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# Scaling and renormalization in two dimensional quantum gravity

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We study scaling and renormalization in two dimensional quantum gravity in a covariant framework. After reviewing the definition of a proper path integral measure, we use scaling arguments to rederive the Knizhnik-Polyakov-Zamolodchikov relations, the fractal dimension of the theory and the scaling of the reparametrization-invariant two point function. Then we compute the scaling exponents entering in these relations by means of the functional renormalization group. We show that a key ingredient to obtain the correct results already known from Liouville theory is the use of the exponential parametrization for metric fluctuations. We also show that with this parametrization we can recover the correct finite part of the effective action as the $\epsilon \to 0$ limit of quantum gravity in $d = 2 + \epsilon$.

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## I. INTRODUCTION

The study of the quantum nature of gravity still lacks a fundamental understanding. We now know that the first nontrivial universal quantum effects, which one can compute considering a free theory with weakly coupled interactions, are only valid up to an energy scale not exceeding the Planck scale, and that properly investigating gravity at higher energies requires either the use of nonperturbative methods or the introduction of new physics.

The past decades have witnessed the birth of different nonperturbative methods that can be used to probe physics beyond the weakly coupled regime. One of these methods is functional renormalization group (FRG) techniques [1,2] realizing Wilson's approach to the renormalization group (RG). Thanks to this new conceptual paradigm, we now better understand what steps are needed to solve a theory. In the Wilsonian RG, the strengths of the coupling constants in a quantum field theory (QFT) form a generalized theory space, and their running describes a trajectory in this space. If we want a theory that describes physics at all possible energy scales, then the end points of the RG trajectory cannot sit at any finite scale, and thus have to represent scale invariant theories. At a scale invariant point the symmetry of the system gets in many cases enhanced to the full conformal group, which can be thought of as generated from Poincarè transformation, dilations, and spacetime inversions. This is strictly true for unitary two dimensional QFTs and is probably also true in four dimensions under suitable assumptions.

Critical properties are associated with a scale invariant phase of the system, since any nonzero mass scale would define a finite correlation length. Thus critical properties are determined by the fixed points of the RG flow. Two trajectories with different starting points but ending at the same fixed point will be associated with the same critical properties. This fact is called universality. Universal properties are the only observable ones and are associated with a fixed point and its basin of attraction, that is, with a neighborhood of it in theory space. Knowing the critical theory that sits at the conformal point, one can perturb it and study how it evolves in a neighborhood of that point. The critical exponents, which dictate the universal, observable properties of the theory, are encoded in the conformal data around the fixed point, namely, for a given representation of the Lorentz group (spin), in the spectrum of scaling dimensions of the fields of the theory. This spectrum uniquely determines the lowest order correlators,[1] from which all higher order correlators can be reconstructed using bootstrap relations. In the weakly coupled regime of standard perturbative QFTs, for example, knowledge of the correlators means that we can in principle compute the $S$-matrix elements for a given process, and thus give the observables. Thus in order to get the physics one does not need to solve the full theory, but just to look at the universal properties. Reconstructing the conformal field theory (CFT) data is one way of accessing the universal properties, and thus have all the physical predictions of the theory at hand.

Even then, though, when gravity is introduced, its backreaction on matter will in general change the critical properties of the theory, and then the CFT data as well, in a nontrivial way. The scaling dimensions will get a gravitational "dressing." The nature of this dressing is a nonperturbative question and is of fundamental importance if we want to address the fundamental universal properties of gravity. If one knows the scaling spectrum at one scale, or

[1]This is only partially true, since one also needs the structure constants $C_{ijk}$. However, the gravitational dressing will only involve the $\Delta$'s.

 

for instance at the fixed point, then in principle one can dress any other scale with RG flows. The problem is then reduced to that of finding the gravitational dressing at the fixed point (or near it).

We find that light is sometimes shed on a complicated problem by investigating a lower dimensional instance of it. In two dimensions, in particular, the last 40 years have seen an incredible progress in our understanding of the mathematical structures involved. We now know that in two dimensions the bootstrap relations are so powerful that they allow for an exact solution of the equations, thus finding the complete spectrum of scaling dimensions. This allows us to completely classify CFTs. This, together with Zamolodchikov's proof that any unitary two dimensional scale invariant QFT is a CFT, implies that we have a complete understanding of the fixed points, and thus the critical phases of (relativistic) matter in low dimensional systems. The last piece of information was given by Knizhnik, Polyakov, and Zamolodchikov (KPZ), who were able to derive an exact formula for the gravitational dressing of scaling dimensions [3]. This tells us the total effect that gravity has on the critical phases of matter and can thus be regarded as a solution of quantum gravity in two dimensions.

In this paper we want to give a unified overview on these aspects, by using the functional RG. We will see that the main physical ideas behind them stem from very simple principles, and that the FRG allows for a very simple derivation of many results that follow from these arguments.

The paper is organized as follows. Section II introduces the quantities of interest in two dimensional quantum gravity, together with the proper definition of path integral measure that we will use, and then rederives the relation between the anomaly coefficient and the beta function of Newton's constant. Section III lays down the scaling arguments for two dimensional quantum gravity, in particular deriving the KPZ relations. Section IV complements the previous one with the dual approach, namely renormalization. In the first part we review the details needed in a background field computation and the basics of functional RG. In the second part we study gravity in $d=2$: we compute the graviton central charge/coefficient anomaly $c_g$, the scaling exponents needed in the scaling relations, and we comment on the relation with the Liouville theory. In the third and final part we study gravity in $d=2+\epsilon$, deriving in a different way the graviton central charge and analyzing the finite part of the effective action. Finally, Sec. V is devoted to conclusions.

## II. TWO DIMENSIONAL QUANTUM GRAVITY

Euclidean two dimensional quantum gravity is loosely defined as the sum over all metrics living in two dimensional manifolds. Since the latter, in the closed orientable case, can be topologically classified by the number of holes $h$, the sum over all metrics becomes a sum of integrals over the functional space of metrics on a manifold with fixed topology. The partition function for $2d$ quantum gravity is then

$$Z(\Lambda, G) = \sum_h \int_h \mathcal{D}g e^{-\Lambda\int\sqrt{g}+\frac{1}{4\pi G}\int\sqrt{g}R}, \qquad (1)$$

where $G$ is Newton's constant and $\Lambda$ is the cosmological constant. Using the Gauss-Bonnet relation $\chi(h) = \frac{1}{4\pi}\int\sqrt{g}R$ we can write

$$Z(\Lambda, G) = \sum_h e^{\chi(h)/G} Z_h(\Lambda),$$
$$Z_h(\Lambda) = \int_h \mathcal{D}g e^{-\Lambda\int\sqrt{g}}, \qquad (2)$$

where $\chi(h) = 2-2h$ is the Euler characteristic. We omit the subscript $h$ when $h=0$.

The major problem in defining the path integral over geometries is the construction or definition of the measure $\mathcal{D}g$ to which now we turn. We will adopt a pragmatic point of view on the problem, and we will not attempt a rigorous construction, which can be found in [4], but instead we want to focus on the basic properties satisfied by the procedure of averaging over metrics with the intent of obtaining general scaling relations for the physical quantities. In other words, we are more interested in computing universal quantities like critical exponents than in solving a particular quantum gravity model.

### A. Fixed-area functionals

It turns out to be very useful to consider the Laplace transform of the partition function

$$Z_h(\Lambda) = \int_0^\infty dA Z_h(A) e^{-\Lambda A},$$
$$Z_h(A) = \int_h \mathcal{D}g\delta\left[\int\sqrt{g}-A\right]. \qquad (3)$$

In fact, these relations show that all we need to do to be able to compute the partition function is to find the average of the delta function containing the composite area operator $\int\sqrt{g}$. Note that this observation is valid for an arbitrary bare action.

Together with the partition function, we can also define other transformed, or "fixed-area," quantities. For instance, the expectation value of a general operator $\mathcal{O}[\Phi; g]$ depending on some matter fields $\Phi$ can be rewritten as

$$\left\langle \int\sqrt{g}\mathcal{O}[\Phi;g]\right\rangle = \sum_h e^{\chi(h)/G}\left\langle\int\sqrt{g}\mathcal{O}[\Phi;g]\right\rangle_h$$
$$= \sum_h e^{\chi(h)/G}\int_0^\infty dA F_h(A)e^{-\Lambda A}, \qquad (4)$$

where $F_h(A)$ is the transformed one-point function,

$$F_h(A) = \frac{1}{Z(A)} \int_h \mathcal{D}g \mathcal{D}_g \Phi \delta \left[ \int \sqrt{g} - A \right] \int \sqrt{g} \mathcal{O}[\Phi; g]. \quad (5)$$

Just as for the partition function, the scaling of $F_h(A)$ will determine the full quantum scaling of the expectation value of the operator and will tell us how its scaling dimension is modified by gravity.

These objects all parametrically depend on the fixed area $A$, whose scaling is still the classical one. However, we can also consider expectation values depending on more interesting geometrical objects, such as the geodesic distance between two points. The geodesic distance is defined as

$$d_g(x, x_0) = \int_0^1 dt \sqrt{g_{\mu\nu}(x(t)) \frac{dx^\mu(t)}{dt} \frac{dx^\nu(t)}{dt}}, \quad (6)$$

where $x^\mu$ is a solution of the geodesic equation

$$\frac{d^2 x^\mu(t)}{dt^2} + \Gamma^\mu_{\rho\sigma}(x(t)) \frac{dx^\rho(t)}{dt} \frac{dx^\sigma(t)}{dt} = 0, \quad (7)$$

with $x(0) = x_0$ and $x(1) = x$. From this we can construct the (geometric) two-point function [4]:

$$G(A, \ell) = \int \mathcal{D}g \delta \left( \int \sqrt{g} - A \right) \int d\xi \sqrt{g} \times \int d\xi' \sqrt{g'} \delta(d_g(\xi - \xi') - \ell) \quad (8)$$

whose scaling will also be studied in the following. The subtle point when studying this quantity is that the parameter $\ell$, which classically scales as a length, in the full quantum regime acquires a nontrivial scaling. We can understand this if we notice that the definition of the geodesic distance involves the Christoffel symbols on the manifold. These in the quantum regime become composite operators of the metric, having their own scaling, which will correct the naive one.

### B. Formal properties of $\mathcal{D}g$

For any quantum field theory in curved space in $d = 2$, a key role is played by the conformal anomaly. This basically says that the standard functional measure for matter fields $\mathcal{D}_g \Phi$ is not invariant under Weyl rescalings of the metric $g_{\mu\nu} \to g_{\mu\nu} e^{2\sigma}$ and the field $\Phi \to \Phi e^{-\Delta\sigma}$,

$$\mathcal{D}_{ge^{2\sigma}}(\Phi e^{-\Delta\sigma}) = \mathcal{D}_g \Phi e^{\frac{c_\Phi}{24\pi} S_L[\sigma; g]}, \quad (9)$$

where $S_L[\sigma; g]$ is the Wess-Zumino or Liouville action (to be defined in a moment) and $c_\Phi$ is the conformal anomaly coefficient, or central charge, of the (UV) matter field theory.

The Liouville action is defined as the difference between the Polyakov action,

$$S_P[g] = -\frac{1}{96\pi} \int d^2x \sqrt{g} R \frac{1}{\Delta} R, \quad (10)$$

evaluated before and after a Weyl rescaling,

$$S_P[ge^{2\sigma}] - S_P[g] = -\frac{1}{24\pi} S_L[\sigma; g]. \quad (11)$$

We thus define the Liouville action as follows:

$$S_L[\sigma; g] = \int d^2x \sqrt{g} [\sigma \Delta \sigma + \sigma R]. \quad (12)$$

The conformal mode can be eliminated by choosing $\sigma(g) = \frac{1}{2\Delta} R$. Note that (11) is invariant modulo a topological term under constant shifts $\sigma \to \sigma + \omega$,

$$S_P[ge^{2\omega}] = S_P[g] - \frac{1}{24\pi} S_L[\omega; g] = S_P[g] - \frac{1}{6} \chi(h) \omega. \quad (13)$$

Following [5] we formally construct a Weyl invariant measure by multiplying the standard one by a factor $e^{c_\Phi S_P[g]}$,

$$\mathcal{D}_g^{Weyl} \Phi = \mathcal{D}_g \Phi e^{c_\Phi S_P[g]}. \quad (14)$$

This may be achieved by multiplying by $1 = e^{c_\Phi S_P[g]} e^{-c_\Phi S_P[g]}$; thus we can write

$$\mathcal{D}_g \Phi = \mathcal{D}_g^{Weyl} \Phi e^{-c_\Phi S_P[g]}. \quad (15)$$

Here we will make the ansatz, or perform the construction, in which the same applies to the gravitational case,

$$\mathcal{D}_g g = \mathcal{D}_g^{Weyl} g e^{-c_g S_P[g]}, \quad (16)$$

where $c_g$ is the (UV) gravitational anomaly coefficient, or central charge, that has to be determined self-consistently. The introduction of the Weyl invariant measure amounts to add to the UV action a Polyakov term $c_g S_P[g]$. Thus we can work as if the measure is Weyl invariant, not caring about the conformal anomaly at all, provided we add the Polyakov action to the bare action with the correct UV central charge $c_g$. Notice that in order to have a well-defined path integral, we also need a gauge fixing procedure. We will leave this implicit until Sec. IV, where this issue will be discussed in greater detail.

From now on we will use the Weyl invariant measure and we will drop the label *Weyl*, unless otherwise specified (as in Sec. IV. C. 3).

### C. Relation between $c_g$ and $\partial_t G_k$

How do we compute $c_g$? The standard way [6,7] is to use Liouville theory (see Sec. IV. C. 3). Another way is to link $c_g$ to the beta function of Newton's constant. Consider the partition function (2) at $\Lambda = 0$ in which we make a rescaling of the metric $g_{\mu\nu} \to \lambda^{-2} g_{\mu\nu}$,

$$\begin{aligned} Z &= \sum_h e^{\frac{\chi(h)}{G}} Z_h \\ &= \sum_h e^{\frac{\chi(h)}{G}} \int_h \mathcal{D}(\lambda^{-2} g) e^{-c_g S_P[\lambda^{-2} g]} \\ &= \sum_h e^{\frac{\chi(h)}{G}} \int_h \mathcal{D}g e^{-c_g S_P[g] - \frac{c_g}{6}\chi(h)\log\lambda} \\ &= \sum_h e^{(\frac{1}{G} - \frac{c_g}{6}\log\lambda)\chi(h)} Z_h, \end{aligned} \tag{17}$$

where in the first step we made a dummy relabeling and in the second we used the Weyl invariance of the measure and (13). This expression is consistent if Newton's constant is scale dependent, $G = G_k$, evaluated at $\lambda^{-1}k$. This is to be since we have to keep the physical scale $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ invariant: the rescaling $g_{\mu\nu} \to \lambda^{-2} g_{\mu\nu}$ means that the coordinates transform as $x \to \lambda x$, which implies that $k \to \lambda^{-1} k$. Since the partition function in this case did not change it must be that

$$\frac{1}{G_{\lambda^{-1}k}} - \frac{c_g}{6}\log\lambda = \frac{1}{G_k}. \tag{18}$$

Expanding (18) leads to

$$\frac{1}{G_{\lambda^{-1}k}} - \frac{c_g}{6}\log\lambda = \frac{1}{G_k} - \left[\partial_t\left(\frac{1}{G_k}\right) + \frac{c_g}{6}\right]\log\lambda + \cdots, \tag{19}$$

where $\partial_t = k\partial_k$. This immediately implies the one-loop exact relation

$$\partial_t\left(\frac{1}{G_k}\right) = -\frac{c_g}{6}, \tag{20}$$

or equivalently

$$\partial_t G_k = \frac{c_g}{6} G_k^2. \tag{21}$$

These relations show that from the computation of Newton's beta function we can thus determine the central charge [5].

## III. SCALING

In this section we will review the scaling arguments as they follow from our construction. The key one is the scaling of the expectation value of a matter (composite) operator, which gives the famous KPZ dressing [3], telling us how flat scaling dimensions are modified by gravity. In order to derive this we will have to start by looking at the scaling of the partition function.

### A. Partition function

We will start by presenting the scaling argument for the partition function $Z_h(A)$. We have

$$Z_h(A) = \int_h \mathcal{D}g e^{-c_g S_P[g]} \delta(I_0[g] - A), \tag{22}$$

where $I_0[g] \equiv \int \sqrt{g}$ is the area composite operator. We want to enquire the scaling $A \to \lambda A$ remembering that the measure is invariant and the Polyakov action satisfies (13). The scaling of the area composite operator can be affected by nontrivial quantum corrections; hence we will leave it general, $I_0[\lambda g] = \lambda^\alpha I_0[g]$. We then have

$$\begin{aligned} Z_h(\lambda A) &= \int_h \mathcal{D}g e^{-c_g S_P[g]} \delta(I_0[g] - \lambda A) \\ &= \frac{1}{\lambda}\int_h \mathcal{D}g e^{-c_g S_P[g]} \delta(\lambda^{-1} I_0[g] - A) \\ &= \frac{1}{\lambda}\int_h \mathcal{D}(\lambda^{1/\alpha} g) e^{-c_g S_P[\lambda^{1/\alpha} g]} \delta(\lambda^{-1} I_0[\lambda^{1/\alpha} g] - A) \\ &= \frac{1}{\lambda}\int_h \mathcal{D}g e^{-c_g S_P[g] + \frac{c_g}{6}\chi(h)\frac{1}{2\alpha}\log\lambda} \delta(I_0[g] - A) \\ &= \lambda^{\frac{c_g}{12\alpha}\chi(h) - 1} Z_h(A). \end{aligned} \tag{23}$$

If we now choose $\lambda = 1/A$, we find the following scaling form:

$$Z_h(A) = C_h A^{\frac{c_g}{6\alpha}(1-h) - 1}, \tag{24}$$

where $C_h = Z_h(A = 1)$. This is the general form of the fixed-area partition function, which can now be transformed back to give the full form of the standard $Z(\Lambda, G)$.

To make further contact with the literature, we can define the string susceptibility $\gamma$ through

$$Z_h(A) = C_h A^{\gamma - 3}, \tag{25}$$

so that we find

$$\gamma = \frac{c_g}{6\alpha}(1 - h) + 2. \tag{26}$$

In the following sections we will see that $c_g = c_\Phi - 25$, and we will compute the scaling exponent $\alpha$, obtaining the value

$$\alpha = \frac{25 - c_\Phi - \sqrt{(1 - c_\Phi)(25 - c_\Phi)}}{12}. \tag{27}$$

Thus we have

$$\gamma = -\frac{25 - c_\Phi + \sqrt{(1-c_\Phi)(25-c_\Phi)}}{12}(1-h) + 2. \tag{28}$$

In the absence of matter we find

$$\gamma = -\frac{5}{2}(1-h) + 2, \tag{29}$$

and on the sphere[2] $\gamma = -\frac{1}{2}$.

The partition function for two dimensional quantum gravity is then found to be

$$Z_h(\Lambda) = C_h \int_0^\infty dA e^{-\Lambda A} A^{\gamma-3} = \tau_h \Lambda^{2-\gamma},$$

where we defined the model dependent constants $\tau_h = C_h \Gamma(\gamma - 2)$ (see [4] for an explicit evaluation of these constants in dynamically triangulated gravity). After combining with (2) we finally arrive at

$$Z(\Lambda, G) = \sum_h \tau_h (e^{\frac{1}{G}} \Lambda^{-\frac{c_g}{12\alpha}})^{\chi(h)}, \tag{30}$$

showing that Newton's constant contributes as a "topological term" $e^{1/G}$ and that the partition function depends only on the variable $\kappa = e^{\frac{1}{G}} \Lambda^{-\frac{c_g}{12\alpha}}$. In the case of no matter we have, more explicitly, $\kappa = e^{\frac{1}{G}} \Lambda^{\frac{5}{2}}$.

### B. KPZ

We can apply the same logic to the expectation value of a matter operator $I_\mathcal{O}[\Phi; g] = \int \sqrt{g}\mathcal{O}[\Phi; g]$, defined through the fixed-area functional $F_h(A)$. Suppose the scaling of this operator is $I_\mathcal{O}[\Phi, \lambda g] = \lambda^\beta I_\mathcal{O}[\Phi, g]$. Again, this takes into account all possible quantum corrections to a composite operator. Then we have

$$\begin{aligned}
F_h(\lambda A) &= \lambda^{-\frac{c_g}{12\alpha}\chi(h)+1} \frac{1}{Z(A)} \int_h \mathcal{D}g\mathcal{D}_g\Phi e^{-c_g S_P[g]} \delta(I_0[g] - \lambda A) I_\mathcal{O}[\Phi; g] \\
&= \lambda^{-\frac{c_g}{12\alpha}\chi(h)} \frac{1}{Z(A)} \int_h \mathcal{D}g\mathcal{D}_g\Phi e^{-c_g S_P[g]} \delta(\lambda^{-1} I_0[g] - A) I_\mathcal{O}[\Phi; g] \\
&= \lambda^{-\frac{c_g}{12\alpha}\chi(h)} \frac{1}{Z(A)} \int_h \mathcal{D}(\lambda^{1/\alpha}g)\mathcal{D}_g\Phi e^{-c_g S_P[\lambda^{1/\alpha}g]} \delta(\lambda^{-1} I_0[\lambda^{1/\alpha}g] - A) I_\mathcal{O}[\Phi, \lambda^{1/\alpha}g] \\
&= \lambda^{-\frac{c_g}{12\alpha}\chi(h)} \lambda^{\beta/\alpha} \frac{1}{Z(A)} \int_h \mathcal{D}g\mathcal{D}_g\Phi e^{-c_g S_P[g] + \frac{c_g}{6}\chi(h)\frac{1}{2\alpha}\log\lambda} \delta(I_0[g] - A) I_\mathcal{O}[\Phi, g] \\
&= \lambda^{\beta/\alpha} \frac{1}{Z(A)} \int_h \mathcal{D}g\mathcal{D}_g\Phi e^{-c_g S_P[g]} \delta(I_0[g] - A) I_\mathcal{O}[\Phi, g] \\
&= \lambda^{\beta/\alpha} F_h(A).
\end{aligned}$$

Choosing again $\lambda = 1/A$, we find the scaling form of the expectation value

$$F_h(A) = A^{\beta/\alpha} F_h(1). \tag{31}$$

The physical meaning of this scaling is found by noticing that the gravitational scaling dimension $\Delta$ can be defined from a one-point function as $F_h(A) \propto A^{1-\Delta}$, while $\beta$ is related to the flat scaling dimension $\Delta_0$ of the operator $\mathcal{O}$, as we will prove later, by

$$\beta = \frac{25 - c_\Phi - \sqrt{(1 + 24\Delta_0 - c_\Phi)(25 - c_\Phi)}}{12}. \tag{32}$$

This means that the scaling dimension $\Delta_0$ of an operator receives a gravitational dressing which changes it into

$$\Delta = 1 - \frac{\beta}{\alpha} = \frac{\sqrt{1-c_\Phi} - \sqrt{1 + 24\Delta_0 - c_\Phi}}{\sqrt{1-c_\Phi} - \sqrt{25 - c_\Phi}}, \tag{33}$$

which is the KPZ formula. This relation can be recast in the equivalent form (see Sec. IV. C. 2),

$$\Delta - \Delta_0 = \frac{6\alpha^2}{25 - c_\Phi} \Delta(\Delta - 1), \tag{34}$$

also known in the literature as the KPZ relation, which shows clearly that all the effect of gravity is encoded in the scaling $\alpha$ of the area operator.

[2]For a complementary approach based on matrix models see [8].

### C. Fractal properties of spacetime

The previous considerations only required the scaling of a fixed area, which is dictated by its classical scaling. However, if the partition function starts to depend on less trivial geometrical quantities such as the geodesic distance between two points, as in the case of the two point function (8), the scaling of these quantities can get a nontrivial modification with respect to the classical one, as we here briefly review [9].

The effective scaling dimension in a quantum spacetime can be probed by considering a random walk, or a diffusion process, and studying its properties. The scaling dimension is related to the return probability, which in our case can be expressed as

$$\begin{aligned} P(A;s) &= \left\langle \frac{1}{A}\mathrm{Tr}e^{-s\Delta} \right\rangle \\ &= \frac{1}{Z(A)}\int \mathcal{D}g e^{-c_g S_P[g]}\delta(I_0[g]-A) \\ &\quad \times \frac{1}{A}\mathrm{Tr}e^{-s\Delta}, \end{aligned} \tag{35}$$

in which $s$ is the diffusion time, and $K_s = e^{-s\Delta}$ is the heat kernel, which is a solution of the heat equation [10],

$$\partial_s K_s(x,x_0) + \Delta_x K_s(x,x_0) = 0, \tag{36}$$

with boundary condition $K_0(x,x_0) = \delta(x-x_0)/\sqrt{g}$. The scaling dimension in the UV is related to the way in which $P(A;s)$ scales as a function of $s$ for $s \to 0$. We immediately notice that at $s = 0$ we have

$$P(A;0) = 1 = P(\lambda A;0). \tag{37}$$

If we assume that this holds also for small finite $s$, whose scaling is still unknown, by repeating the same manipulations of the previous sections we find the following relation:

$$\begin{aligned} &P(\lambda A;\lambda^\omega s) \\ &= \frac{1}{Z(A)}\int \mathcal{D}g e^{-c_g S_P[g]}\delta(I_0[g]-A)\frac{1}{A}\mathrm{Tr}e^{-\lambda^\omega s\Delta(\lambda^{1/\alpha}g)} \\ &= \frac{1}{Z(A)}\int \mathcal{D}g e^{-c_g S_P[g]}\delta(I_0[g]-A)\frac{1}{A}\mathrm{Tr}e^{-\lambda^{\omega+\tilde{\alpha}/\alpha}s\Delta(g)} \\ &= P(A;s), \end{aligned}$$

where we used the fact that $\mathrm{Tr}K_s(x,y) = \int\sqrt{g}\mathrm{tr}K_s(x,x)$, so it scales as $I_0[g]$, and the Laplacian has its own scaling $\Delta(\lambda g) = \lambda^{\tilde{\alpha}}\Delta(g)$, with $\tilde{\alpha}$ a new scaling exponent. The only way to fulfill this condition is that $\omega = -\tilde{\alpha}/\alpha$. Thus the diffusion time $s$ scales as $A^{-\tilde{\alpha}/\alpha}$. The average, in the diffusion process, of the squared geodesic distance from a starting point $x_0$ is given by (the subscript $s$ indicates that this is the diffusion average, not the quantum one)

$$\langle d_g^2(x,x_0)\rangle_s = \int d^2x\sqrt{g}d_g^2(x,x_0)K_s(x,x_0). \tag{38}$$

The scaling is determined by the small $s$ behavior of the average. Expanding the heat kernel for small $s$, using $d_g(x_0,x_0) = 0$, we see that $\langle d_g^2(x,x_0)\rangle_s$ starts at linear order in $s$, and thus scales as well as $A^{-\tilde{\alpha}/\alpha}$. The coefficient $\tilde{\alpha}$ will be determined later like the other exponents encountered previously. We will find that

$$\frac{\tilde{\alpha}}{\alpha} = \frac{\sqrt{49-c_\Phi}-\sqrt{25-c_\Phi}}{\sqrt{1-c_\Phi}-\sqrt{25-c_\Phi}}. \tag{39}$$

This relation (or more precisely an equivalent version of it) was found in [9]. We see that for $c = 0$ we have $\tilde{\alpha}/\alpha = -1/2$, which means that in the full quantum regime the geodesic distance scales like $A^{1/4}$.

A more direct physical way of seeing this can be the following. We know that for a random walk on a fractal, the average square displacement is related to the walking time $T$ by the power law,

$$\langle r^2\rangle \propto T^{2/d_w}, \tag{40}$$

$d_w$ being the walking dimension. Since the walking time scales like an area (this is a general property of random walks in two dimensions: their trajectories have Hausdorff dimension 2), we deduce that the geodesic distance scales like $d_g \sim A^{1/d_w}$. Now we can use the known form of $d_w$ [11], which is

$$d_w = 4 + \text{beta functions} \tag{41}$$

to get that, at a fixed point, $d_w^* = 4$, and thus $d_g$ scales like $A^{1/4}$.

### D. Two point function

Finally, knowing the scaling of the geodesic distance, we can reproduce the previous arguments for the geometric two point function $G(A,\ell)$ defined in (8). Using the scaling we just found for the geodesic length, we find

$$
\begin{aligned}
G(\lambda A, \lambda^{1/4}\ell) &= \int \mathcal{D}g e^{-c_g S_P[g]} \delta(I_0[g] - \lambda A) \int d^2\xi \sqrt{g} \int d^2\xi' \sqrt{g'} \delta(d_g(\xi - \xi') - \lambda^{1/4}\ell) \\
&= \lambda^{-\frac{5}{4}} \int \mathcal{D}g e^{-c_g S_P[g]} \delta(\lambda^{-1} I_0[g] - A) \int d^2\xi \sqrt{g} \int d^2\xi' \sqrt{g'} \delta(\lambda^{-1/4} d_g(\xi - \xi') - \ell) \\
&= \lambda^{-\frac{5}{4}} \int \mathcal{D}(\lambda^{1/\alpha} g) e^{-c_g S_P[\lambda^{1/\alpha} g]} \delta(\lambda^{-1} I_0[\lambda^{1/\alpha} g] - A) \\
&\quad \times \lambda^2 \int d^2\xi \sqrt{g} \int d^2\xi' \sqrt{g'} \delta(\lambda^{-1/4} d_{\lambda^{1/\alpha} g}(\xi - \xi') - \ell) \\
&= \lambda^{\frac{3}{4} + \frac{c_g}{12\alpha}\chi(h)} \int \mathcal{D}g e^{-c_g S_P[g]} \delta(I_0[g] - A) \int d^2\xi \sqrt{g} \int d^2\xi' \sqrt{g'} \delta(d_g(\xi - \xi') - \ell) \\
&= \lambda^{\frac{3}{4} + \frac{c_g}{12\alpha}\chi(h)} G(A, \ell),
\end{aligned}
$$

in which we had to assume that $\lambda^{-1/4} d_{\lambda^{1/\alpha} g}(\xi - \xi') = d_g(\xi - \xi')$, which is required to have a well-defined delta function. Taking again $\lambda = A^{-1}$ we find

$$G(A, \ell) = A^{\frac{3}{4} + \frac{c_g}{12\alpha}\chi(h)} f(\ell A^{-1/4}), \tag{42}$$

with $f(x) \equiv G(1, x)$. The scaling in $\Lambda$ will then be

$$
\begin{aligned}
G(\Lambda, \ell) &= \int_0^\infty dA G(A, \ell) e^{-\Lambda A} \\
&= \int_0^\infty dA A^{\frac{3}{4} + \frac{c_g}{12\alpha}\chi(h)} f(\ell A^{-1/4}) e^{-\Lambda A} \\
&= \Lambda^{-\frac{7}{4} - \frac{c_g}{12\alpha}\chi(h)} g(\ell \Lambda^{1/4}),
\end{aligned}
$$

with

$$g(x) \equiv \int_0^\infty dy e^{-y} y^{3/4 + \frac{c_g}{12\alpha}\chi(h)} f(x y^{-1/4}). \tag{43}$$

This way we recover the known scaling on the sphere [4],

$$G(\Lambda, \ell) = \Lambda^{-\frac{7}{4} + \frac{5}{2}(1-h)} g(\ell \Lambda^{1/4}) \underset{(h=0)}{=} \Lambda^{\frac{3}{4}} g(\ell \Lambda^{1/4}). \tag{44}$$

The scaling function has been computed in [4], and its detailed form is $g(x) = \cosh x / \sinh^3 x$.

## IV. RENORMALIZATION

In this section we will consider renormalization in order to compute the anomaly coefficient $c_g$ and the critical exponents $\alpha, \beta, \ldots,$ that characterize the scaling laws derived in the previous section. In particular we will compute the beta function of Newton's constant since this leads to the computation of $c_g$ via relation (20). The critical exponents are instead related to the scaling dimensions of composite operators, such as $I_0[g] = \int \sqrt{g}$.

We will perform our computations in both $d = 2$ and $d = 2 + \epsilon$ in order to enquire various things: which operator drives the flow, i.e. $S_P[g]$ in $d = 2$ versus $\int \sqrt{g} R$ in $d = 2 + \epsilon$; the connection with Liouville theory; the limit $\epsilon \to 0$; the role of different parametrizations of the metric fluctuation. From now on we will also fix the topology to be spherical, since $c_g$ and the scaling exponents do not depend on the topology.

### A. Background, gauges, and ghosts

We need now to discuss in more detail the construction of the measure. The standard approach that we will follow here is the original Faddeev-Popov method that allows one to factor out the volume of the Diff group via a gauge fixing and at the cost of introducing ghost fields, or better at the cost of introducing an additional functional determinant,

$$\mathcal{D}g \to \mathcal{D}g \delta[f] Z_{gh}[g], \tag{45}$$

where $f = 0$ is the gauge-fixing condition and $Z_{gh}[g]$ is the Fadeev-Popov determinant. A nice and elegant introduction to gravitational functional integrals can be found in [12], to which we refer for more details.

To preserve invariance under diffeomorphisms we employ the background field method where we expand around a background metric $\bar{g}_{\mu\nu}$ and we integrate over the metric fluctuation $h_{\mu\nu}$. Fluctuations can be parametrized in different ways; here we will discuss two of them, the linear (or standard) parametrization,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \tag{46}$$

and the exponential parametrization [13–15],

$$g_{\mu\nu} = \bar{g}_{\mu\lambda} e^{h^\lambda_\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} + \frac{1}{2} h_{\mu\lambda} h^\lambda_\nu + \cdots. \tag{47}$$

Since the flow equations involve the Hessians, or second variations, of the effective action, we will keep track of the parametrization choice by introducing the tensor $H_{\mu\nu} = \xi h_{\mu\lambda} h^{\lambda}_{\nu}$, where $\xi$ is a parameter which can be either zero or one, so that we can write $\delta g_{\mu\nu} = h_{\mu\nu}$ and $\delta^2 g_{\mu\nu} = H_{\mu\nu}$. Now the functional integration over the metric $g_{\mu\nu}$ can be replaced by one over the fluctuation $h_{\mu\nu}$, i.e. $\mathcal{D}g_{\mu\nu} = \mathcal{D}h_{\mu\nu}$, and the Fadeev-Popov operator $\mathcal{M}$, defined by $\det \mathcal{M} = Z_{gh}[g]$, is given by

$$\mathcal{M}[h;\bar{g}] = \frac{\delta f[h^{\epsilon},\bar{g}]}{\delta\epsilon}\bigg|_{\epsilon=0}, \tag{48}$$

where $h^{\epsilon}_{\mu\nu} = h_{\mu\nu} + \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu$ represents an infinitesimal coordinate transformation of the tensor $h_{\mu\nu}$ with respect to the full metric $g_{\mu\nu}$. There are now two possible gauge choices. The conformal gauge (CG)

$$f_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu} h, \tag{49}$$

which fixes the gauge completely only in $d = 2$, and the Feynman gauge (FG)

$$f_\mu = \bar{\nabla}^\nu h_{\mu\nu} - \frac{1}{2}\bar{\nabla}_\mu h, \tag{50}$$

which can be used in any dimension $d \geq 2$. Note that the FG is the gradient of the CG. As usual the strict gauge-fixing condition $\delta[f]$ can be relaxed by exponentiation of the delta function, in this way introducing the gauge-fixing action $S_{gf}[h;\bar{g}]$, that in the background gauge depends on both the fluctuation and the background metric. In CG the gauge-fixing action is

$$S_{gf}[h;\bar{g}] = \frac{1}{2\alpha}\int d^2x\sqrt{\bar{g}}\left(h_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}h\right)\left(h^{\mu\nu} - \frac{1}{2}\bar{g}^{\mu\nu}h\right), \tag{51}$$

while in FG it is

$$S_{gf}[h;\bar{g}] = \frac{1}{2\alpha}\int d^dx\sqrt{\bar{g}}\left(\bar{\nabla}^\alpha h_{\alpha\mu} - \frac{1}{2}\bar{\nabla}_\mu h\right) \times \left(\bar{\nabla}^\beta h^\mu_\beta - \frac{1}{2}\bar{\nabla}^\mu h\right). \tag{52}$$

In both cases $\alpha$ is the gauge-fixing parameter.[3]

The Fadeev-Popov operator $\mathcal{M}$ can be computed given the gauge condition. In CG the variation of the gauge condition leads to (remember that $h = \bar{g}^{\mu\nu}h_{\mu\nu}$)

$$\delta f_{\mu\nu} = \delta h_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\delta h = \nabla_\mu\epsilon_\nu + \nabla_\nu\epsilon_\mu - \bar{g}_{\mu\nu}\nabla\cdot\epsilon \equiv (L\epsilon)_{\mu\nu}, \tag{53}$$

which defines the vector to symmetric traceless rank two tensor differential operator $L^\alpha_{\mu\nu} \equiv \delta^\alpha_\nu\nabla_\mu + \delta^\alpha_\mu\nabla_\nu - \bar{g}_{\mu\nu}\nabla^\alpha$. Introducing the adjoint operator,

$$\begin{aligned}\int\sqrt{\bar{g}}\chi^{\mu\nu}(L\epsilon)_{\mu\nu} &= 2\int\sqrt{\bar{g}}\chi^{\mu\nu}\nabla_\mu\epsilon_\nu \\ &= -2\int\sqrt{\bar{g}}\nabla_\mu\chi^{\mu\nu}\epsilon_\nu \equiv \int\sqrt{\bar{g}}(L^\dagger\chi)^\nu\epsilon_\nu,\end{aligned} \tag{54}$$

we find $(L^\dagger)^{\mu\nu}_\alpha = -(\delta^\mu_\alpha\nabla^\nu + \delta^\nu_\alpha\nabla^\mu)$. We can handle the FP determinant more easily using the fact that $L$ and $L^\dagger$ have the same nonzero eigenvalues,

$$\det\mathcal{M} = \det{}'L = (\det{}'L^\dagger L)^{\frac{1}{2}}, \tag{55}$$

where we exclude the zero modes from the determinant, which are actually the zero modes of $L^\dagger$. It is easy to reveal the explicit form of the FP operator when $h_{\mu\nu} = 0$,

$$\begin{aligned}(L^\dagger)^{\mu\nu}_\beta L^\alpha_{\mu\nu} &= -(\delta^\mu_\beta\bar{\nabla}^\nu + \delta^\nu_\beta\bar{\nabla}^\mu)(\delta^\alpha_\nu\bar{\nabla}_\mu + \delta^\alpha_\mu\bar{\nabla}_\nu - \bar{g}_{\mu\nu}\bar{\nabla}^\alpha) \\ &= 2(\bar{\Delta}\delta^\alpha_\beta - \bar{R}^\alpha_\beta) \equiv 2(\bar{\Delta}_1)^\alpha_\beta,\end{aligned} \tag{56}$$

where we introduced the spin one Laplacian $\Delta_1$. In FG we have instead

$$\begin{aligned}\delta f_\mu &= \left(\delta^\alpha_\mu\bar{\nabla}^\beta - \frac{1}{2}\bar{g}^{\alpha\beta}\bar{\nabla}_\mu\right)\delta h_{\alpha\beta} \\ &= \left(\delta^\alpha_\mu\bar{\nabla}^\beta - \frac{1}{2}\bar{g}^{\alpha\beta}\bar{\nabla}_\mu\right)(\nabla_\alpha\epsilon_\beta + \nabla_\beta\epsilon_\alpha),\end{aligned}$$

and thus

$$\det\mathcal{M} = \det(\bar{\nabla}^\alpha g_{\alpha\nu}\nabla_\mu + \bar{\nabla}^\alpha g_{\mu\nu}\nabla_\alpha - \bar{\nabla}_\mu g_{\nu\alpha}\nabla^\alpha). \tag{57}$$

Note that the differential operator in (57) depends nontrivially on $h_{\mu\nu}$ and $g_{\mu\nu}$. It simplifies when $h_{\mu\nu} = 0$,

$$\bar{\nabla}^\alpha\bar{g}_{\alpha\nu}\bar{\nabla}^\mu + \bar{\nabla}^\alpha\delta^\mu_\nu\bar{\nabla}_\alpha - \bar{\nabla}^\mu\bar{g}_{\nu\alpha}\bar{\nabla}^\alpha = \bar{\Delta}\delta^\mu_\nu - \bar{R}^\mu_\nu = (\bar{\Delta}_1)^\alpha_\beta. \tag{58}$$

We see that the CG and FG ghost operators are the same when $h_{\mu\nu} = 0$, but CG has a real ghost while FG has complex ghosts; i.e. the determinant is under a square root in the first case. Note also that in CG we need to exclude zero modes, while in FG we need not (since it is the gradient of the CG).

[3]The gauge-fixing parameter should not be confused with the scaling exponent labeled in the same way.

A final comment on the CG in $d = 2$. In the exponential parametrization, i.e. $\xi = 1$, we have

$$g_{\mu\nu} = \bar{g}_{\mu\lambda} e^{h^\lambda_\nu} = \bar{g}_{\mu\lambda} e^{\frac{1}{2}\delta^\lambda_\nu h} = \bar{g}_{\mu\lambda}\delta^\lambda_\nu e^{\frac{1}{2}h} = \bar{g}_{\mu\nu} e^{\frac{1}{2}h} \equiv \bar{g}_{\mu\nu} e^{2\sigma}, \quad (59)$$

showing that all metrics can be reached from the background metric via a Weyl transformation with factor

$$\sigma = \frac{h}{4}. \quad (60)$$

Note that the Liouville action (12) in terms of $h$ reads

$$\frac{1}{24\pi} S_L[h/4; g] = \frac{1}{96\pi} \int d^2x \sqrt{g} \left[\frac{1}{4} h\Delta h + hR\right]. \quad (61)$$

Having set up the background, the gauges, and the ghosts, we can now turn to the discussion of the RG flow equations.

### B. Flow equations

To study the renormalizaion group flow and to compute the beta function and the critical or scaling exponents we will employ the functional RG approach based on the exact RG flow equation satisfied by the effective action[4] [2,16].

Using the background field method to preserve gauge invariance along the flow leads to the following flow equation first derived in [2]:

$$\partial_t \Gamma_k[h;\bar{g}] = \frac{1}{2} \mathrm{Tr}(\Gamma_k^{(2;0)}[h;\bar{g}] + R_k[\bar{g}])^{-1} \partial_t R_k[\bar{g}] + \text{ghost}, \quad (62)$$

where the scale dependent, or running, effective action $\Gamma_k[h;\bar{g}]$ depends on the fluctuation and background metric. The ghost terms will be discussed in a moment. Here $R_k[\bar{g}]$ is the cutoff kernel, responsible for the regulation and for the coarse graining. The flow equation (62) can be derived by the RG improvement of the regularized one-loop background gauge effective action, and as shown in [2], this improvement leads to an exact equation. The gauge invariant part of the effective action is[5] $\Gamma_k[\bar{g}] \equiv \Gamma_k[0;\bar{g}]$ and satisfies a flow equation given by setting $h_{\mu\nu} = 0$ in (62). In general the effective action can be written as

$$\Gamma_k[h;\bar{g}] = \Gamma_k[\bar{g}+h] + S_{gf}[h;\bar{g}], \quad (63)$$

where we introduced the gauge-fixing action. The Hessian in (62) is then

$$\Gamma_k^{(2;0)}[h;\bar{g}] = \Gamma_k^{(2)}[\bar{g}+h] + S_{gf}^{(2;0)}[h;\bar{g}]. \quad (64)$$

[4]The effective action, when evaluated on shell, is related to the partition function by $\Gamma_* = -\log Z$.

[5]We use $\Gamma_k[g]$ in place of the standard notation $\bar{\Gamma}_k[g]$ to simplify the notation.

In this way, the flow equation for the gauge invariant part of the effective action becomes

$$\partial_t \Gamma_k[\bar{g}] = \frac{1}{2} \mathrm{Tr}(\Gamma_k^{(2)}[\bar{g}] + S_{gf}^{(2;0)}[0;\bar{g}] + R_k[\bar{g}])^{-1} \partial_t R_k[\bar{g}] + \text{ghost} \quad (65)$$

and will be used in the next sections to compute the beta functions of $\Lambda_k$ and $G_k$. More specifically, to extract the beta functions of a set of couplings $\lambda_k^i$, we expand both sides of Eq. (65) on the relative operator basis $I_i[\bar{g}]$,

$$\partial_t \Gamma_k[\bar{g}] = \sum_i \partial_t \lambda_k^i I_i[\bar{g}],$$
$$\frac{1}{2} \mathrm{Tr}\left(\Gamma_k^{(2)}[\bar{g}] + S_{gf}^{(2;0)}[0;\bar{g}] + R_k[\bar{g}]\right)^{-1} \partial_t R_k[\bar{g}] = \sum_i \beta^i I_i[\bar{g}], \quad (66)$$

and by comparison we obtain the equations $\partial_t \lambda_k^i = \beta^i$; i.e. the beta functions are the coefficients of the expansion of the functional trace on the right-hand side (rhs) of the flow equation. In the context of quantum gravity, the expansion of the functional trace is performed with the fundamental aid of the heat kernel expansion, in both its local and nonlocal realizations [10]. These techniques allow us to work covariantly at any step of the computations.

The ghost contribution in (62) or (65) depends on the gauge; in CG we have

$$\text{ghost} = -\frac{1}{2} \mathrm{Tr}' \frac{\partial_t R_k(\Delta_1)}{\Delta_1 + R_k(\Delta_1)}, \quad (67)$$

where we used the fact that $L^\dagger L = \Delta_1$ and the excluded zero modes are those of $L^\dagger$. In FG one instead finds

$$\text{ghost} = -\mathrm{Tr} \frac{\partial_t R_k(\Delta_1)}{\Delta_1 + R_k(\Delta_1)}, \quad (68)$$

still involving the spin one Laplacian but counted twice and with no zero modes excluded. One of the virtues of Eq. (65) is that it holds in any dimension and allows, via the expansion (66), the computation of the beta functions of any set of couplings. We are going to exploit these properties in the next two sections, IV. C and IV. D, to covariantly derive the RG flow of quantum gravity in, respectively, $d = 2$ and $d = 2 + \epsilon$.

### C. Quantum gravity in $d = 2$

In this section we discuss the renormalization group flow in strictly two dimensions. Since the invariant $\int \sqrt{g} R$ is topological, it cannot be driving the RG flow, as instead it does in $d \geq 2$, and another invariant must take its place in order to have nontrivial beta functions. From the discussion of Secs. II and III we know that the natural candidate is the

Polyakov action. Generally we are then led to consider the following ansatz for the gauge invariant part of the running effective action:

$$\Gamma_k[g] = \int d^2x\sqrt{g}\left\{\Lambda_k - \frac{1}{4\pi G_k}R - \frac{c_k}{96\pi}R\frac{1}{\Delta}R\right\} = \Lambda_k I_0[g] - \frac{1}{4\pi G_k}I_1[g] + c_k S_P[g]. \quad (69)$$

Here $\Lambda_k$, $G_k$, and $c_k$ are running couplings, the scale dependence of which contains the information about the RG flow. We will soon see that the conformal anomaly does not renormalize, at least within the set of operators that we are considering in (69) (see [5] for a deeper analysis of this point), and we can thus set $c_k = c_g$ without any loss of generality. We will also drop the bar over the background metric when its presence is understood.

### 1. Beta functions

We will extract the beta functions for the couplings in (69) from the flow equation (65). The first thing we need to do is to compute the Hessian of the action (69). To obtain the quadratic action from which we can extract the Hessian, we need the second variations of the operators $I_0[g]$, $I_1[g]$, and $S_P[g]$. The details of these computations are given in the Appendix. We will also employ the traceless-trace decomposition,

$$h_{\mu\nu} = \hat{h}_{\mu\nu} + \frac{1}{d}\bar{g}_{\mu\nu}h, \qquad \bar{g}^{\mu\nu}\hat{h}_{\mu\nu} = 0, \quad (70)$$

both to simplify the second variations and to separate the gauge part (traceless) from the physical part (trace). For $I_0[g]$ we have from (A6)

$$\delta^2 I_0[g] = \int d^2x\sqrt{g}\left\{\frac{\xi}{4}h^2 + \frac{\xi-1}{2}\hat{h}^{\alpha\beta}\hat{h}_{\alpha\beta}\right\}, \quad (71)$$

showing in the standard parametrization we have only a traceless contribution, while in the exponential parametrization we have only a trace contribution. The second variation of $I_1[g]$ (A10) can be written as

$$\delta^2 I_1[g] = -\int d^2x\sqrt{g}\left\{\frac{1}{2}\hat{h}^{\mu\nu}(\Delta + R)\hat{h}_{\mu\nu} + \hat{h}^{\mu\nu}\nabla_\nu\nabla_\alpha\hat{h}^\alpha_\mu\right\}, \quad (72)$$

showing no dependence on $\xi$. We also note that this variation has only a traceless part and will thus vanish when we employ the strict CG forcing $\hat{h}_{\mu\nu} = 0$, as expected from the topological nature of the invariant. In a general CG with $\alpha \neq 0$ we will see that the traceless contributions are clearly pure gauge. In the Appendix we also report the details that lead to the second variation of the Polyakov action (A30),

$$\delta^2 S_P[g] = -\frac{1}{96\pi}\frac{1}{2}\int d^2x\sqrt{g}\{h[\Delta - (\xi-1)R]h + 2hA^{\mu\nu}\hat{h}_{\mu\nu} + \hat{h}_{\mu\nu}B^{\mu\nu}_{\alpha\beta}\hat{h}^{\alpha\beta}\}, \quad (73)$$

where both $A^{\mu\nu}$ and $B^{\mu\nu}_{\alpha\beta}$ are known tensors of which we do not need the explicit expression. Since they are part of the traceless and traceless-trace sectors, in strict CG they vanish, while, as before, in general CG they will be pure gauge.

Before explicitly computing the Hessian we make an important point: we recover the Liouville action (61) only if we use the exponential parametrization $\xi = 1$. Using the first (A26) and second (73) variations of the Polyakov action we find the following relation:

$$\delta S_P[g] + \frac{1}{2}\delta^2 S_P[g] = -\frac{1}{24\pi}\int d^2x\sqrt{g}\left\{\frac{h}{4}R + \frac{h}{4}\Delta\frac{h}{4}\right\} = -\frac{1}{24\pi}S_L\left[\frac{h}{4};g\right], \quad (74)$$

which we expected to hold given (60). We will argue that this fact strongly supports the use of the exponential parametrization.

The gauge-fixing action (51) is purely traceless,

$$S_{gf}[h;g] = \frac{1}{2\alpha}\int d^2x\sqrt{g}\hat{h}_{\mu\nu}\hat{h}^{\mu\nu},$$

and when added to the other variations (71), (72), and (73) gives the quadratic part of the action (69),

$$\frac{1}{2}\Lambda_k\delta^2 I_0[g] - \frac{1}{8\pi G_k}\delta^2 I_1[g] + \frac{1}{2}\delta^2 S_P[g] + S_{gf}[h;g] = \frac{1}{2}\int d^2x\sqrt{g}\left\{h\Gamma h + 2h\Gamma^{\mu\nu}\hat{h}_{\mu\nu} + \hat{h}_{\mu\nu}\left[\frac{1}{\alpha}\hat{\delta}^{\mu\nu}_{\alpha\beta} + \Gamma^{\mu\nu}_{\alpha\beta}\right]\hat{h}^{\alpha\beta}\right\}, \quad (75)$$

where we defined the following tensors:

$$\Gamma = \frac{Q^2}{8}\left[\Delta + (\xi-1)R + \frac{2\Lambda_k\xi}{Q^2}\right],$$
$$\Gamma^{\mu\nu} = A^{\mu\nu},$$
$$\Gamma^{\mu\nu}_{\alpha\beta} = \Lambda_k\frac{\xi-1}{2}\delta^{\mu\nu}_{\alpha\beta} + \frac{1}{4\pi G_k}\left[\frac{1}{2}\delta^{\mu\nu}_{\alpha\beta}(\Delta + R) + \nabla^\nu\nabla_\alpha\delta^\mu_\beta\right] + \frac{Q^2}{8}B^{\mu\nu}_{\alpha\beta}, \quad (76)$$

and $\hat{\delta}^{\mu\nu}_{\alpha\beta}$ is the identity in the space of symmetric rank two tensors. We also conventionally define

$$Q = \sqrt{\frac{-c_g}{24\pi}}, \quad (77)$$

preparing for discussing the connection with the Liouville theory approach to two dimensional quantum gravity that we will make in a later section. Using the notation $\bar{\Gamma}$ to represent $\Gamma^{\mu\nu}$ and $\Gamma$ to represent $\Gamma^{\mu\nu}_{\alpha\beta}$, we can write and perform the multiplet trace implicit in the flow equation (69),

$$\mathrm{tr}\left[\begin{pmatrix}\frac{1}{\alpha}\mathbf{1}+\mathbf{\Gamma}+\mathbf{R}_k & \bar{\Gamma}\\ \bar{\Gamma}^T & \Gamma+R_k\end{pmatrix}^{-1}\begin{pmatrix}\partial_t\mathbf{R}_k & 0\\ 0 & \partial_t R_k\end{pmatrix}\right]$$
$$=\frac{\partial_t R_k}{\Gamma+R_k}+\bar{\Gamma}^T\frac{\alpha}{\mathbf{1}+\alpha(\mathbf{\Gamma}+\mathbf{R}_k+\bar{\Gamma}^T\frac{1}{\Gamma+R_k}\bar{\Gamma})}\bar{\Gamma}\frac{\partial_t R_k}{\Gamma+R_k}+\frac{\alpha\partial_t\mathbf{R}_k}{\mathbf{1}+\alpha(\mathbf{\Gamma}+\mathbf{R}_k+\bar{\Gamma}^T\frac{1}{\Gamma+R_k}\bar{\Gamma})}. \tag{78}$$

We see from (78) that after the inversion we can safely go to the strict CG $\alpha=0$. Note also that Eq. (78) implicitly defines the tensor structure of the cutoff kernel. When we insert (78) in the flow equation (65), we obtain the following form:

$$\partial_t\Gamma_k[g]=\frac{1}{2}\mathrm{Tr}\frac{\partial_t R_k(\Delta_0)}{\Delta_0+R_k(\Delta_0)+\frac{2\Lambda_k\xi}{Q^2}}-\frac{1}{2}\mathrm{Tr}'\frac{\partial_t R_k(\Delta_1)}{\Delta_1+R_k(\Delta_1)}, \tag{79}$$

where we defined $\Delta_0=\Delta+(\xi-1)R$ as the spin zero operator and rescaled the cutoff $R_k\to\frac{Q^2}{8}R_k$. This is the flow equation for two dimensional quantum gravity in CG from which now we will extract the beta function of $\Lambda_k$ and $G_k$.

The functional traces of functions of Laplacian operators of the general form $\Delta=-\nabla^2\mathbf{1}+\mathbf{U}$, like those in (79), can be computed with the standard local heat kernel expansion,

$$\mathrm{Tr}\mathbf{M}f(\Delta)=\frac{1}{(4\pi)^{d/2}}\sum_{n=0}^{\infty}Q_{\frac{d}{2}-n}[f]\int d^dx\,\mathrm{tr}[\mathbf{M}\mathbf{b}_{2n}(\Delta)], \tag{80}$$

where $\mathbf{M}$ is a possible matrix insertion. The first two heat kernel coefficients, the only ones we will use, are

$$\mathbf{b}_0(\Delta)=\mathbf{1},\qquad \mathbf{b}_2(\Delta)=\mathbf{1}\frac{R}{6}-\mathbf{U}. \tag{81}$$

The $Q$ functionals appearing in (80) are defined as $Q_n[f]=\frac{1}{\Gamma(n)}\int dz z^{n-1}f(z)$ if $n>0$ and as $Q_n[f]=(-1)^{|n|}f^{(|n|)}(0)$ if $n\le 0$ (see the Appendix in [16] for more details).

We can now prove the nonrenormalization of the anomaly following [17]: the heat kernel expansion does not contain the invariant $\int\sqrt{g}R\frac{1}{\Delta}R$, and thus the beta function of the anomaly coefficient is zero. As we will see in Sec. IV. D. 2, only in the $k\to 0$ limit will this operator be produced. This justifies the substitution $c_k\to c_g$ we previously made.

Using (80) we can immediately evaluate the traces in (79),

$$\mathrm{Tr}\frac{\partial_t R_k(\Delta_0)}{\Delta_0+R_k(\Delta_0)+\frac{2\Lambda_k\xi}{Q^2}}=\frac{1}{4\pi}\int d^2x\sqrt{g}\left\{Q_1\left[h_k\left(\frac{2\Lambda_k\xi}{Q^2}\right)\right]b_0(\Delta_0)+Q_0\left[h_k\left(\frac{2\Lambda_k\xi}{Q^2}\right)\right]b_2(\Delta_0)+O(R^2)\right\}, \tag{82}$$

where the explicit values for the heat kernel coefficients are the following:

$$b_0(\Delta_0)=1,\qquad b_2(\Delta_0)=\frac{R}{6}+(1-\xi)R=\frac{7-6\xi}{6}R. \tag{83}$$

We also introduced the notation $h_k(\omega)=\frac{\partial_t R_k}{z+R_k+\omega}$. The ghost trace in (79) is

$$\mathrm{Tr}'\frac{\partial_t R_k(\Delta_1)}{\Delta_1+R_k(\Delta_1)}=\frac{1}{4\pi}\int d^2x\sqrt{g}\{Q_1[h_k(0)]b'_0(\Delta_1)^\mu_\mu+Q_0[h_k(0)]b'_2(\Delta_1)^\mu_\mu+O(R^2)\}, \tag{84}$$

where the heat kernel coefficients with the zero mode extracted are[6]

$$b'_0(\Delta_1)^\mu_\mu=b_0(\Delta_1)^\mu_\mu=\delta^\mu_\mu=2,$$
$$b'_2(\Delta_1)^\mu_\mu=b_2(\Delta_1)^\mu_\mu+3R=2\frac{R}{6}+R+3R=\frac{13}{3}R. \tag{85}$$

The flow equation (79) then becomes

$$\partial_t\Gamma_k[g]=\frac{1}{8\pi}\left\{-Q_1\left[h_k\left(\frac{2\Lambda_k\xi}{Q^2}\right)\right]+2Q_1[h_k(0)]\right\}\int d^2x\sqrt{g}+\frac{1}{8\pi}\left\{\frac{7-6\xi}{6}Q_0\left[h_k\left(\frac{2\Lambda_k\xi}{Q^2}\right)\right]-\frac{26}{3}\right\}\times\int d^2x\sqrt{g}R+O(R^2),$$

[6]The zero modes of $\Delta_1\equiv L^\dagger L$ are those of $L^\dagger$; one finds $N_0(L^\dagger)=-3\chi$ for the number of zero modes, and thus the heat kernel coefficients satisfy $b_2(\Delta_1)=b'_2(\Delta_1)+N_0(L^\dagger)$.

where we used the cutoff independent fact that $Q_0[h_k(0)] = 2$. A comparison with (69) gives the beta functions

$$\partial_t \Lambda_k = \frac{1}{8\pi}\left\{-Q_1\left[h_k\left(\frac{2\Lambda_k \xi}{Q^2}\right)\right] + 2Q_1[h_k(0)]\right\},$$
$$\partial_t\left(-\frac{1}{G_k}\right) = \frac{1}{2}\left\{\frac{7-6\xi}{6}Q_0\left[h_k\left(\frac{2\Lambda_k \xi}{Q^2}\right)\right] - \frac{26}{3}\right\}. \tag{86}$$

In the case $\Lambda_k = 0$ this last relation becomes

$$\partial_t\left(-\frac{1}{G_k}\right) = \underbrace{\frac{7-6\xi}{6}}_{\text{spin } 0}\underbrace{-\frac{26}{6}}_{\text{ghost}} = -\frac{19+6\xi}{6}, \tag{87}$$

or more explicitly

$$\partial_t G_k = \begin{cases} -\frac{1}{6}19G_k^2 & \xi = 0 \\ -\frac{1}{6}25G_k^2 & \xi = 1 \end{cases}, \tag{88}$$

which shows that the gravitational contribution to the beta function of Newton's constant is $-19$ in the standard parametrization, while it is $-25$ in the exponential parametrization. This result is new and shows that there is a dependence on the field parametrization, at least in the computation we have done, also in strictly two dimensional quantum gravity. Later we will compare this with the relative computation in $d = 2 + \epsilon$. Using now the relation (20) between $c_g$ and the beta function of Newton's constant leads to the following value for the total gravitational anomaly coefficient:

$$c_g = \underbrace{c_\Phi}_{\text{matter}}\underbrace{+7-6\xi}_{\text{spin } 0}\underbrace{-26}_{\text{ghost}} = \begin{cases} c_\Phi - 19 & \xi = 0 \\ c_\Phi - 25 & \xi = 1 \end{cases}, \tag{89}$$

where we made the breakdown of the various contributions and added the matter contribution. This is, in the $\xi = 1$ case, the result that we preannounced in Sec. III. As we see, the ghost contributes the well-known $-26$, while the trace spin zero part of the metric contributes like a standard scalar in the exponential parametrization and like seven scalars in the standard parametrization. From many other computations and constructions (see [4] for a comprehensive discussion of the literature) we know that the correct value is the one found in the exponential parametrization. Why the standard parametrization fails is to be understood. We note finally that these values are scheme independent since their derivation relied on only the fact that $Q_0[h_k(0)] = 2$, which is true for any admissible cutoff shape.

In the general case $\Lambda_k \neq 0$ we find, employing Litim's cutoff, i.e. $R_k(z) = (k^2 - z)\theta(k^2 - z)$, the following beta functions:

$$\partial_t \tilde{\Lambda}_k = -2\tilde{\Lambda}_k + \frac{1}{4\pi}\left\{\frac{1}{1+\frac{2\tilde{\Lambda}_k\xi}{Q^2}} - 2\right\},$$
$$\partial_t G_k = \left\{\frac{7-6\xi}{6}\frac{1}{1+\frac{2\tilde{\Lambda}_k\xi}{Q^2}} - \frac{26}{6}\right\}G_k^2. \tag{90}$$

These beta functions show that two dimensional quantum gravity is asymptotically free. We refer to the literature for more on this point [16].

### 2. Scaling exponents

Having computed $c_g$ we now turn to the computation of the scaling exponents $\alpha, \beta, \ldots$, in this way completing the determination of the scaling laws of Sec. III.

The area operator scales classically as $I_0[\lambda g] = \int\sqrt{\det[\lambda g]} = \lambda I_0[g]$. Fluctuations will generically change this by adding a nontrivial anomalous dimension $I_0[\lambda g] = \lambda^\alpha I_0[g]$. To account for this we will consider the generalized composite operator

$$I_0[g] = \int d^2x\sqrt{g}^\alpha, \tag{91}$$

where $\alpha$ is the scaling exponent we want to determine. One can attribute a dimensionality either to the coordinates or to the metric. Let us assume the last case, so $[g_{\mu\nu}] = k^{-2}$. The exponent $\alpha$ is determined self-consistently by requiring that the operator dimension of the area operator $I_0[g]$ is two. In this section we will work strictly in CG, and we will employ the exponential parametrization $\xi = 1$ only, since this is the one that leads to the correct scaling exponents.

To determine the anomalous scaling dimension of $I_0[g]$ we add to the effective action (69) the term[7] $-Z_k\int\sqrt{g}^\alpha$, where $Z_k$ is the wave-function renormalization constant of $I_0[g]$. The Hessian to insert in the flow equation (62) (note that we are considering the full bimetric action $\Gamma_k[h; g]$) is of the general form (76). Since we work in CG, we need only the trace part of the Hessian, which now reads

$$\Gamma = \frac{Q^2}{8}\Delta - \frac{\alpha^2}{4}Z_k e^{\frac{\alpha}{2}h}, \tag{92}$$

where we remember $Q = \frac{-c_g}{24\pi} = \frac{25-c_\Phi}{24\pi}$. Using (92) in the flow equation (62) after performing a rescaling of the cutoff $R_k \to \frac{Q^2}{8}R_k$ as before, we find the following expression:

[7]Remember that composite operators are introduced as $\int\mathcal{D}\phi e^{-S[\phi]+\int J\mathcal{O}(\phi)}$. In any case the sign in front of $Z_k$ drops out from the final formula for the scaling dimension.

$$\partial_t\Gamma_k[h;g] = \frac{1}{2}\mathrm{Tr}\frac{\frac{Q^2}{8}\partial_t R_k(\Delta)}{\frac{Q^2}{8}\Delta - \frac{\alpha^2}{4}Z_k e^{\frac{\alpha}{2}h} + \frac{Q^2}{8}R_k(\Delta)}$$
$$= \frac{1}{2}\mathrm{Tr}\frac{\partial_t R_k(\Delta)}{\Delta + R_k(\Delta)}$$
$$+ \frac{\alpha^2}{Q^2}Z_k\mathrm{Tr}\left\{\frac{\partial_t R_k(\Delta)}{[\Delta + R_k(\Delta)]^2}e^{\frac{\alpha}{2}h}\right\} + \cdots. \quad (93)$$

Without loss of generality we can set the background metric to be the flat metric in order to simplify the computations. The left-hand side (lhs) of (93) is then $\partial_t\Gamma_k[h;\delta] = -(\int d^2x)\partial_t Z_k e^{\frac{\alpha}{2}h}$. Using the heat kernel expansion (80) to its lowest order to evaluate the functional trace and comparing both sides leads to the following equation for the wave-function renormalization:

$$\partial_t Z_k = -\frac{\alpha^2}{Q^2}Z_k\frac{1}{4\pi}Q_1[g_k], \quad (94)$$

where $g_k(z) = \frac{\partial_t R_k(z)}{[z+R_k(z)]^2}$. The $Q$ functional in (94) is scheme independent $Q_1[g_k] = 2$, and thus we find

$$\partial_t Z_k = -\frac{\alpha^2}{2\pi Q^2}Z_k. \quad (95)$$

In terms of dimensionless variables, $[\sqrt{g}^\alpha] = k^{-2\alpha}$ and $[Z_k] = k^{2\alpha-\eta}$, where

$$\eta = -\partial_t \log Z_k = \frac{\alpha^2}{2\pi Q^2} \quad (96)$$

is the anomalous dimension of the operator $I_0[g]$. The request that this operator scales like an area leads to $2\alpha - \eta = 2$, or more explicitly using (96)

$$2\alpha - \frac{\alpha^2}{2\pi Q^2} = 2, \quad (97)$$

which, knowing that $Q = \frac{25-c_\Phi}{24\pi}$, is equivalent to

$$2 - 2\alpha + 2\frac{6}{25 - c_\Phi}\alpha^2 = 0. \quad (98)$$

The solution to (98) is

$$\alpha = \frac{25 - c_\Phi - \sqrt{(1 - c_\Phi)(25 - c_\Phi)}}{12}, \quad (99)$$

where we picked the branch that leads to $\alpha \to 1$ in the classical limit $c_\Phi \to -\infty$.

This can now be applied to a general composite operator $I_{\mathcal{O}}[\Phi;g] = \int\sqrt{g}\mathcal{O}[\Phi;g]$; the only difference will be the classical scaling dimension, which we generally call $\Delta_0$ (we had $\Delta_0 = 0$ in the case of $I_0[g]$), and Eq. (98) generalizes to

$$1 - \Delta_0 - \beta + \frac{6}{25 - c_\Phi}\beta^2 = 0. \quad (100)$$

This immediately gives the scaling exponent $\beta$ as

$$\beta = \frac{25 - c_\Phi - \sqrt{(1 + 24\Delta_0 - c_\Phi)(25 - c_\Phi)}}{12}. \quad (101)$$

This is the scaling that enters in the KPZ relation. To find the alternative form of this scaling, simply use $\beta = \alpha(1-\Delta)$ in the equation that defines $\beta$, to find

$$\Delta - \Delta_0 = 1 - \Delta_0 - (1 - \Delta)$$
$$= (1 - \Delta)\left[-1 + \alpha - \frac{6\alpha^2}{25 - c_\Phi}(1 - \Delta)\right]$$
$$= \frac{6\alpha^2}{25 - c_\Phi}\Delta(1 - \Delta). \quad (102)$$

Likewise, the exponent $\tilde{\alpha}$ needed in the scaling of the geodesic distance is determined as the scaling of the Laplacian. Since in $d = 2$ the operator $\sqrt{g}\Delta$ is scale invariant, the classical scaling of the Laplacian is fixed to be that of an inverse area and is thus found by solving

$$-1 - \tilde{\alpha} + \frac{6}{25 - c_\Phi}\tilde{\alpha}^2 = 0, \quad (103)$$

which gives

$$\tilde{\alpha} = \frac{25 - c_\Phi - \sqrt{(49 - c_\Phi)(25 - c_\Phi)}}{12}, \quad (104)$$

another relation that we used before.

### *3. Connection with Liouville theory*

In this section we want to make contact with the standard way to determine $c_g$ which is via Liouville theory [6,7,18]. In CG the partition function, in terms of the standard non-Weyl invariant measure, takes the form

$$Z = \int \mathcal{D}_g\hat{h}_{\mu\nu}\mathcal{D}_g h\,\delta[\hat{h}_{\mu\nu}]Z_\Phi[g]Z_{gh}[g], \quad (105)$$

where $g_{\mu\nu} = \bar{g}_{\mu\lambda}e^{h^\lambda_\nu}$. The integral over traceless metric fluctuations $\hat{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}g_{\mu\nu}h$ can be performed directly, imposing in this way the CG gauge-fixing condition strictly. The partition function then becomes Gaussian,

$$Z = \int \mathcal{D}_{\bar{g}e^{h/2}}h Z_\Phi[\bar{g}e^{h/2}]Z_{gh}[\bar{g}e^{h/2}], \quad (106)$$

where we already know the matter and ghost partition functions $Z_\Phi$ and $Z_{gh}$ as a function of $g$. Under a Weyl rescaling they transform as follows:

$$
Z_\Phi[\bar{g}e^{h/2}] = Z_\Phi[\bar{g}]e^{\frac{c_\Phi}{24\pi}S_L[h/4;\bar{g}]},
\\
Z_{gh}[\bar{g}e^{h/2}] = Z_{gh}[\bar{g}]e^{-\frac{26}{24\pi}S_L[h/4;\bar{g}]}. \tag{107}
$$

Assuming also that $h$ behaves as a standard scalar of weight zero, so that its measure is subject to (9), and leads to the following form for the partition function:

$$
Z = Z_\Phi[\bar{g}]Z_{gh}[\bar{g}]\int \mathcal{D}_{\bar{g}}h e^{\frac{c_\Phi+c_h-26}{24\pi}S_L[h/4;\bar{g}]}, \tag{108}
$$

where $c_h$ is in principle unknown and will be fixed in a moment (even if we expect it to be one). We can now work with the more standard Liouville variable $\sigma = \frac{h}{4}$. Since $Z$ is not affected by a Weyl rescaling, we must check that the same is true for the rhs of (108); thus it must be independent of the shift $\bar{g} \to \bar{g}e^{2\chi}$ and $\sigma \to \sigma - \chi$ if the conformal factor measure is translation invariant $\mathcal{D}_{\bar{g}}(\sigma - \chi) = \mathcal{D}_{\bar{g}}\sigma$. We have

$$
S_L[\sigma - \chi; \bar{g}e^{2\chi}] = \int d^2x\sqrt{\bar{g}}e^{2\chi}[2(\sigma-\chi)e^{-2\chi}\bar{\Delta}(\sigma-\chi) + 2(\sigma-\chi)e^{-2\chi}(\bar{R}+2\bar{\Delta}\chi)]
\\
= S_L[\sigma;\bar{g}] - S_L[\chi;\bar{g}], \tag{109}
$$

so we find that indeed

$$
\mathcal{D}_{\bar{g}e^{2\chi}}\sigma Z_\Phi[\bar{g}e^{2\chi}]Z_{gh}[\bar{g}e^{2\chi}]e^{\frac{c_\Phi+c_h-26}{24\pi}S_L[\sigma-\chi;\bar{g}e^{2\chi}]} = \mathcal{D}_{\bar{g}}\sigma Z_\Phi[\bar{g}]Z_{gh}[\bar{g}]e^{\frac{c_\Phi+c_h-26}{24\pi}S_L[\sigma;\bar{g}]}.
$$

Thus (108) is well defined. We define conventionally $Q = \sqrt{\frac{26-c_\Phi-c_h}{24\pi}}$, implicitly assuming that $c_\Phi + c_h < 26$ in order to have a positive definite action. We need to evaluate the following Gaussian integral:

$$
I = \int \mathcal{D}_g\sigma e^{-Q^2\int\sqrt{g}(\sigma\Delta\sigma+\sigma R)}. \tag{110}
$$

This is easily performed by just completing the square

$$
\frac{1}{2}\int d^2x\sqrt{g}\left(\sigma+\frac{1}{\Delta}R\right)\Delta\left(\sigma+\frac{1}{\Delta}R\right)
\\
= \int d^2x\sqrt{g}\left[\frac{1}{2}\sigma\Delta\sigma + \sigma R + \frac{1}{2}R\frac{1}{\Delta}R\right], \tag{111}
$$

where we integrated by parts. Inserting (111) into (110) and shifting the integration variable as $\sigma \to Q(\sigma + \frac{1}{\Delta}R)$ gives

$$
I = e^{\frac{Q^2}{2}\int\sqrt{g}R\frac{1}{\Delta}R}\int\mathcal{D}_g\sigma e^{-\frac{1}{2}\int\sqrt{g}\sigma\Delta\sigma}
\\
= e^{\frac{Q^2}{2}\int\sqrt{g}R\frac{1}{\Delta}R}e^{-\frac{1}{2}\mathrm{Tr}\log\Delta}
\\
= e^{\frac{1+48\pi Q^2}{96\pi}\int\sqrt{g}R\frac{1}{\Delta}R}, \tag{112}
$$

where we evaluated the Gaussian integral and we collected all terms. Incidentally this shows the Liouville action has central charge $c_L = 1 + 48\pi Q^2$.

Using (112) in (108) and remembering the form of $Z_m[\bar{g}]$ and of $Z_{gh}[\bar{g}]$ gives

$$
Z = e^{\frac{c_\Phi-26}{96\pi}\int\sqrt{g}R\frac{1}{\Delta}R}e^{\frac{1+26-c_\Phi-c_h}{96\pi}\int d^2x\sqrt{g}R\frac{1}{\Delta}R} = e^{\frac{1-c_h}{96\pi}\int d^2x\sqrt{g}R\frac{1}{\Delta}R}. \tag{113}
$$

Demanding that $Z$ is Weyl invariant leads to $c_h = 1$ since the only (on-shell) effective action which is Weyl invariant in $d = 2$ is $\Gamma[g] = 0$, which in turn implies $Z = 1$. Said in an equivalent way, to ask for $c_h = 1$ is equivalent to ask that conformal invariance is restored. This was basically the original argument of [6,7]; see also [12]. We then have

$$
Q = \sqrt{\frac{25 - c_\Phi}{24\pi}}, \tag{114}
$$

which establishes again the fundamental result $c_g = c_\Phi - 25$ and agrees with our previous definitions. In the context of Liouville theory the exponents $\alpha, \beta, \ldots,$ are now related to the scaling dimensions of the so-called vertex operators, $V_\alpha = e^{\alpha\sigma}$, and can be computed by standard CFT methods (we refer to [12] for more details). Needless to say, the results are the same as those we gave in the previous sections. With the knowledge of $c_g$ and the scaling exponents one then derives the scaling relations for various observables within Liouville theory and recovers the results we presented in Sec. III. We want to remark that our derivation shows that it is possible to respect covariance and that Liouville theory is not the only way, or the fundamental way, to establish scaling relations in the continuum. It is just one way to perform the analysis; more precisely, it is the way to exploit the fact that in CG quantum gravity in $d = 2$ is a Gaussian theory. In fact, all results of Liouville theory derive from the use of the Gaussian integral (112) or generalizations to include vertex operators, i.e. the integral (112) in the presence of external currents.

### D. Quantum gravity in $d = 2 + \epsilon$

In this section we consider quantum gravity in dimensions greater than two. This is the case considered in almost all studies of renormalization in the context of quantum gravity, starting from [19] to the works of [13,18] and later to the studies of gravity in the context of asymptotic safety [11,16]. A recent study of the dependence of the beta functions on field parametrizations in this last context has been presented recently [15]; for an application in the context of unimodular quantum gravity see [14].

Two things happen in $d > 2$: the Polyakov action is no longer induced by matter and ghost fluctuations, and the operator $\int \sqrt{g}R$ ceases to be topological. Somehow the latter start playing the role of the former, but we cannot in general expect the arguments and scaling relations of Secs. II and III to still be valid, since they were genuine to $d = 2$. In any case, we may hope to obtain a continuous $\epsilon \to 0$ limit by employing $\int \sqrt{g}R$ in place of $S_P[g]$. As we will explain more precisely later in this section, this requires a careful choice of the regulator in order to suppress the pathological limit the Hessan of $\int \sqrt{g}R$ has when $d \to 2$. We will also see that, when we consider the finite part of the effective action, only within the exponential parametrization will we be able to take the limit $\epsilon \to 0$. We thus consider the following ansatz for the gauge invariant part of the effective action:

$$\Gamma_k[g] = \Lambda_k \int d^dx\sqrt{g} - \frac{1}{4\pi G_k}\int d^dx\sqrt{g}R = \Lambda_k I_0[g] - \frac{1}{4\pi G_k} I_1[g], \tag{115}$$

and we compute the beta functions of the cosmological constant and of Newton's constant from those terms proportional to the invariants $I_0[g]$ and $I_1[g]$ stemming from the expansion of functional traces on the rhs of the flow equation (62).

#### 1. Beta functions

As in Sec. IV. C. 1, to compute the Hessian needed in the RG flow equation (65) we first derive the quadratic action. We will consider only the gauge $\alpha = 1$ since this allows us to employ heat kernel methods to compute the functional traces. We also employ the traceless-trace decomposition (70). The second variation of $I_0[g]$ in an arbitrary dimension is given in Eq. (A6) of the Appendix and reads

$$\frac{1}{2}\Lambda_k \delta^2 I_0[g] = \frac{1}{2}\Lambda_k \int d^dx\sqrt{g}\left(\frac{d-2+2\xi}{4d}h^2 + \frac{\xi-1}{2}\hat{h}^{\alpha\beta}\hat{h}_{\alpha\beta}\right), \tag{116}$$

while the second variation of $I_1[g]$, when summed to the FG gauge-fixing action (52) and evaluated on a spherical background, is given in Eq. (A13) of the Appendix, or

$$-\frac{1}{2}\frac{1}{4\pi G_k}\delta^2 I_1[g] + S_{gf}[h;g] = \frac{1}{2}\int d^dx\sqrt{g}\left\{\frac{1}{2}\hat{h}^{\mu\nu}\left(\Delta + \frac{d^2-3d+4}{d(d-1)}R - \frac{d-2}{d}\xi R\right)\hat{h}_{\mu\nu} - \frac{d-2}{4d}h\left(\Delta + \frac{d-4+2\xi}{d}R\right)h\right\}. \tag{117}$$

We will perform the replacement [20]

$$h_{\mu\nu} \to \sqrt{8\pi G_k}h_{\mu\nu}, \tag{118}$$

in the expansions (116) and (117) and rescale $\Lambda_k$ appropriately. We can define the symmetric spin two tensor identity $\delta^{\mu\nu}_{\rho\sigma} = \frac{1}{2}(\delta^\mu_\rho\delta^\nu_\sigma + \delta^\mu_\sigma\delta^\nu_\rho)$ and the trace projector $P^{\mu\nu}_{\rho\sigma} = \frac{1}{d}g^{\mu\nu}g_{\rho\sigma}$; then we have

$$\hat{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}h = \left(\delta^{\mu\nu}_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}g^{\mu\nu}\right)h_{\mu\nu} \equiv (\mathbf{1}-\mathbf{P})^{\mu\nu}_{\alpha\beta}h_{\mu\nu},$$
$$\frac{1}{2}g_{\alpha\beta}h = \frac{1}{2}g_{\alpha\beta}g^{\mu\nu}h_{\mu\nu} \equiv \mathbf{P}^{\mu\nu}_{\alpha\beta}h_{\mu\nu}. \tag{119}$$

Note that $\mathbf{1}-\mathbf{P}$ and $\mathbf{P}$ are orthogonal projectors into the trace and traceless subspaces in the space of symmetric tensors. In terms of these projectors we can now write the gravitational Hessian in the following way:

$$\mathbf{\Gamma}^{(2;0)}_k[0;g] = \frac{1}{2}(\mathbf{1}-\mathbf{P})[\Delta_2 + (\xi-1)\Lambda_k] - \frac{d-2}{4}\mathbf{P}\left[\Delta_0 - \frac{d-2+2\xi}{d(d-2)}\Lambda_k\right], \tag{120}$$

where we defined the spin two and spin zero differential operators,

$$\Delta_2 = \Delta + \left(\frac{d^2-3d+4}{d(d-1)} - \frac{d-2}{d}\xi\right)R,$$
$$\Delta_0 = \Delta + \frac{d-4+2\xi}{d}R, \tag{121}$$

while the ghost Hessian is the spin one differential operator $\Delta_1$ given in (58). We need now to choose the cutoff kernel, and the structure of the inverse propagator (120) suggests the following:

$$\mathbf{R}_k = (\mathbf{1}-\mathbf{P})R_k(\mathbf{\Delta}_2) - \frac{d-2}{2}\mathbf{P}R_k(\Delta_0). \qquad (122)$$

This natural choice is actually nontrivial and is ultimately responsible for the continuity, which we will discuss in a moment, of the $\epsilon \to 0$ limit and for the taming of the "wrong" sign of the spin zero inverse propagator. It was first introduced by [13,18] and later proposed in the context we are considering by [2]. It is easy now to write down explicitly the full regularized graviton propagator,

$$\left[(1-P)(\mathbf{\Delta}_2 + R_k(\mathbf{\Delta}_2) + (\xi-1)\Lambda_k) - \frac{d-2}{2}\mathbf{P}\left(\Delta_0 + R_k(\Delta_0) - \frac{d-2+2\xi}{d(d-2)}\Lambda_k\right)\right]^{-1}$$
$$= (\mathbf{1}-\mathbf{P})\frac{1}{\Delta_2 + R_k(\Delta_2) + (\xi-1)\Lambda_k} - \frac{2}{d-2}\mathbf{P}\frac{1}{\Delta_0 + R_k(\Delta_0) - \frac{d-2+2\xi}{d(d-2)}\Lambda_k}. \qquad (123)$$

Now when we multiply (123) with $\partial_t \mathbf{R}_k$, both the $d=2$ pole and the minus sign disappear, and the flow equation, and lately the beta functions, will not suffer from these problems in the $\epsilon \to 0$ limit, problems that are related to the topological nature of the invariant $\int \sqrt{g}R$ when $d=2$. As said, this choice of regulator is the one responsible for the good behavior of the $\epsilon \to 0$ limit.

To proceed, we insert in the graviton part of the flow equation the identity in the space of symmetric rank two tensor in the form $\mathbf{1} = (\mathbf{1}-\mathbf{P}) + \mathbf{P}$. After adding the ghost contribution (68) this gives the following flow equation:

$$\partial_t \Gamma_k[g] = \frac{1}{2}\mathrm{Tr}(\mathbf{1}-\mathbf{P})\frac{\partial_t R_k(\Delta_2)}{\Delta_2 + R_k(\Delta_2) + (\xi-1)\Lambda_k} + \frac{1}{2}\mathrm{Tr}\mathbf{P}\frac{\partial_t R_k(\Delta_0)}{\Delta_0 + R_k(\Delta_0) - \frac{d-2+2\xi}{d(d-2)}} - \mathrm{Tr}\delta^\mu_\nu \frac{\partial_t R_k(\Delta_1)}{\Delta_1 + R_k(\Delta_1)}. \qquad (124)$$

It is now easy to evaluate the traces using the local heat kernel expansion (80). We find the following heat kernel coefficients:

$$\mathrm{tr}[(\mathbf{1}-\mathbf{P})\mathbf{b}_2(\mathbf{\Delta}_2)] = \frac{d^2+d-2}{2}\left[\frac{R}{6} - \left(\frac{d^2-3d+4}{d(d-1)} - \frac{d-2}{2d}\xi\right)R\right] \underset{d=2}{=} -\frac{5}{3}R,$$

where we used $\mathrm{tr}(\mathbf{1}-\mathbf{P}) = \frac{d^2+d-2}{2}$;

$$\mathrm{tr}[\mathbf{P}\mathbf{b}_2(\Delta_0)] = \frac{R}{6} - \frac{d-4+2\xi}{d}R \underset{d=2}{=} \frac{7-6\xi}{6}R,$$

showing that only the conformal mode is $\xi$ dependent; and

$$\mathrm{tr}b_2(\Delta_1) = \delta^\mu_\mu \frac{R}{6} + R^\mu_\mu = \frac{d+6}{6}R \underset{d=2}{=} \frac{4}{3}R,$$

showing that there is nothing universal in the ghost trace. Then, to linear order in the curvature and for $d=2$, the flow equation (124) is

$$\partial_t \Gamma_k[g] = \frac{1}{8\pi}\left\{2Q_1[h_k((\xi-1)\Lambda_k)] + Q_1\left[h_k\left(-\frac{\xi}{\epsilon}\Lambda_k\right)\right] - 2Q_1[h_k(0)]\right\}\int d^2x\sqrt{g}$$
$$+ \frac{1}{8\pi}\left\{-\frac{5}{3}Q_0[h_k((\xi-1)\Lambda_k)] + \frac{7-6\xi}{6}Q_0\left[h_k\left(-\frac{\xi}{\epsilon}\Lambda_k\right)\right] - \frac{8}{3}Q_0[h_k(0)]\right\}\int d^2x\sqrt{g}R + O(R^2). \qquad (125)$$

This leads to the following beta functions:

$$\partial_t \Lambda_k = \frac{1}{8\pi}\left\{2Q_1[h_k((\xi-1)\Lambda_k)] + Q_1\left[h_k\left(-\frac{\xi}{\epsilon}\Lambda_k\right)\right] - 2Q_1[h_k(0)]\right\},$$
$$\partial_t\left(-\frac{1}{G_k}\right) = \frac{1}{2}\left\{-\frac{5}{3}Q_0[h_k((\xi-1)\Lambda_k)] + \frac{7-6\xi}{6}Q_0\left[h_k\left(-\frac{\xi}{\epsilon}\Lambda_k\right)\right] - \frac{8}{3}Q_0[h_k(0)]\right\}. \qquad (126)$$

In the case $\Lambda_k = 0$ we find, as in the previous section, the universal beta function for Newton's constant (using the scheme independent value $Q_0[h_k(0)] = 2$),

$$\partial_t\left(-\frac{1}{G_k}\right) = \underbrace{-\frac{5}{3}}_{\text{spin 2}} + \underbrace{\frac{7-6\xi}{6}}_{\text{spin 0}} \underbrace{-\frac{8}{3}}_{\text{ghost}} = -\frac{19+6\xi}{6}, \quad (127)$$

where we tagged the various contributions explicitly. The amazing fact is that the total universal beta function of $G_k$ is the same as in the strictly two dimensional case (87), but now the contributions are the following:

$$c_g = \underbrace{c_\Phi}_{\text{matter}} \underbrace{-10}_{\text{spin 2}} \underbrace{+7-6\xi}_{\text{spin 0}} \underbrace{-16}_{\text{ghost}} = \begin{cases} c_\Phi - 19 & \xi = 0 \\ c_\Phi - 25 & \xi = 1 \end{cases}. \quad (128)$$

Thus again $c_g = c_\Phi - 25$ if $\xi = 1$. Note that now the $-\frac{13}{3}$ contribution of the CG ghost is split up in a $-\frac{5}{3}$ from "gravitons" and a $-\frac{8}{3}$ from the FG ghost, and this clearly shows that the ghost contribution to $c_g$ alone has no universal meaning. In both gauges, the trace behaves as a standard scalar only in the exponential parametrization. The same ghost contribution was found in the extrinsic approach, where one computes the RG flow in the theory of two dimensional surfaces embedded in $D$-dimensional Euclidean space and then takes the limit $D \to 0$ in the equivalent of Newton's constant beta function [21].

In terms of dimensionless variables, $\Lambda_k = k^2\tilde{\Lambda}_k$, and employing Litim's cutoff, the beta functions (126) become

$$\partial_t\tilde{\Lambda}_k = -2\tilde{\Lambda}_k + \frac{1}{4\pi}\left\{\frac{2}{1-(\xi-1)\tilde{\Lambda}_k} + \frac{1}{1-\frac{\xi}{\epsilon}\tilde{\Lambda}_k} - 2\right\},$$
$$\partial_t\left(-\frac{1}{G_k}\right) = -\frac{5}{3}\frac{1}{1-(\xi-1)\tilde{\Lambda}_k} + \frac{7-6\xi}{2}\frac{1}{1-\frac{\xi}{\epsilon}\tilde{\Lambda}_k} - \frac{8}{3}. \quad (129)$$

Note that only if the cosmological constant is zero can we take the limit $d \to 2$ when $\xi = 1$. A more detailed discussion of these beta functions can be found in the literature [15,16].

### 2. *Finite part of the effective action*

In this section we compute the finite parts of the effective action. We will follow the methods of [17] which employ the nonlocal heat kernel expansion; in particular we compute the $R^2$ terms. As we have seen in the previous section, the Polyakov coupling does not run; i.e. $\partial_t c_k = 0$ (we need terms proportional to beta functions to have a nonzero running). In this section we will show that it is generated in the $k \to 0$ limit (as explained in [17]).

The nonlocal ansatz for the gauge invariant part of the effective action to use on the lhs of the flow equation is

$$\Gamma_k[g] = \int d^2x\sqrt{g}[a_k + b_kR + Rc_k(\Delta)R] + O(R^3). \quad (130)$$

In two dimensions the Ricci tensor is proportional to the Ricci scalar $R_{\mu\nu} = \frac{1}{2}g_{\mu\nu}R$ so there is only one nonlocal heat kernel structure function at the order curvature square [10], and this is given by the following linear combination:

$$\begin{aligned} f_{R^2}(x) &= \mathrm{tr}\mathbf{1} f_{R2d}(x) + \mathrm{tr}\mathbf{1} f_{UR}(x)\left(\frac{d^2-3d+4}{d(d-1)} - \frac{d-2}{4d}\xi\right) \\ &+ \mathrm{tr}\mathbf{1} f_U(x)\left(\frac{d^2-3d+4}{d(d-1)} - \frac{d-2}{4d}\xi\right)^2 + \left(1-\frac{4}{d}\right)(d+2)f_\Omega(x) \\ &+ f_{R2d}(x) + f_{UR}(x)\frac{d-4+2\xi}{d} + f_U(x)\left(\frac{d-4+2\xi}{d}\right)^2 \\ &+ \delta^\mu_\mu f_{R2d}(x) - f_{UR}(x)U + \frac{1}{d}f_U(x) + \left(1-\frac{4}{d}\right)f_\Omega(x). \end{aligned} \quad (131)$$

The first two lines are the spin two contribution, the third line is the spin zero contribution, while the last line is the spin one contribution. When we write (131) in terms of the basic nonlocal structure function and set $d = 2$, we find

$$\begin{aligned} f_{R^2} &= \frac{9f}{8x^2} + \frac{15f}{8x} + \frac{43f}{32} - \frac{9}{8x^2} - \frac{27}{16x} - \frac{5f}{4}\xi + \frac{1}{2x}\xi - \frac{f}{2x}\xi + \frac{f}{2}\xi^2 \\ &+ \frac{3f}{4x^2} + \frac{5f}{4x} + \frac{9f}{16} - \frac{3}{4x^2} - \frac{9}{8x}, \end{aligned} \quad (132)$$

where the first line is the gravitational contribution and the second line is the ghost contribution. The flow equation for $c_k(x)$ can be written as

$$\partial_t c_k(x) = \frac{1}{8\pi k^2} g\left(\frac{x}{k^2}\right). \tag{133}$$

Note the overall power of $k^{-2}$ in (133). If we employ Litim's cutoff shape function, then we find ($u = x/k^2$)

$$\begin{aligned} g(u) = &-\frac{1}{8u^2}\left\{[12 + (27 - 8\xi)u]\sqrt{1 - \frac{4}{u}} - (43 - 40\xi + 16\xi^2)u\sqrt{\frac{u}{u-4}}\right\}\theta(u-4) \\ &+ \frac{1}{2u^2}\left[(4 + 9u)\sqrt{1 - \frac{4}{u}} - 9u\sqrt{\frac{u}{u-4}}\right]\theta(u-4), \end{aligned} \tag{134}$$

where the first line is the graviton contribution and the second line is the ghost contribution. An expansion around $u = \infty$,

$$g(u) = \frac{2\xi^2 - 4\xi + 2}{u} + \frac{4\xi^2 - 12\xi + 16}{u^2} + O\left(\frac{1}{u^3}\right), \tag{135}$$

shows the important point that in the exponential parametrization the coefficient of the leading term is zero, i.e. $2\xi^2 - 4\xi + 2 = 0$ if $\xi = 1$. This fact was also noticed in the original covariant perturbation theory literature in the case of a scalar field [22]. We will see in a moment that this behavior makes the integral of $g(u)$ finite in the $k \to 0$ limit. This cancellation can be seen directly in (134) since the coefficients of the $u$ times square roots terms agree when $\xi = 1$.

Integrating the flow from the UV scale $\Lambda$ to the IR scale $k$ and shifting to the variable $u = x/k^2$ gives

$$c_k(x) = c_\Lambda(x) - \frac{1}{16\pi x}\int_{x/\Lambda^2}^{x/k^2} du\, g(u). \tag{136}$$

The integral in (136) is finite for $\Lambda \to \infty$—i.e. there are no UV divergences—so we can take the UV cutoff to infinity. This is related to the theta functions in (134) which imply that we have to compute the integral between 4 and $x/k^2$, explicitly showing that the high energy part does not contribute. The integral in (136) can be performed analytically, but the result is not very revealing and in any case is scheme dependent. Instead we report the small $k$ expansion,

$$c_k(x) - c_\infty(x) = -\frac{1}{16\pi x}\left[\frac{13}{2} - 2\xi + 2(1-\xi)^2 \log\frac{x}{k^2} - 4(\xi^2 - 3\xi + 4)\frac{k^2}{x} - \frac{26}{3} + 16\frac{k^2}{x} + O\left(\frac{k^4}{x^2}\right)\right], \tag{137}$$

where again we separated the gravitational from the ghost contributions. This expression shows clearly that the limit $k \to 0$ is obstructed by the diverging logarithm term if $\xi \neq 1$; i.e. there is an IR divergence if we use the standard parametrization. In the exponential parametrization, instead, we can safely take the IR limit to find

$$c_0(x) = c_\infty(x) + \frac{25}{96\pi x} \equiv -\frac{c_\infty - 25}{96\pi x}, \tag{138}$$

and thus $c_0 = c_\infty - 25$. This checks explicitly our assumptions that the gravitational effective action has the same form as the Polyakov effective action for matter fields, but with the proper coefficient $c_g = c_\Phi - 25$ (with matter included as in [17]).

## V. CONCLUSIONS

In this paper we have explored the quantum properties of two dimensional quantum gravity by putting together two complementary approaches: scaling arguments and the renormalization group analysis. In both cases we pursued a fully covariant formulation to prepare the ground for a future study of four dimensional quantum gravity along the same lines.

The full quantum properties of a theory are accessible only when we have a well-defined quantization procedure. In the path integral approach, which is the most useful one to set up the RG analysis, this translates into correctly identifying the measure of integration. We found that by using the prescription given in [5], standard results in two dimensional quantum gravity can be reproduced in a simple and clear way. Once the quantization is understood, a shortcut to study the UV properties of gravity is to consider the partition function at a fixed volume, which is an area in $d = 2$, and study how this scales when we rescale the area.

The path integral will generate a nontrivial quantum scaling on top of the classical one, which determines the quantum properties of the theory. The standard partition function then can be recovered from this by a Laplace transform, and it will pick up these contributions. The advantage of this construction is that once this "reduced" partition function is well defined, exact scaling relations can be derived from it which translate into exact properties of the full partition function. In this way one is able to find out how gravity modifies the spectrum of scaling dimensions of matter and essentially solve quantum gravity in two dimensions.

Scaling relations are a natural hint for RG arguments. Paradigmatic examples are those one finds in a statistical physics context. First observed in purely phenomenological terms, they relate the various critical exponents of a statistical system by assuming that the free energy (read partition function) has some definite scaling form, in terms of one dimensionful quantity, say the temperature, and a function of dimensionless ratios. The exponent of the temperature is one of the critical exponents. Other quantities derived from the free energy can be put in similar scaling form, and by comparing these different quantities scaling relations between different exponents are found. The RG gives an intuitive reason for this. Since at the critical point correlation lengths diverge, the system reaches a scale invariant phase, which is associated with a fixed point of the RG flow. At the fixed point all dimensionless quantities approach a finite value (including zero), so physical observables acquire a definite scaling with respect to a dimensionful scale. Moreover, by considering the linearized RG flow in the neighborhood of the fixed point, it is actually possible to calculate the value of the critical exponents, whereas scaling arguments alone are not sufficient to determine them.

This is conceptually the same approach that was followed here. To calculate the gravitational scaling exponents, we used the functional RG. The scaling exponents can then be easily found as the wave function renormalization of composite operators. Still, up to this point there is one piece of information missing, which is the value of the gravitational central charge. Here we found that to reproduce the correct value, consistent with what one finds from Liouville theory, one needs to use the exponential parametrization for the metric fluctuations. We performed the calculation in $d = 2$ and $d = 2 + \epsilon$. In the last case one also looks at the finite part of the effective action and recovers the same result. It is crucial there to use the exponential parametrization in order to be able to recover the $k \to 0$ limit. The result also agrees with the relation found in [5] between the central charge and the beta function of Newton's constant, which was briefly rederived, in a slightly different way, in the first part of the paper. This relation allows us to use the known form of the beta function to compute a universal quantity like the $c$ function.

Probably the most interesting aspect of this work was that Liouville theory, which is peculiar to two dimensions, was never really needed. All we needed were scaling arguments and RG calculations. As we said, the whole approach seems to work in covariant form. Thus there is the hope that the same analysis can be carried through in $d = 4$. We will investigate this in a future paper.

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## APPENDIX: VARIATIONS

In this appendix we collect the basic variations, and their derivations, which are needed in the main text.

### 1. Variations of $\sqrt{g}$ and $\sqrt{g}R$

We start considering the variations of the two invariants

$$I_0[g] = \int \sqrt{g}, \qquad I_1[g] = \int \sqrt{g}R. \tag{A1}$$

To compute the Hessians entering the flow equation we need the second variations of the invariants (A1). These read, taken for example from [16] or computed using xTensor,

$$\delta^2 I_0[g] = \int d^d x \sqrt{g} \left( \frac{1}{4} h^2 - \frac{1}{2} h^{\alpha\beta} h_{\alpha\beta} + \frac{1}{2} H \right), \tag{A2}$$

where we remember that $H_{\mu\nu} = \delta^2 g_{\mu\nu} = \xi h_{\mu\lambda} h^\lambda_\mu$, and

$$\begin{aligned} \delta^2 I_1[g] = \int d^d x \sqrt{g} \Big[ & -\frac{1}{2} h^{\mu\nu} \Delta h_{\mu\nu} + \frac{1}{2} h \Delta h - h^{\mu\nu} \nabla_\nu \nabla_\alpha h^\alpha_\mu + h \nabla^\mu \nabla^\nu h_{\mu\nu} + h^{\mu\nu} h^\alpha_\mu R_{\nu\alpha} \\ & + h^{\mu\nu} h^{\alpha\beta} R_{\alpha\mu\beta\nu} - h R^{\mu\nu} h_{\mu\nu} + \left( \frac{1}{4} h^2 - \frac{1}{2} h^{\alpha\beta} h_{\alpha\beta} \right) R \\ & - H_{\mu\nu} R^{\mu\nu} + \frac{1}{2} H R + \nabla_\mu \nabla_\nu H^{\mu\nu} + \Delta H \Big]. \end{aligned} \tag{A3}$$

We note that the last two terms are total derivatives; thus the use of the exponential parametrization will not change the kinetic terms of the standard parametrization, and so the difference will be in the curvature terms.

It is useful now to perform the trace-traceless decomposition,

$$h_{\mu\nu} = \hat{h}_{\mu\nu} + \frac{1}{d} h \bar{g}_{\mu\nu}, \qquad \bar{g}^{\mu\nu}\hat{h}_{\mu\nu} = 0, \tag{A4}$$

so that

$$\begin{aligned} h^{\mu\nu} h_{\mu\nu} &= \hat{h}^{\mu\nu}\hat{h}_{\mu\nu} + \frac{1}{d} h^2, \\ h^{\mu\nu}\Delta h_{\mu\nu} &= \hat{h}^{\mu\nu}\Delta\hat{h}_{\mu\nu} + \frac{1}{d} h\Delta h, \\ h\nabla^\mu\nabla^\nu h_{\mu\nu} &= h\nabla^\mu\nabla^\nu\hat{h}_{\mu\nu} - \frac{1}{d} h\Delta h, \\ h^{\mu\nu}\nabla_\nu\nabla_\alpha h^\alpha_\mu &= \hat{h}^{\mu\nu}\nabla_\nu\nabla_\alpha\hat{h}^\alpha_\mu + \frac{2}{d} h\nabla^\mu\nabla^\nu\hat{h}_{\mu\nu} - \frac{1}{d^2} h\Delta h, \end{aligned} \tag{A5}$$

where an integration by parts is implicit in the last relation. Using these relations in (A2) gives

$$\begin{aligned} \delta^2 I_0[g] &= \int d^dx\sqrt{g}\left(\frac{1}{4}h^2 + \frac{\xi-1}{2} h^{\alpha\beta}h_{\alpha\beta}\right) \\ &= \int d^dx\sqrt{g}\left(\frac{d-2+2\xi}{4d} h^2 + \frac{\xi-1}{2}\hat{h}^{\alpha\beta}\hat{h}_{\alpha\beta}\right). \end{aligned} \tag{A6}$$

Thus in two dimensions the second variation of $I_0[g]$ is purely traceless in the standard parametrization $\xi = 0$, while it is purely trace in the exponential parametrization $\xi = 1$. Note that in the standard parametrization there is a dangerous $d-2$ pole term that must be properly treated in the $d \to 2$ limit.

Using (A5) in (minus) the derivative terms of (A3) gives

$$\frac{1}{2} h^{\mu\nu}\Delta h_{\mu\nu} - \frac{1}{2} h\Delta h + h^{\mu\nu}\nabla_\nu\nabla_\alpha h^\alpha_\mu - h\nabla^\mu\nabla^\nu h_{\mu\nu} = \frac{1}{2}\hat{h}^{\mu\nu}\Delta\hat{h}_{\mu\nu} - \frac{(d-2)(d-1)}{2d^2} h\Delta h + \hat{h}^{\mu\nu}\nabla_\nu\nabla_\alpha\hat{h}^\alpha_\mu - \frac{d-2}{d} h\nabla^\mu\nabla^\nu\hat{h}_{\mu\nu}. \tag{A7}$$

We will now choose the background metric to be maximally symmetric in order to simplify the curvature terms of (A3); in two dimensions this is no restriction at all, while in $d \geq 2$ we lose no generality since we are interested in expansions of the effective action up to linear order in the curvature. The Riemann and Ricci tensors are then proportional to the Ricci scalar,

$$R_{\mu\nu} = \frac{R}{d} g_{\mu\nu}, \qquad R_{\mu\nu\rho\sigma} = \frac{R}{d(d-1)}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}). \tag{A8}$$

Inserting these relations and performing the trace-traceless decomposition of (minus) the curvature terms of (A3) gives

$$\begin{aligned} &- h^{\mu\nu}h^\alpha_\mu R_{\nu\alpha} - h^{\mu\nu}h^{\rho\alpha}R_{\rho\nu\alpha\mu} + hh_{\mu\nu}R^{\mu\nu} - \left(\frac{1}{4}h^2 - \frac{1}{2}h^{\alpha\beta}h_{\alpha\beta}\right)R + H_{\mu\nu}R^{\mu\nu} - \frac{1}{2}HR \\ &= \frac{1}{2}\hat{h}^{\mu\nu}\hat{h}_{\mu\nu}\left(\frac{d^2-3d+4}{d(d-1)} - \frac{d-2}{d}\xi\right)R + h^2\left(\frac{d^2-3d+4}{2d^2(d-1)} - \frac{d-2}{4d^2}\xi - \frac{d^2-5d+8}{4d(d-1)}\right)R. \end{aligned} \tag{A9}$$

Finally we are led to

$$\begin{aligned} \delta^2 I_1[g] = -\int d^dx\sqrt{g}\Bigg\{&\frac{1}{2}\hat{h}^{\mu\nu}\left(\Delta + \frac{d^2-3d+4}{d(d-1)}R - \frac{d-2}{d}\xi R\right)\hat{h}_{\mu\nu} + \hat{h}^{\mu\nu}\nabla_\nu\nabla_\alpha\hat{h}^\alpha_\mu - \frac{d-2}{d} h\nabla^\mu\nabla^\nu\hat{h}_{\mu\nu} \\ &- \frac{d-2}{2d^2} h\left[(d-1)\Delta + \frac{d-4+2\xi}{2}R\right]h\Bigg\}. \end{aligned} \tag{A10}$$

This relation is interesting for several reasons: first, it shows that the trace part vanishes in two dimensions making the Hessian noninvertible; second, it shows that $\xi$ terms make no difference when again $d = 2$. These are both signs of the topological nature of the operator in this dimension and imply that in CG the contribution of the invariant $I_1[g]$ is purely gauge and will not contribute when we enforce the gauge strictly (as expected). In FG, as for the other invariant, the fact that the inverse of the trace part is singular when $d \to 2$ calls for a careful definition of the regulator.

The background FG gauge fixing action (52) is already quadratic in $h_{\mu\nu}$; when expanded it reads

$$S_{gf}[h;g] = \frac{1}{2\alpha}\int d^dx\sqrt{g}\left(-h^{\mu\nu}\nabla_\nu\nabla_\alpha h^\alpha_\mu + h\nabla^\mu\nabla^\nu h_{\mu\nu} + \frac{1}{4}h\Delta h\right)$$
$$= \frac{1}{2\alpha}\int d^dx\sqrt{g}\left(-\hat{h}^{\mu\nu}\nabla_\nu\nabla_\alpha\hat{h}^\alpha_\mu + \frac{d-2}{d}h\nabla^\mu\nabla^\nu\hat{h}_{\mu\nu} + \frac{(d-2)^2}{4d^2}h\Delta h\right). \tag{A11}$$

Combining now (A13) with (A11) gives

$$-\frac{1}{2}\delta^2 I_1[g] + S_{gf}[h;g] = \frac{1}{2}\int d^dx\sqrt{g}\Bigg[\frac{1}{2}\hat{h}^{\mu\nu}\Delta\hat{h}_{\mu\nu} - \frac{d-2}{2d^2}\left(d-1-\frac{d-2}{2\alpha}\right)h\Delta h + \left(1-\frac{1}{\alpha}\right)\hat{h}^{\mu\nu}\nabla_\nu\nabla_\alpha\hat{h}^\alpha_\mu$$
$$-\frac{d-2}{d}\left(1-\frac{1}{\alpha}\right)h\nabla^\mu\nabla^\nu\hat{h}_{\mu\nu} + \hat{h}^{\mu\nu}\hat{h}_{\mu\nu}\left(\frac{d^2-3d+4}{2d(d-1)} - \frac{d-2}{2d}\xi\right)R - h^2\frac{(d-2)(d-4+2\xi)}{4d^2}R\Bigg], \tag{A12}$$

which shows that the choice $\alpha = 1$ leads to the diagonalization of the quadratic action

$$-\frac{1}{2}\delta^2 I_1[g] + S_{gf}[h;g] = \frac{1}{2}\int d^dx\sqrt{g}\left\{\frac{1}{2}\hat{h}^{\mu\nu}\left[\Delta + \left(\frac{d^2-3d+4}{d(d-1)}R - \frac{d-2}{d}\xi\right)R\right]\hat{h}_{\mu\nu} - \frac{d-2}{4d}h\left[\Delta + \frac{d-4+2\xi}{d}R\right]h\right\}. \tag{A13}$$

We stress again that the Hessian of this quadratic action will not be invertible in two dimensions since the trace part vanishes.

### 2. Variations of the Polyakov action

We will now determine the Hessian of the Polyakov action (10),

$$S_P[g] = -\frac{1}{96\pi}\int\sqrt{g}R\frac{1}{\Delta}R. \tag{A14}$$

First we will need some basic variations,

$$\delta g^{\mu\nu} = -h^{\mu\nu},$$
$$\delta\sqrt{g} = \frac{1}{2}\sqrt{g}h,$$
$$\delta\Gamma^\alpha_{\mu\nu} = \frac{1}{2}(\nabla_\mu h^\alpha_\nu + \nabla_\nu h^\alpha_\mu - \nabla^\alpha h_{\mu\nu}),$$
$$g^{\mu\nu}\delta\Gamma^\alpha_{\mu\nu} = \nabla^\mu h^\alpha_\mu - \frac{1}{2}\nabla^\alpha h,$$
$$\delta R = \Delta h + \nabla^\mu\nabla^\nu h_{\mu\nu} - \frac{1}{2}hR, \tag{A15}$$

where we used $R_{\mu\nu} = \frac{1}{2}g_{\mu\nu}R$ in the last line, the relation valid only in two dimensions. We start with

$$\delta\left(\sqrt{g}R\frac{1}{\Delta}R\right) = \frac{1}{2}hR\frac{1}{\Delta}R + 2\delta R\frac{1}{\Delta}R + R\delta\frac{1}{\Delta}R, \tag{A16}$$

where we integrated by parts one of the terms with the variation of the Ricci scalar. From $\frac{1}{\Delta}\Delta = 1$ we find

$$\delta\frac{1}{\Delta} = -\frac{1}{\Delta}\delta\Delta\frac{1}{\Delta}, \tag{A17}$$

where the variation of the Laplacian acting on a scalar, using variations from (A15), is

$$\delta\Delta\phi = -\delta(g^{\mu\nu}\nabla_\nu\nabla_\mu\phi)$$
$$= -\delta g^{\mu\nu}\nabla_\nu\nabla_\mu\phi - g^{\mu\nu}\delta\nabla_\mu\partial_\nu\phi$$
$$= h^{\mu\nu}\nabla_\nu\nabla_\mu\phi + g^{\mu\nu}\delta\Gamma^\alpha_{\mu\nu}\nabla_\alpha\phi$$
$$= h^{\mu\nu}\nabla_\mu\nabla_\nu\phi + \nabla^\mu h^\nu_\mu\nabla_\nu\phi - \frac{1}{2}\nabla^\alpha h\nabla_\alpha\phi. \tag{A18}$$

Inserting (A18) in (A17) and then (A17) in (A16) gives

$$\frac{1}{\sqrt{g}}\delta\left(\sqrt{g}R\frac{1}{\Delta}R\right) = -\frac{1}{2}hR\frac{1}{\Delta}R + 2hR + 2h_{\mu\nu}\nabla^\mu\nabla^\nu\frac{1}{\Delta}R + h^{\mu\nu}\left(\nabla_\mu\frac{1}{\Delta}R\right)\left(\nabla_\nu\frac{1}{\Delta}R\right)$$
$$-\frac{1}{2}h\left(\nabla^\alpha\frac{1}{\Delta}R\right)\left(\nabla_\alpha\frac{1}{\Delta}R\right) + \frac{1}{2}hR\frac{1}{\Delta}R. \tag{A19}$$

Equation (A19) gives us directly the first variation of the Polyakov action,

$$-96\pi\delta S_P[g] = \int d^2x\sqrt{g}\left\{hR + \hat{h}_{\mu\nu}\left[2\nabla^\mu\nabla^\nu\frac{1}{\Delta}R + \left(\nabla^\mu\frac{1}{\Delta}R\right)\left(\nabla^\nu\frac{1}{\Delta}R\right)\right]\right\}, \tag{A20}$$

where we have separated the trace part from trace-free part. Defining the tensor

$$t^{\mu\nu} = 2\nabla^\mu\nabla^\nu\frac{1}{\Delta}R + \left(\nabla^\mu\frac{1}{\Delta}R\right)\left(\nabla^\nu\frac{1}{\Delta}R\right), \tag{A21}$$

allows us to write (A20) in the following simple way:

$$\delta S_P[g] = -\frac{1}{96\pi}\int d^2x\sqrt{g}\{hR + \hat{h}_{\mu\nu}t^{\mu\nu}\}. \tag{A22}$$

From this relation we can compute the energy-momentum tensor,

$$\langle T^{\mu\nu}\rangle = \frac{2}{\sqrt{g}}\frac{\delta S_P[g]}{\delta g_{\mu\nu}} = -\frac{1}{48\pi}\left\{g^{\mu\nu}R + t^{\mu\nu} - \frac{1}{2}g^{\mu\nu}t\right\}, \tag{A23}$$

where $t \equiv t^\alpha{}_\alpha = -2R + (\nabla^\mu\frac{1}{\Delta}R)(\nabla_\mu\frac{1}{\Delta}R)$, which shows directly the anomalous trace

$$g_{\mu\nu}\langle T^{\mu\nu}\rangle = -\frac{R}{24\pi}. \tag{A24}$$

Now note that we could have written (A16) as

$$\frac{1}{\sqrt{g}}\delta\left(\sqrt{g}R\frac{1}{\Delta}R\right) = \frac{1}{2}hR\frac{1}{\Delta}R + \delta R\frac{1}{\Delta}R + R\delta\left(\frac{1}{\Delta}R\right);$$

since

$$\frac{1}{2}hR\frac{1}{\Delta}R + \delta R\frac{1}{\Delta}R = \frac{1}{2}hR + \hat{h}_{\mu\nu}\nabla^\mu\nabla^\nu\frac{1}{\Delta}R,$$

it must be that

$$\delta\left(\frac{1}{\Delta}R\right) = \frac{1}{2}h + \frac{1}{R}\hat{h}_{\mu\nu}\left[\nabla^\mu\nabla^\nu\frac{1}{\Delta}R + \left(\nabla^\mu\frac{1}{\Delta}R\right)\left(\nabla^\nu\frac{1}{\Delta}R\right)\right]. \tag{A25}$$

Thus $\sigma(g) = \frac{1}{2\Delta}R$ in strict CG ($\hat{h}_{\mu\nu} = 0$) is such that $\delta\sigma = \frac{1}{4}h$. Note also that in strict CG the first variation (A22) becomes

$$\delta S_P[g] = -\frac{1}{96\pi}\int d^2x\sqrt{g}hR. \tag{A26}$$

We can now perform the second variation of (A26) directly (we leave to the next section the second variation in general gauge). Using the basic variations (A15), the traceless-trace decomposition relations (A5), and $\delta(\sqrt{g}\hat{h}_{\mu\nu}) = \frac{1}{2}\hat{h}^{\mu\nu}\hat{h}_{\mu\nu}g_{\mu\nu}$ we find

$$\delta^2 S_P[g] = -\frac{1}{96\pi}\int d^2x\sqrt{g}\left\{\frac{1}{2}h\Delta h - \frac{1}{2}h^2R + h\nabla^\mu\nabla^\nu\hat{h}_{\mu\nu} + \hat{h}^{\mu\nu}\hat{h}_{\mu\nu}\left(\frac{1}{2}t - R\right) + \hat{h}_{\mu\nu}\delta t^{\mu\nu}\right\}. \tag{A27}$$

We have not written explicitly the variation $\delta t^{\mu\nu}$ since it is not very illuminating, and in any case we will not need the explicit expression for it; we use the fact that $\delta t^{\mu\nu}$ has both traceless and trace parts to write (A27) as

$$\delta^2 S_P[g] = -\frac{1}{96\pi}\frac{1}{2}\int d^2x\sqrt{g}\{h(\Delta - R)h + 2hA^{\mu\nu}\hat{h}_{\mu\nu} + \hat{h}_{\mu\nu}B^{\mu\nu}_{\alpha\beta}\hat{h}^{\alpha\beta}\}, \tag{A28}$$

where $A^{\mu\nu}$ and $B^{\mu\nu}_{\alpha\beta}$ can be read off from (A27) and the knowledge of $\delta t^{\mu\nu}$.

We now consider the variation leading to terms proportional to $H_{\mu\nu} \equiv \delta^2 g_{\mu\nu} = \xi h_{\mu\lambda}h^{\lambda}_{\nu}$,

$$\begin{aligned}\delta\int d^2x\sqrt{g}R\frac{1}{\Delta}R &= 2\int d^2x\sqrt{g}R\frac{1}{\Delta}\delta R + O(R^2)\\ &= 2\int d^2x\sqrt{g}R\frac{1}{\Delta}\left(\Delta H + \nabla^\mu\nabla^\nu H_{\mu\nu} - \frac{1}{2}HR\right) + O(R^2)\\ &= 2\int d^2x\sqrt{g}\left(RH + R\frac{1}{\Delta}\nabla^\mu\nabla^\nu H_{\mu\nu}\right) + O(R^2)\\ &= 2\int d^2x\sqrt{g}H_{\mu\nu}\left(Rg^{\mu\nu} + \nabla^\mu\nabla^\nu\frac{1}{\Delta}R\right) + O(R^2).\end{aligned}$$

The trace part is

$$2\int d^2x\sqrt{g}H_{\mu\nu}\left(Rg^{\mu\nu} + \nabla^\mu\nabla^\nu\frac{1}{\Delta}R\right) = \xi\int d^2x\sqrt{g}\frac{1}{2}h^2R. \tag{A29}$$

Combining this with (A28) finally gives

$$\delta^2 S_P[g] = -\frac{1}{96\pi}\frac{1}{2}\int d^2x\sqrt{g}\{h[\Delta + (\xi - 1)R]h + 2hA^{\mu\nu}\hat{h}_{\mu\nu} + \hat{h}_{\mu\nu}B^{\mu\nu}_{\alpha\beta}\hat{h}^{\alpha\beta}\}, \tag{A30}$$

with new $A^{\mu\nu}$ and $B^{\mu\nu}_{\alpha\beta}$. This completes the collection of variations that we need.

As a final check of (A26) and (A30), we correctly find that in strict CG and for $\xi = 1$ the following relation between variations of the Polyakov and Liouville actions is fulfilled:

$$\delta S_P[g] + \frac{1}{2}\delta^2 S_P[g] = -\frac{1}{24\pi}\int d^2x\sqrt{g}\left\{\frac{h}{4}R + \frac{h}{4}\Delta\frac{h}{4}\right\} = -\frac{1}{24\pi}S_L\left[\frac{h}{4};g\right]. \tag{A31}$$

This relation strongly supports the use of the exponential parametrization.

Finally, for completeness we report the variation of $\delta t^{\mu\nu}$ for which we will need the following relations:

$$\begin{aligned}\delta\left(\nabla_\mu\nabla_\nu\frac{1}{\Delta}R\right) &= \delta\left(\nabla_\mu\partial_\nu\frac{1}{\Delta}R\right)\\ &= \delta\left(\partial_\mu\partial_\nu\frac{1}{\Delta}R + \Gamma^\lambda_{\mu\nu}\partial_\lambda\frac{1}{\Delta}R\right)\\ &= \partial_\mu\partial_\nu\delta\left(\frac{1}{\Delta}R\right) + \delta\Gamma^\lambda_{\mu\nu}\partial_\lambda\left(\frac{1}{\Delta}R\right) + \Gamma^\lambda_{\mu\nu}\partial_\lambda\delta\left(\frac{1}{\Delta}R\right)\\ &= \nabla_\mu\nabla_\nu\delta\left(\frac{1}{\Delta}R\right) + \frac{1}{2}(\nabla_\mu h^\lambda_\nu + \nabla_\nu h^\lambda_\mu - \nabla^\lambda h_{\mu\nu})\nabla_\lambda\left(\frac{1}{\Delta}R\right)\end{aligned} \tag{A32}$$

and

$$\delta\left(\nabla_\mu\frac{1}{\Delta}R\right) = \partial_\mu\delta\left(\frac{1}{\Delta}R\right) = \nabla_\mu\delta\left(\frac{1}{\Delta}R\right). \tag{A33}$$

Using these relations and (A25) in the definition (A21) gives

$$\begin{aligned}\hat{h}_{\mu\nu}\delta t^{\mu\nu} &= \hat{h}^{\mu\nu}\delta\left(\nabla_\mu\nabla_\nu\frac{1}{\Delta}R\right) + \hat{h}_{\mu\nu}\left(\nabla^\mu\frac{1}{\Delta}R\right)\nabla^\nu\delta\left(\frac{1}{\Delta}R\right)\\ &= \frac{1}{2}\hat{h}^{\mu\nu}\nabla_\mu\nabla_\nu h + \hat{h}^{\mu\nu}\nabla_\mu\nabla_\nu\left\{\frac{1}{R}\hat{h}_{\mu\nu}\left[\nabla^\mu\nabla^\nu\frac{1}{\Delta}R + \left(\nabla^\mu\frac{1}{\Delta}R\right)\left(\nabla^\nu\frac{1}{\Delta}R\right)\right]\right\}\\ &\quad + \hat{h}^{\mu\nu}\left(\nabla_\mu\hat{h}^\lambda_\nu - \frac{1}{2}\nabla_\mu\delta^\lambda_\nu h\right)\nabla_\lambda\left(\frac{1}{\Delta}R\right) + \frac{1}{2}\hat{h}_{\mu\nu}\left(\nabla^\mu\frac{1}{\Delta}R\right)\nabla^\nu h\\ &\quad + \hat{h}_{\mu\nu}\left(\nabla^\mu\frac{1}{\Delta}R\right)\nabla^\nu\left\{\frac{1}{R}\hat{h}_{\mu\nu}\left[\nabla^\mu\nabla^\nu\frac{1}{\Delta}R + \left(\nabla^\mu\frac{1}{\Delta}R\right)\left(\nabla^\nu\frac{1}{\Delta}R\right)\right]\right\},\end{aligned} \tag{A34}$$

from which one can extract the explicit form of the tensors $A^{\mu\nu}$ and $B^{\mu\nu}_{\alpha\beta}$ appearing in (A30), but their form is not very illuminating.


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