduced correspondingly, and details of the handling of data and their uncertainties is presently irrelevant.

6 High-energy asymptotic contribution: $a_\mu^{(4)}$

A numerical integration of the high-energy contribution, Eq. (12), using Eq. (15), is straightforward. We set $\mu^2 = s$ and use the running α_s from `RunDec` [36] with $\alpha_s(M_Z) = 0.1181$. We find

$$a_\mu^{(4)} = 35.37 \times 10^{-10} \tag{24}$$

with a negligible error of $\pm 0.05 \times 10^{-10}$ due to the uncertainty in α_s. Varying the renormalization scale μ in Eq. (15) up and down by a factor of two, increases $a_\mu^{(4)}$ by $+0.61$ and $+0.44$, respectively.

We have checked that at high energies the kernel $K(s)$ can be safely approximated by its asymptotic form $K(s) \simeq \frac{m_\mu^2}{3s}$, with a precision of better than 2 %. Hence, $a_\mu^{(4)}$ can be calculated analytically and expressed in terms of a PQCD moment:

$$\begin{aligned} a_\mu^{(4)} &= \left(\frac{\alpha_{\mathrm{EM}} m_\mu}{3\,\pi}\right)^2 2 \int_{s_0}^{\infty} \frac{ds}{s^2}\, 4\pi^2 \frac{1}{\pi} \operatorname{Im} \Pi^{\mathrm{PQCD}}(s) \\ &= -\left(\frac{\alpha_{\mathrm{EM}} m_\mu}{3\,\pi}\right)^2 \frac{2}{s_0} M_{-2}(s_0) \\ &= 36.05 \times 10^{-10}\,. \end{aligned} \tag{25}$$

The use of PQCD at squared energies above $s_0 = 4$ GeV^2 is well justified. For instance, the excellent agreement of $R(s)$ with PQCD is supported by the recent KEDR data [37,38] in the range $\sqrt{s} = 1.84 - 3.05$ GeV. Moreover, since $a_\mu^{(4)}$ is only a small contribution to the total a_μ^{hvp}, one can expect corrections due to condensates and duality violations to be completely negligible. In particular, the imaginary part of $\Pi_V^{\mathrm{GG}}(s)$, see Eq. (18), is generated only by the logarithmic scale dependence of α_s, hence it is suppressed by two powers of the strong coupling.

7 Combined results and impact of kernel variations

For $s_0 = 4$ GeV^2 and using the estimate for $\Pi'_{\mathrm{EM}}(0)$ derived from the LQCD results in [22] we find $a_\mu^{(u,d,s)} = (620.1 \pm 34.6) \times 10^{-10}$, while with $\Pi'_{\mathrm{EM}}(0)$ from [21] we have $a_\mu^{(u,d,s)} = (664.9 \pm 17.7) \times 10^{-10}$. These results fall in the right ballpark, but are obviously not yet competitive with other determinations. The slight disagreement between the two values shows that a careful assessment of the error contributions to the LQCD results for $\Pi'_{\mathrm{EM}}(0)$ is necessary. At any rate, the error in the final result is completely dominated by the uncertainty in the LQCD determination of the slope $\Pi'_{\mathrm{EM}}(0)$.

Case	$a_\mu^{(1)}$	$a_\mu^{(2)}$ ([21], [22])	$a_\mu^{(3)}$	$a_\mu^{(4)}$	$a_\mu^{(u,d,s)}$ ([21], [22])
0	9.56	$(564.5 \pm 17.7, \quad 519.7 \pm 34.6)$	55.5 ± 0.6	35.37	$(664.9 \pm 17.7, \quad 620.1 \pm 34.6)$
1	36.48	$(555.7 \pm 17.4, \quad 511.5 \pm 34.2)$	56.6 ± 0.6	15.64	$(664.4 \pm 17.4, \quad 620.2 \pm 34.2)$
2	-47.13	$(482.3 \pm 15.1, \quad 443.9 \pm 29.6)$	195.0 ± 2.0	35.37	$(665.5 \pm 15.2, \quad 627.1 \pm 29.7)$
3	-57.26	$(586.0 \pm 18.4, \quad 539.3 \pm 36.0)$	96.9 ± 1.0	35.37	$(661.0 \pm 18.4, \quad 614.3 \pm 36.0)$
4	-31.30	$(660.1 \pm 20.6, \quad 607.8 \pm 40.7)$	0	35.37	$(664.2 \pm 20.6, \quad 611.9 \pm 40.7)$

Table 2: The light-quark hadronic contribution to the muon anomalous magnetic moment and its split-up as defined in the text, in units of 10^{-10}. In the line labeled 'Case 0' we collect the numerical results discussed in previous sections while cases 1 – 4 are modifications of the kernel $K_1(s)$ described in the text. For the residue part we show two results based on input from LQCD of Refs. [22] and [21]. Uncertainties for the perturbative parts are not shown since they are sub-dominant.

The final result for $a_\mu^{(u,d,s)}$ should not depend on the specific choice of the approximate kernel $K_1(s)$, if all contributions could be calculated with the same exact current correlator. However, differences can occur to the extent that the different pieces of $\Pi_{\rm EM}$ are affected by different errors and uncertainties. In fact, (i) the PQCD parts are not exact because higher-order terms are missing, and the known parts depend on the choice of the renormalization scale. (ii) The LQCD contribution has its own specific statistical and systematic uncertainties associated with the lattice approach. Furthermore, strong and electromagnetic sources of isospin breaking are unaccounted for in Refs. [21, 22]. Recent calculations [39, 40] have provided indications that isospin-breaking effects in $a_\mu^{\rm hvp}$ are of the order of -1%. (iii) PQCD does not take into account QED corrections. (iv) The data are obviously affected by experimental uncertainties. It is therefore interesting to study variations in the approximate kernel $K_1(s)$. Resulting shifts of $a_\mu^{(u,d,s)}$ can then be taken as an indication of the presence of unknown systematic errors in the various ingredients of our approach.

To this end we consider four modifications of $K_1(s)$, to wit:

1. Choosing the larger value for $s_0 = 9$ GeV^2 (instead of $s_0 = 4$ GeV^2) will extend the range of energies for which the data integral $a_\mu^{(3)}$ has to be evaluated. The PQCD contribution is expected to be even more reliable. The fit is given by Eq. (4) with

$$c_{-2} = 2.7195{\times}10^{-3}\,{\rm GeV}^2, \quad c_0 = 2.9618{\times}10^{-4}\,{\rm GeV}^{-2}, \quad c_1 = -3.6775{\times}10^{-5}\,{\rm GeV}^{-4}\,. \tag{26}$$

Both the data integral and the residue term change only little: $a_\mu^{(1)} = (36.48{\pm}0.19)\times 10^{-10}$, $a_\mu^{(2)} = (511.5 \pm 34.2) \times 10^{-10}$ (we use the LQCD result of [22] for the four cases with modified $K_1(s)$), $a_\mu^{(3)} = (56.6 \pm 0.6) \times 10^{-10}$, $a_\mu^{(4)} = (15.64 \pm 0.19) \times 10^{-10}$, and the total for $s_0 = 9$ GeV^2 is

$$a_\mu^{(u,d,s)} = (620.2 \pm 34.3) \times 10^{-10}\,. \tag{27}$$

2. The ansatz
$$K_1(s) = \frac{c_{-2}}{s}\left(1 - \frac{s^2}{s_0^2}\right) \tag{28}$$
provides pinching at $s = s_0$ which we choose again as $s_0 = 4$ GeV^2. We choose $c_{-2} = 2.36 \times 10^{-3}$ GeV^2 such that $K_1(s)$ and $K(s)$ agree at threshold and find $a_\mu^{(1)} = (-47.13 \pm 0.14) \times 10^{-10}$, $a_\mu^{(2)} = (443.9 \pm 29.6) \times 10^{-10}$, $a_\mu^{(3)} = (195.0 \pm 2.0) \times 10^{-10}$, $a_\mu^{(4)} = (35.36 \pm 0.61) \times 10^{-10}$, and for the total:
$$a_\mu^{(u,d,s)} = (627.1 \pm 31.2) \times 10^{-10}. \tag{29}$$

3. The same form, Eq. (28), with pinching at $s = s_0 = 4$ GeV^2, but restricting the energy range to $s \geq 0.2$ GeV^2 leads to a larger value for the parameter $c_{-2} = 2.87 \times 10^{-3}$ GeV^2. This results in $a_\mu^{(1)} = (-57.26 \pm 0.17) \times 10^{-10}$ for fixed-order perturbation theory (or $a_\mu^{(1)} = (-57.19 \pm 0.17) \times 10^{-10}$ for CIPT), $a_\mu^{(2)} = (539.3 \pm 36.0) \times 10^{-10}$, $a_\mu^{(3)} = (96.9 \pm 1.0) \times 10^{-10}$, $a_\mu^{(4)} = (35.36 \pm 0.61) \times 10^{-10}$, and for the total:
$$a_\mu^{(u,d,s)} = (614.3 \pm 36.3) \times 10^{-10}. \tag{30}$$

4. Finally, an extreme option could be to choose $c_0 = c_1 = 0$ and determine the coefficient c_{-2} as in Eq. (5), but including $R(s)$ in the integral. This renders the data integral $a_\mu^{(3)}$ equal to zero by definition and the total $a_\mu^{(u,d,s)}$ as well as its error is dominated by the LQCD part. We find $c_{-2} = 3.23 \times 10^{-3}$ GeV^2 and $a_\mu^{(u,d,s)} = (611.9 \pm 40.7) \times 10^{-10}$.

The changes of individual contributions to $a_\mu^{(u,d,s)}$ (see Table 2) compensate each other in the total and the final result varies little within the given uncertainties. In Table 2 we also show results where the LQCD input is taken from [21].

The total error is dominated by the LQCD calculation, while uncertainties arising from variations in the treatment of the kernel function and the data integral are clearly subdominant. Case 2 based on a kernel with pinching is preferred over the other scenarios since it minimizes the LQCD contribution, as well as the uncertainty. With a future improved determination of $\Pi'_{\rm EM}(0)$ on can use

$$a_\mu^{(u,d,s)} = \left(183.2 \pm 2.1 + 5027\,\Pi'_{\rm EM}(0)\,{\rm GeV}^2\right) \times 10^{-10} \tag{31}$$

to calculate directly the light-quark contribution to the muon anomalous magnetic moment.

At this point heavy-quark contributions, which can be calculated from PQCD, have to be added. There is no need to invoke model assumptions or to use LQCD results. Charm and bottom contributions can be described in the same way as above. As shown in Ref. [12] excellent approximations for the kernel $K(s)$ can be found in the respective energy ranges for the charm and bottom sector. Therefore we take numerical results from [12], for charm quarks $a_\mu^{(c)} = (14.4 \pm 0.1) \times 10^{-10}$ and for bottom quarks $a_\mu^{(b)} = (0.29 \pm 0.01) \times 10^{-10}$. The sum, $a_\mu^{(c+b)} = (14.7 \pm 0.1) \times 10^{-10}$, has to be added to the total light-quark contribution shown in the last columns of Table 2. We note that the above value of $a_\mu^{(c)}$ agrees very well with the recent determination in LQCD [22], i.e. $a_\mu^{(c)} = (14.3 \pm 0.2 \pm 0.1) \times 10^{-10}$.

8 Summary

In this paper we discussed a new QCD sum rule determination of the leading order hadronic vacuum polarization contribution to the anomalous magnetic moment of the muon, a_μ. This determination combines theoretical input from both perturbative as well as lattice QCD. The splitting into different contributions depends on an approximation to the original kernel function, $K(s)$, entering the expression of the anomaly, $a_\mu^{\rm hvp}$, Eq. (1), involving the electromagnetic current correlator. This approximate kernel, $K_1(s)$, is chosen so as to suppress the contribution of the experimental data, and their uncertainties. It is also designed to allow for the main contribution to the anomaly to be determined by the first derivative of the electromagnetic current correlator at zero-momentum, $\Pi'_{\rm EM}(0)$. The latter can be obtained from LQCD, so that the final result becomes essentially a QCD prediction. Several options for the approximate kernel $K_1(s)$ have been considered, leading to practically the same results for the anomaly. Regarding the uncertainties involved in this approach, they are essentially of two types. The first, and by far the largest, is due to the LQCD input for $\Pi'_{\rm EM}(0)$. Secondly, the e^+e^- data errors, the PQCD term, and the heavy-quark contribution to the sum rule involve tiny uncertainties which are comparable, if not smaller than those from the best data-driven determinations of $a_\mu^{\rm hvp}$. It should be mentioned that uncertainty sources in the PQCD part such as e.g. isospin breaking of order $(m_u - m_d)^2/s_0$, the singlet contribution of order m_s^2, and potential duality violations are all negligible.

Currently, LQCD uncertainties on $\Pi'_{\rm EM}(0)$ are large, so that predictions from this method do not yet compete in accuracy with results using e^+e^- data. Continuous improvement in the accuracy of LQCD results for $\Pi'_{\rm EM}(0)$ are likely to render this method competitive, thus contributing to settle the issue of whether a_μ opens a window beyond the Standard Model.

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References

[1] M. Davier, A. Hoecker, B. Malaescu and Z. Zhang, arXiv:1706.09436 [hep-ph].

[2] C. Patrignani *et al.* [Particle Data Group], Chin. Phys. C **40** (2016) no.10, 100001.

[3] K. Hagiwara, R. Liao, A. D. Martin, D. Nomura and T. Teubner, J. Phys. G **38** (2011) 085003 doi:10.1088/0954-3899/38/8/085003 [arXiv:1105.3149 [hep-ph]].

[4] S. Eidelman and F. Jegerlehner, Z. Phys. C **67** (1995) 585 [hep-ph/9502298].

[5] S. Groote, J. G. Körner and J. Maul, hep-ph/0309226.

[6] M. Davier, A. Hoecker, B. Malaescu and Z. Zhang, Eur. Phys. J. C **71** (2011) 1515 Erratum: [Eur. Phys. J. C **72** (2012) 1874] [arXiv:1010.4180 [hep-ph]].

[7] E. de Rafael, Phys. Lett. B **322** (1994) 239 [hep-ph/9311316].

[8] T. Blum, Phys. Rev. Lett. **91** (2003) 052001 [hep-lat/0212018].

[9] M. Della Morte, B. Jäger, A. Jüttner and H. Wittig, JHEP **1203** (2012) 055 [arXiv:1112.2894 [hep-lat]].

[10] C. Aubin, T. Blum, M. Golterman and S. Peris, Phys. Rev. D **86** (2012) 054509 [arXiv:1205.3695 [hep-lat]].

[11] M. Golterman, K. Maltman and S. Peris, Phys. Rev. D **90** (2014) no.7, 074508 [arXiv:1405.2389 [hep-lat]].

[12] S. Bodenstein, C. A. Dominguez and K. Schilcher, Phys. Rev. D **85** (2012) 014029 Erratum: [Phys. Rev. D **87** (2013) no.7, 079902] [arXiv:1106.0427 [hep-ph]].

[13] C. A. Dominguez, K. Schilcher and H. Spiesberger, arXiv:1704.02843 [hep-ph].

[14] C. Aubin and T. Blum, Phys. Rev. D **75** (2007) 114502 [hep-lat/0608011].

[15] P. Boyle, L. Del Debbio, E. Kerrane and J. Zanotti, Phys. Rev. D **85** (2012) 074504 [arXiv:1107.1497 [hep-lat]].

[16] X. Feng, K. Jansen, M. Petschlies and D. B. Renner, Phys. Rev. Lett. **107** (2011) 081802 [arXiv:1103.4818 [hep-lat]].

[17] A. Francis, B. Jäger, H. B. Meyer and H. Wittig, Phys. Rev. D **88** (2013) 054502 [arXiv:1306.2532 [hep-lat]].

[18] F. Burger *et al.* [ETM Collaboration], JHEP **1402** (2014) 099 doi:10.1007/JHEP02(2014)099 [arXiv:1308.4327 [hep-lat]].

[19] B. Chakraborty *et al.* [HPQCD Collaboration], Phys. Rev. D **89** (2014) no.11, 114501 [arXiv:1403.1778 [hep-lat]].

[20] B. Chakraborty, C. T. H. Davies, P. G. de Oliviera, J. Koponen and G. P. Lepage, arXiv:1601.03071 [hep-lat].

[21] S. Borsanyi *et al.*, arXiv:1612.02364 [hep-lat].

[22] M. Della Morte *et al.*, arXiv:1705.01775 [hep-lat].

[23] E. de Rafael, Phys. Lett. B **736** (2014) 522 [arXiv:1406.4671 [hep-lat]].

[24] E. de Rafael, arXiv:1702.06783 [hep-ph].

[25] M. Benayoun, P. David, L. DelBuono and F. Jegerlehner, arXiv:1605.04474 [hep-ph].

[26] F. Jegerlehner and A. Nyffeler, Phys. Rept. **477** (2009) 1 [arXiv:0902.3360 [hep-ph]].

[27] P. A. Baikov, K. G. Chetyrkin and J. H. Kühn, Phys. Rev. Lett. **101** (2008) 012002 [arXiv:0801.1821 [hep-ph]].

[28] J. Penarrocha and K. Schilcher, Phys. Lett. B **515** (2001) 291 [hep-ph/0105222].

[29] M. Beneke and M. Jamin, JHEP **0809** (2008) 044 [arXiv:0806.3156 [hep-ph]].

[30] F. Le Diberder and A. Pich, Phys. Lett. B **286** (1992) 147.

[31] A. A. Pivovarov, Z. Phys. C **53** (1992) 461 [Sov. J. Nucl. Phys. **54** (1991) 676] [Yad. Fiz. **54** (1991) 1114] [hep-ph/0302003].

[32] C. A. Dominguez, L. A. Hernandez, K. Schilcher and H. Spiesberger, JHEP **1503** (2015) 053 [arXiv:1410.3779 [hep-ph]].

[33] A. Pich and A. Rodrguez-Snchez, Phys. Rev. D **94** (2016) no.3, 034027 [arXiv:1605.06830 [hep-ph]].

[34] S. Bodenstein, C. A. Dominguez, S. I. Eidelman, H. Spiesberger and K. Schilcher, JHEP **1201** (2012) 039 [arXiv:1110.2026 [hep-ph]].

[35] S. Bodenstein, C. A. Dominguez, K. Schilcher and H. Spiesberger, Phys. Rev. D **88** (2013) no.1, 014005 [arXiv:1302.1735 [hep-ph]].

[36] K. G. Chetyrkin, J. H. Kuhn and M. Steinhauser, Comput. Phys. Commun. **133** (2000) 43 [hep-ph/0004189].

[37] V. V. Anashin *et al.*, Phys. Lett. B **753** (2016) 533 [arXiv:1510.02667 [hep-ex]].

[38] V. V. Anashin *et al.*, arXiv:1610.02827 [hep-ex].

[39] P. Boyle, V. Gülpers, J. Harrison, A. Jüttner, C. Lehner, A. Portelli and C. T. Sachrajda, arXiv:1706.05293 [hep-lat].

[40] D. Giusti, V. Lubicz, G. Martinelli, F. Sanfilippo and S. Simula, arXiv:1707.03019 [hep-lat].