Braüner’s article surveys present research on intuitionistic hybrid logic (henceforth IHL). In order to set the stage for this, the first section of the article provides a brief introduction to classical hybrid logic. Two model-theoretic semantics for IHL are described then in the second section, which is the main part of the article. This section mainly relies on the work of the author and his collaborators while some other works on IHL are briefly summarized in the third section. The short final section lists some open problems within the metatheory of IHL. Since the last two sections of the article themselves have the character of review-like summaries, the present review concentrates on the first two sections.

Classical hybrid logic extends propositional modal logic by the addition of a second sort of propositional variables called “nominals”. Nominals a, b, . . . stand for maximal proposition which are true in exactly one single world. Hence, in a model $\mathfrak{M}$ of modal logic, the nominal a is true at world $w$ under assignment $g$ (of worlds to nominals), i. e., $\mathfrak{M}, g, w \models a$, iff $g(a) = w$. This basic framework may be enriched by other modes of expression making use of the newly introduced nominals. All the systems discussed by Braüner have a satisfaction operator $@$ which takes nominals a and formulas $\varphi$ as its operands in order to build up new formulas $@a \varphi$. The semantic clause for formulas of that type is: $\mathfrak{M}, g, w \models @a \varphi$ iff $\mathfrak{M}, g, g(a) \models \varphi$.

In IHL intuitionistic logic is used instead of classical logic in order to reason about possible worlds and their relationships. As is well-known, the framework of Kripke semantics can also be used for interpreting intuitionistic logic. The “possible worlds” used for this sake are then interpreted as “epistemic states” of a reasoner. Hence, when turning to IHL, we have to distinguish two kinds of “possible worlds”: (1) the just mentioned epistemic states and (2) the worlds which are the objects of our modal reasoning. This distinction is central for Braüner’s approach to modal intuitionistic logic and sets it apart from other (“birelational”) approaches which only use a single realm of worlds but two accessibility relations: (1) the epistemic states and (2) the worlds which are the objects of our modal reasoning. This distinction is central for Braüner’s approach to modal intuitionistic logic and sets it apart from other (“birelational”) approaches which only use a single realm of worlds but two accessibility relations: one for the epistemic states and one for the worlds used for the interpretation of the modalities. Braüner defines intuitionistic models for IHL as 6-tupels $(W, \leq, \{D_w\}_{w \in W}, \{\neg w\}_{w \in W}, \{R_w\}_{w \in W}, \{V_w\}_{w \in W})$ in which $W$ is the set of epistemic states and $\leq$ is their accessibility relation, which is, as to be expected for intuitionistic logic, a partial order of $W$. $D_w$ is the set of worlds known in the epistemic state $w$, $R_w$ is that fragment of the entire modal accessibility relation which is known in state $w$, and $V_w$ assigns propositions, i. e., sets of possible worlds, to propositional variables. If $a \sim_w b$, then the worlds $a$ and $b$ are indistinguishable with respect to the epistemic state $w$. Obviously, the relations $\sim_w (w \in W)$ have to be equivalence relations; furthermore, they have to be compatible with the corresponding accessibility relations $R_w$ and with membership in propositions (available at state $w$). Hence we have $dR_wd'$ if it holds true that both $d \sim_w d'$, $e \sim_w e'$ and $dRe$, and we have $d \in V_w(p)$ if it holds true that both $d \sim_w d$ and $d \in V_w(p)$. As is usual in the semantics of intuitionistic logic, knowledge is assumed to “grow monotonously”. Hence it is postulated for a model that $w \leq v$ implies $D_w \subseteq D_v$, $R_w \subseteq R_v$, and $\sim_w \subseteq \sim_v$. Given that apparatus, the semantic clauses for nominals, the modalities, and
the @-operator take on the following form:

1. \( M, g, w, d \models a \iff d \sim_w g(a) \)
2. \( M, g, w, d \models \Box \varphi \iff \text{for all } v \succeq w, \text{ for all } e \in D_v, \) \( dR_v e \) implies \( M, g, v, e \models \varphi \)
3. \( M, g, w, d \models \Diamond \varphi \iff \text{for some } e \in D_w, \) \( dR_w e \) and \( M, g, w, e \models \varphi \)
4. \( M, g, w, d \models @_a \varphi \iff M, g, w, g(a) \models \varphi \)

There is a bijective correspondence between the intuitionistic models of IHL just described and models of intuitionistic predicate logic with identity. With respect to this correspondence, a translation method \( ST \) is specified for converting a formula \( \varphi \) of IHL into a formula \( ST_a(\varphi) \) (a a nominal) of intuitionistic predicate logic such that \( M, g, w, g(a) \models \varphi \iff M^*, g, w \models ST_a(\varphi) \), where \( M^* \) is the model corresponding to \( M \).

Besides the possible world semantics an alternative, “many-valued” semantic for IHL is described. The truth-values of that second semantics are the elements of some Heyting algebra \( T = (\top, \bot, \land, \lor, \Rightarrow) \). A many-valued model (based upon \( T \)) is a triple \( (W, R, V) \), where \( W \) is a set (of worlds), \( R : W \times W \to T \) and \( V \) is an assignment function mapping pairs of worlds and propositional variables to elements of \( T \). Of course, the intuitionistic connectives \( \bot, \land, \lor, \Rightarrow \) are respectively interpreted by the operations \( \bot, \land, \lor, \Rightarrow \) of \( T \). The semantic clauses for nominals, the modalities and the @-operator are:

\[
V(w, a) = \begin{cases} 
\top & \text{if } g(a) = w \\
\bot & \text{else} 
\end{cases}
\]

\[
V(w, \Box \varphi) = \bigcap \{ R(w, v) \Rightarrow V(v, \varphi) \mid v \in W \}
\]

\[
V(w, \Diamond \varphi) = \bigcup \{ R(w, v) \land V(v, \varphi) \mid v \in W \}
\]

\[
V(w, @_a \varphi) = V(g(a), \varphi)
\]

It can be shown that many-valued models are equivalent to restricted intuitionistic models. These are intuitionistic models where (1) \( D \) is constant (i. e., \( D_w = D_{w'} \) for all \( w, w' \in W \)) and (2) \( \sim_w \) is the identity relation for each \( w \in W \).

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