



University of Southern Denmark

## Magnetic S-parameter

Sannino, Francesco

*Published in:*  
Physical Review Letters

*DOI:*  
[10.1103/PhysRevLett.105.232002](https://doi.org/10.1103/PhysRevLett.105.232002)

*Publication date:*  
2010

*Citation for published version (APA):*  
Sannino, F. (2010). Magnetic S-parameter. *Physical Review Letters*, 105, 232002.  
<https://doi.org/10.1103/PhysRevLett.105.232002>

### Terms of use

This work is brought to you by the University of Southern Denmark through the SDU Research Portal. Unless otherwise specified it has been shared according to the terms for self-archiving. If no other license is stated, these terms apply:

- You may download this work for personal use only.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying this open access version

If you believe that this document breaches copyright please contact us providing details and we will investigate your claim. Please direct all enquiries to [puresupport@bib.sdu.dk](mailto:puresupport@bib.sdu.dk)

## Magnetic S-parameter

Francesco SANNINO<sup>♥\*</sup>

<sup>♥</sup> CP<sup>3</sup>-Origins, Campusvej 55, DK-5230 Odense M, Denmark.<sup>†</sup>

We propose a direct test of the existence of gauge duals for nonsupersymmetric asymptotically free gauge theories developing an infrared fixed point by computing the S-parameter in the electric and dual magnetic description. In particular we show that at the lower bound of the conformal window the magnetic S-parameter, i.e. the one determined via the dual magnetic gauge theory, assumes a simple expression in terms of the elementary magnetic degrees of freedom. The results further support our recent conjecture of the existence of a universal lower bound on the S parameter and indicates that it is an ideal operator for *counting* the active physical degrees of freedom within the conformal window. Our results can be directly used to unveil possible four dimensional gauge duals and constitute the first explicit computation of a nonperturbative quantity, in the electric variables, via nonsupersymmetric gauge duality.

One of the most fascinating possibilities is that generic asymptotically free gauge theories have magnetic duals. In fact, in the late nineties, in a series of ground breaking papers Seiberg [1] provided strong support for the existence of a consistent picture of such a duality within a supersymmetric framework. Using such a duality, Seiberg has been able to identify the boundary of the conformal window for supersymmetric QCD as function of the number of flavors and colors.

Arguably the existence of a possible dual of a generic nonsupersymmetric asymptotically free gauge theory able to reproduce its infrared dynamics must match the 't Hooft anomaly conditions [2]. We have exhibited several solutions of these conditions for QCD in [3] and for certain gauge theories with higher dimensional representations in [4]. The novelty with respect to these earlier results [5] are: i) The request that the gauge singlet operators associated to the magnetic baryons should be interpreted as bound states of ordinary baryons [3]; ii) The fact that the asymptotically free condition for the dual theory matches the lower bound on the conformal window obtained using the all orders beta function [6]. These extra constraints help restricting further the number of possible gauge duals without diminishing the exactness of the associate solutions with respect to the 't Hooft anomaly conditions.

In this work we suggest a direct test of the possible existence of gauge duals using the  $VV - AA$  two-point function determined in the conformal window of the underlying gauge theory upon the introduction of a mass term for the fermions and in the limit in which the external momentum vanishes at a nonzero value of the fermion mass [7].

We have shown in [7] that this parameter is exactly calculable, using the electric theory, near the upper limit of the conformal window. The reason is that there the electric theory is in a perturbative regime. We argued that the

results are important for the explorations of the conformal window since they shed light on relevant properties in this region. The results are also directly applicable to unparticle extensions of the standard model (SM) [8, 9]. Near the lower bound of the conformal window we cannot compute analytically in a controllable way this parameter but we expect the magnetic dual to be weakly coupled and hence we can derive a closed form expression for the S there via the gauge dual. We will refer to it as the *magnetic S* parameter ( $S_m$ ).

The oblique [10–13] parameter we use here is the one defined in [14] and adapted for a theory developing an infrared fixed point in [7]. As we have demonstrated in [7], within the conformal window, one must distinguish between two non-commuting limits once we introduce a mass term for the fermions. To be precise we have shown that if we take the fermion mass to zero, at finite external momentum, then the associated S-parameter vanishes, viceversa, if we send the external momentum to zero first the S-parameter never vanishes. We have also argued that this is the limit which smoothly connects to the S-parameter in the chirally broken phase relevant for beyond SM applications. We will therefore concentrate on:

$$\lim_{\frac{q^2}{m^2} \rightarrow 0} S. \quad (1)$$

The *electric S*-parameter ( $S_e$ ) is defined as the one computed using the electrical variables. Of course, if the magnetic and the electric theory are gauge duals of each others then  $S_m = S_e$ . Near the electric (or magnetic) Banks-Zaks [15] infrared fixed point IRFP this parameter can be computed reliably by means of perturbation theory [7]. We found that for an electric  $SU(N)$  gauge theory with  $N_f$  Dirac fermions transforming according to the representation  $r$  of the  $SU(N)$  gauge group, and a sufficiently large number of flavors to be near the upper line of the conformal window, the leading terms in the  $q^2/m^2$  expansion and at the leading perturbative order

<sup>†</sup>CP<sup>3</sup>-Origins: 2010-25

in the gauge coupling constant:

$$\lim_{\frac{q^2}{m^2} \rightarrow 0} S_e = \frac{\sharp}{6\pi} \left[ 1 + \frac{1}{10x} + \frac{1}{70x^2} + \mathcal{O}(x^{-3}) \right], \quad (2)$$

with  $x = \frac{m^2}{q^2}$ . The associated quantum global symmetries of the underlying gauge theory are  $SU_L(N_f) \times SU_R(N_f) \times U_V(1)$  if the fermion representation is complex or  $SU(2N_f)$  if real or pseudoreal. Here  $\sharp = N_D d[r]$  counts the number of doublets times the dimension of the representation  $d[r]$  under which the fermions transform. For example for the fundamental representation  $d[F] = N$ , for an  $SU(N)$  gauge group and  $d[S] = N(N+1)/2$  for the two-index symmetric representation of the gauge group. Note that given that we are in the conformal window the mass to the fermions is given via the standard Higgs mechanism.

Consider the case of an underlying gauge group  $SU(3)$ . The quantum flavor group of the massless theory is:

$$SU_L(N_f) \times SU_R(N_f) \times U_V(1). \quad (3)$$

The classical  $U_A(1)$  symmetry is destroyed at the quantum level by the Adler-Bell-Jackiw anomaly. We indicate with  $Q_{\alpha c}^i$  the two component left spinor where  $\alpha = 1, 2$  is the spin index,  $c = 1, \dots, 3$  is the color index while  $i = 1, \dots, N_f$  represents the flavor.  $\bar{Q}_i^{\alpha c}$  is the two component conjugated right spinor. We summarize the transformation properties in the following table. The global

Fields	$[SU(3)]$	$SU_L(N_f)$	$SU_R(N_f)$	$U_V(1)$
$Q$	$\square$	$\square$	$1$	$1$
$\bar{Q}$	$\bar{\square}$	$1$	$\bar{\square}$	$-1$

TABLE I: Fermion field content of an  $SU(3)$  gauge theory with quantum global symmetry  $SU_L(N_f) \times SU_R(N_f) \times U_V(1)$ .

anomalies are associated to the triangle diagrams featuring at the vertices three  $SU(N_f)$  generators (either all right or all left), or two  $SU(N_f)$  generators (all right or all left) and one  $U_V(1)$  charge. We indicate these anomalies for short with:

$$SU_{L/R}(N_f)^3, \quad SU_{L/R}(N_f)^2 U_V(1). \quad (4)$$

For a vector like theory there are no further global anomalies. The cubic anomaly factor, for fermions in fundamental representations, is 1 for  $Q$  and  $-1$  for  $\bar{Q}$  while the quadratic anomaly factor is 1 for both leading to

$$SU_{L/R}(N_f)^3 \propto \pm 3, \quad SU_{L/R}(N_f)^2 U_V(1) \propto \pm 3. \quad (5)$$

We have computed the  $S$ -parameter in the perturbative regime of the conformal window, however we

would like now to determine this parameter near the lower bound of the conformal window. Here perturbation theory fails, in the electric variables, and one has to resort to other methods. However, if a *magnetic* gauge dual exists one expects it to be weakly coupled near the critical number of flavors below which one breaks large distance conformality in the electric variables. We can then determine  $S$  near the lower boundary of the conformal window using perturbation theory in the magnetic variables. Determining a possible unique dual theory for QCD is, however, not simple given the few mathematical constraints at our disposal. The saturation of the global anomalies is an important tool but is not able to select out a unique solution. The goal is to find the explicit expression for  $S_m$  in terms of the magnetic variables by means of the most general expectation for the structure of the gauge dual.

As argued in [3–5] a candidate gauge dual theory within the conformal window, saturating the 't Hooft anomaly conditions, would be constituted by an  $SU(X)$  gauge group with global symmetry group  $SU_L(N_f) \times SU_R(N_f) \times U_V(1)$  featuring *magnetic* quarks  $q$  and  $\bar{q}$  together with  $SU(X)$  gauge singlet fermions identifiable as baryons built out of the *electric* quarks  $Q$ . Since mesons do not affect directly global anomaly matching conditions we can add them to the spectrum of the dual theory. In particular they are needed to let the magnetic quarks and the gauge singlet fermions interact with each others. The new mesons will be massless and have no-self potential to respect the conformal invariance of the model at large distances. We add to the *magnetic* quarks gauge singlet Weyl fermions which can be identified with the baryons of QCD but are, in fact, massless. The generic dual spectrum is summarized in table II. The wave functions for the gauge singlet fields  $A$ ,  $C$  and  $S$  are obtained by projecting the flavor indices of the following operator

$$\epsilon^{c_1 c_2 c_3} Q_{c_1}^{i_1} Q_{c_2}^{i_2} Q_{c_3}^{i_3}, \quad (6)$$

over the three irreducible representations of  $SU_L(N_f)$  as indicated in the table II. These states are all singlets under the  $SU_R(N_f)$  flavor group. Similarly one can construct the only right-transforming baryons  $\bar{A}$ ,  $\bar{C}$  and  $\bar{S}$  via  $\bar{Q}$ . The  $B$  states are made by two  $Q$  fields and one right  $\bar{Q}$  field while the  $D$  fields are made by one  $Q$  and two  $\bar{Q}$  fermions.  $y$  is the, yet to be determined, baryon charge of the *magnetic* quarks while the baryon charge of composite states is fixed in units of the QCD quark one. The  $\ell$ s count the number of times the same baryonic matter representation appears as part of the spectrum of the theory. Invariance under parity and charge conjugation of the underlying theory requires  $\ell_J = \ell_{\bar{J}}$  with  $J = A, S, \dots, C$  and  $\ell_B = -\ell_D$ .

The simplest mesonic operator is  $M_i^j$  and transforms simultaneously according to the antifundamental representation of  $SU_L(N_f)$  and the fundamental represen-

Fields	[SU(X)]	SU <sub>L</sub> (N <sub>f</sub> )	SU <sub>R</sub> (N <sub>f</sub> )	U <sub>V</sub> (1)	# of copies
$q$	$\square$	$\square$	$1$	$y$	$1$
$\bar{q}$	$\bar{\square}$	$1$	$\bar{\square}$	$-y$	$1$
$A$	$1$	$\begin{array}{c} \square \\ \square \\ \square \end{array}$	$1$	$3$	$\ell_A$
$S$	$1$	$\begin{array}{c} \square \\ \square \\ \square \end{array}$	$1$	$3$	$\ell_S$
$C$	$1$	$\begin{array}{c} \square \\ \square \end{array}$	$1$	$3$	$\ell_C$
$B_A$	$1$	$\begin{array}{c} \square \\ \square \end{array}$	$\square$	$3$	$\ell_{B_A}$
$B_S$	$1$	$\begin{array}{c} \square \\ \square \end{array}$	$\square$	$3$	$\ell_{B_S}$
$D_A$	$1$	$\square$	$\begin{array}{c} \square \\ \square \end{array}$	$3$	$\ell_{D_A}$
$D_S$	$1$	$\square$	$\begin{array}{c} \square \\ \square \end{array}$	$3$	$\ell_{D_S}$
$\bar{A}$	$1$	$1$	$\begin{array}{c} \square \\ \square \\ \square \end{array}$	$-3$	$\ell_{\bar{A}}$
$\bar{S}$	$1$	$1$	$\begin{array}{c} \square \\ \square \\ \square \end{array}$	$-3$	$\ell_{\bar{S}}$
$\bar{C}$	$1$	$1$	$\begin{array}{c} \square \\ \square \end{array}$	$-3$	$\ell_{\bar{C}}$
$M_j^i$	$1$	$\square$	$\square$	$0$	$1$

TABLE II: Massless spectrum of *magnetic* quarks and baryons and their transformation properties under the global symmetry group. The last column represents the multiplicity of each state and each state is a Weyl fermion.

tation of  $SU_R(N_f)$ . These states are not constrained by anomaly matching conditions and they mediate the interactions between the magnetic quarks and the gauge singlet fermions via Yukawa-type interactions.

To probe the chiral properties of the theory requires adding a mass term for the fermions. Near the lower end of the conformal window the dual theory is expected to be weakly coupled yielding the following expression for the magnetic  $S$ -parameter:

$$S_m = S_q + S_B + S_M, \quad (7)$$

with

$$S_q = \frac{N_D}{6\pi} X. \quad (8)$$

We will, however, consider here the case in which we gauge, with respect to the electroweak interactions, only the  $SU_L(2) \times SU_R(2)$  subgroup where the hypercharge is the diagonal generator of  $SU(2)_R$ . In this case only one doublet contributes directly to the  $S$  parameter, i.e. we can set  $N_D = 1$ . This parameter is still sensitive to the whole dynamics. The spectrum of the magnetic quarks, baryons and mesons naturally splits into representations of  $SU_L(2) \times SU_L(N_f - 2) \times SU_R(2) \times SU_R(N_f - 2) \times U_V(1)$ . The magnetic quark  $q$ , with respect to this group, transforms according to:

$$q \rightarrow \left[ (\square, 1, 1, 1)_y \oplus (1, \square, 1, 1)_y \right]. \quad (9)$$

The baryons have the following decomposition under

$SU_L(2) \times SU_L(N_f - 2) \times SU_R(2) \times SU_R(N_f - 2) \times U_V(1)$ :

$$\begin{aligned} A &\rightarrow \left[ (1, \square, 1, 1)_3 \oplus (\square, \square, 1, 1)_3 \oplus (1, \square, 1, 1)_3 \right], \\ S &\rightarrow \left[ (\square, \square, 1, 1)_3 \oplus (\square, \square, 1, 1)_3 \oplus (\square, \square, 1, 1)_3 \oplus \right. \\ &\quad \left. \oplus (1, \square, 1, 1)_3 \right] \\ C &\rightarrow \left[ (\square, 1, 1, 1)_3 \oplus (1, \square, 1, 1)_3 \oplus (\square, \square, 1, 1)_3 \oplus \right. \\ &\quad \left. \oplus (\square, \square, 1, 1)_3 \oplus (1, \square, 1, 1)_3 \oplus (\square, \square, 1, 1)_3 \right] \\ B_A &\rightarrow \left[ (1, 1, \square, 1)_3 \oplus (1, 1, 1, \square)_3 \oplus (\square, \square, 1, 1)_3 \oplus \right. \\ &\quad \left. \oplus (\square, \square, 1, \square)_3 \oplus (1, \square, 1, \square)_3 \oplus (1, \square, 1, \square)_3 \right] \\ B_S &\rightarrow \left[ (\square, 1, \square, 1)_3 \oplus (\square, 1, 1, \square)_3 \oplus (\square, \square, 1, 1)_3 \oplus \right. \\ &\quad \left. \oplus (\square, \square, 1, \square)_3 \oplus (1, \square, 1, \square)_3 \oplus (1, \square, 1, \square)_3 \right]. \end{aligned} \quad (10)$$

The decomposition of the charged conjugated baryons is obtained from the one above by exchanging left with right.

Since we are gauging with respect to the electroweak theory the first two flavors we provide a mass term to them as done in [16], i.e. via the introduction of a SM Higgs-type interaction. Since we are operating within the conformal window this is the direct way to provide a mass to the fermions. By symmetry arguments we can pair only the states which do not transform with respect to  $SU_L(N_f - 2) \times SU_R(N_f - 2)$  but still transform nontrivially under  $SU_L(2) \times SU_R(2)$ . These states are  $(\square, \square, 1, 1)_3$  for the baryon  $S$ ;  $(\square, 1, 1, 1)_3$  for  $C$ ;  $(1, 1, \square, 1)_3$  for  $B_A$  and for  $B_S$  the state  $(\square, 1, \square, 1)_3$ . We need to consider the charge conjugated states as well. In terms of the spinorial representations of  $SU_L(2) \otimes SU_R(2)$  the states above are  $\ell_S(\frac{3}{2}, 0)_3 \oplus \ell_C(\frac{1}{2}, 0)_3 \oplus \ell_{B_A}(0, \frac{1}{2})_3 \oplus \ell_{B_S}(1, \frac{1}{2})_3$  with the  $\ell$  prefactor taking into account the multiplicity of each state. They will pair with their charged conjugated fermion via the mass term operator of the type  $\psi H \psi$  with  $H$  the standard model Higgs field which transforms according to the  $(\frac{1}{2}, \frac{1}{2})$  representation. Note that we can only pair states with  $j_2 = j_1 \pm \frac{1}{2}$ .

Each pair of conjugated fermions transforming according to  $(j_1, j_2)_\lambda$  under  $SU_L(2) \times SU_R(2) \times U_V(1)$  leads to the following contribution to the  $S_m$  parameter [16]:

$$\begin{aligned} S_b &= \frac{2d_b}{3\pi} \sum_{J'} X_{J,J'} \left[ 2f(m_{J'}^2, m_{J'}^2) + g(m_{J'}^2, m_{J'}^2) \right] + \\ &\quad + \left( \frac{[j^-(j^+ + 1)]^2}{9\pi} \sum_J \frac{2J + 1}{J(J + 1)} \right), \end{aligned} \quad (11)$$

with the index  $b$  indicating the specific baryon and  $d_b$  its degeneracy. We also have  $j^- = |j_1 - j_2|$ ,  $j^+ = j_1 + j_2$  and  $j^- \leq J \leq j^+$  the total spin for each baryon contribution. If more than one spinorial representation belongs to the same baryon  $b$  the contributions of all the states must

be taken into account. The nonvanishing components of the group theoretical factor  $X_{JJ'}$  are:

$$X_{JJ} = \left[ 1 - \left( \frac{j^-(j^+ + 1)}{J(J+1)} \right)^2 \right] \frac{J(J+1)(2J+1)}{12},$$

$$X_{JJ-1} = X_{J-1,J} = \frac{-1}{12} \left( (j^+ + 1)^2 - J^2 \right) (J^2 - j^{-2}). \quad (12)$$

The functions  $f$  and  $g$  read [16]:

$$f(m_J^2, m_{J'}^2) = -6 \int_0^1 dx x(1-x) \log \left( \frac{xm_J^2 + (1-x)m_{J'}^2}{\mu^2} \right),$$

$$g(m_J^2, m_{J'}^2) = 6 \int_0^1 dx \frac{x(1-x)m_J m_{J'}}{xm_J^2 + (1-x)m_{J'}^2}. \quad (13)$$

The mass of each fermion is directly proportional to the electric fermion mass  $m$  and depends on the representation according to the formula  $m_J = -m \frac{J+\frac{1}{2}}{j_1(j_1+\frac{1}{2})}$ . We have chosen as a reference energy scale  $\mu = m$ . The contribution of the baryon sector is then:

$$S_B = \sum_b S_b. \quad (14)$$

The complex scalar meson  $M$  decomposes as:

$$M \rightarrow [(\square, 1, \bar{\square}, 1) \oplus (\square, 1, 1, \bar{\square}) \oplus (1, \square, \bar{\square}, 1) \oplus (1, \square, 1, \bar{\square})]. \quad (15)$$

Only the first state,  $(\frac{1}{2}, \frac{1}{2})$ , contributes to  $S_M$  and leads to:

$$S_M = \frac{1}{3\pi} \sum_{JJ'} f(m_J^2, m_{J'}^2). \quad (16)$$

with  $J, J' = 1, 0$ ,  $m_J^2 = m_0^2(1 + J(J+1))$ . This is a different mass parameterization than the one given in [16]. We also have  $m_0^2 \propto m^2$ . All factors of order unity have been set to unity and finally set the scale  $\mu = m_0$  in the function  $f$  for the scalars. The contribution to  $S_M$  vanishes unless there is a mass splitting between the different multiplets of the unbroken  $SU(2)_V$  symmetry.

Putting together the various terms we have for the normalized  $S_m$ :

$$\frac{6\pi}{3} S_m = \frac{X}{3} + \frac{\ell_C + \ell_{B_A}}{3} + \frac{25}{729} \ell_{B_s} (32 \log 2 - 39) - 0.14. \quad (17)$$

The explicit dependence on the quark masses disappear for the  $S_m$  parameter in agreement with the expectation from the leading contribution in  $q^2/m^2$  to the  $S_e$  parameter. The above is the general expression for  $S_m$  near the lower end of the conformal window corresponding to the nonperturbative regime in the electric variables. From this expression is evident that the present definition of the normalized  $S$ -parameter counts the relevant

degrees of freedom as function of the number of flavors. We estimate  $S_m$  using the possible dual provided in [3] for which  $X = 2N_f - 15$ ,  $\ell_A = 2$ ,  $\ell_{B_A} = -2$  (we take +2 since we are simply counting the states) with the other  $\ell$ s vanishing. Asymptotic freedom for the magnetic dual requires at least  $N_f = 9$  for which  $6\pi S_m/3 = 1.523$  while if the lower bound of the conformal window occurs for  $N_f = 10$  we obtain  $6\pi S_m/3 = 2.19$ . Of course, only one of these two values should be considered as the actual value of the normalized magnetic  $S$  parameter near the lower end of the electric conformal window. Both values are such that the normalized  $S_m$  is always larger than the electrical one near the upper end of the conformal window and are close to the one for two flavors QCD which is around two [17].

These results support our recent conjecture [7] according to which the normalized  $S$  parameter, obtained in the limit when the external momentum vanishes at a nonzero value of the quark mass, is a nondecreasing function of the number of flavors with respect to the underlying electric theory satisfying a universal lower bound corresponding to unity.

---

\* Electronic address: sannino@cp3.sdu.dk

- [1] N. Seiberg, Nucl. Phys. B **435**, 129 (1995) [arXiv:hep-th/9411149].
- [2] G. 't Hooft, *Recent Developments in Gauge Theories*, Plenum Press, 1980, 135; reprinted in *Unity of Forces in the Universe Vol. II*, A. Zee ed., World Scientific 1982, 1004.
- [3] F. Sannino, Phys. Rev. D **80**, 065011 (2009) [arXiv:0907.1364 [hep-th]].
- [4] F. Sannino, Nucl. Phys. B **830**, 179 (2010) [arXiv:0909.4584 [hep-th]].
- [5] J. Terning, Phys. Rev. Lett. **80**, 2517 (1998) [arXiv:hep-th/9706074].
- [6] T. A. Ryttov and F. Sannino, Phys. Rev. D **78**, 065001 (2008) [arXiv:0711.3745 [hep-th]].
- [7] F. Sannino, arXiv:1006.0207 [hep-lat].
- [8] H. Georgi, Phys. Rev. Lett. **98**, 221601 (2007) [arXiv:hep-ph/0703260].
- [9] F. Sannino and R. Zwicky, Phys. Rev. D **79**, 015016 (2009) [arXiv:0810.2686 [hep-ph]].
- [10] M. E. Peskin and T. Takeuchi, Phys. Rev. Lett. **65**, 964 (1990).
- [11] M. E. Peskin and T. Takeuchi, Phys. Rev. D **46**, 381 (1992).
- [12] D. C. Kennedy and P. Langacker, Phys. Rev. Lett. **65**, 2967 (1990) [Erratum-ibid. **66**, 395 (1991)].
- [13] G. Altarelli and R. Barbieri, Phys. Lett. B **253**, 161 (1991).
- [14] H. J. He, N. Polonsky and S. f. Su, Phys. Rev. D **64**, 053004 (2001) [arXiv:hep-ph/0102144].
- [15] T. Banks and A. Zaks, Nucl. Phys. B **196**, 189 (1982).
- [16] M. J. Dugan and L. Randall, Phys. Lett. B **264**, 154 (1991).
- [17] P. A. Boyle, L. Del Debbio, J. Wennekers and J. M. Zanotti [RBC Collaborations and UKQCD Collaborations], Phys. Rev. D **81**, 014504 (2010) [arXiv:0909.4931 [hep-lat]].