Dynamic Pricing and Inventory Control with Delivery Flexibility

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Received: date / Accepted: date

Abstract We study a multi-period inventory system with price-sensitive demand and uncertain supplier, focusing on the advantage of delivery flexibility. The optimal pricing and inventory replenishment decisions are explored. We also investigate the changes of marginal profit, optimal order quantities and optimal prices over time horizon with additive demand noise. By comparing our system with delivery flexibility with the other two traditional systems numerically in different scenarios, we show that the delivery flexibility can improve the total profit and mitigate the supply risk.

Keywords Inventory Control · Dynamic Pricing · Delivery Flexibility

1 Introduction

Delivery flexibility has become an increasing important issue in supply chain management, especially for online sales. It is usually acceptable for customers

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to wait a short period between the time of placing an order and the time of shipping the order, which gives retailer a flexibility in determining when to ship the order. Such a flexibility is called *Delivery Flexibility* in this paper.

*Delivery flexibility* has been observed not only in in-store sales but also in online sales. The online shoppers usually do not care too much about whether their orders are shipped immediately or delayed for a short period. For instance, many online retailers only promise shipping the order within a few working days. Most shoppers feel waiting for this period is totally acceptable. Therefore, most online retailers usually do not reveal information about the current inventory level so that they will not lose customers because of the temporary stock out of the commodity.

For example, after placing an order at www.acehardware.com, customers are notified that the order will be shipped as soon as it is ready within 1 to 7 working days. By continuing placing the order, a customer agrees with this type of shipping term, which gives acehardware.com a type of *delivery flexibility*. If the local warehouse of acehardware.com that is close to the customer’s address has enough on hand inventory, it can ship the order immediately to save the holding cost by reducing the inventory level. Otherwise, if the local warehouse run out of its inventory because a large amount of demand depletes the inventory suddenly, the 7-working-day period allows acehardware.com to ship the ordered items from another warehouse in another county or state, or even order it from its supplier. In doing so, the acehardware.com reduces the chance of inducing penalty cost from lost demand.

Intuitively, this delivery flexibility helps the retailer in two ways. When demand is greater than inventory, the flexibility can reduce the penalty cost from not being able to meet demand immediately, because the retailer can defer shipping for a period to use incoming ordered commodities to meet demand. This is exactly why some retailers are still taking order from customers even when the listed items are out of stock. In the other case, shipping an order may require some time for preparation, which induces holding costs during this period. The delivery flexibility means that the retailer has the ability
to prepare the order and ship it immediately. This can reduce the holding cost compared with the situation when a period of preparation is required. In practice, many retailers have been improving their ability to ship the order immediately so as to reduce holding cost. In this paper, we study the optimal pricing and inventory replenishment decisions under this delivery flexibility as well as the benefit from this flexibility.

Besides this delivery flexibility, we also consider uncertainties from both supplier and demand. The supplier uncertainty is characterized by production capacity. The demand uncertainty is characterized by both multiplicative and additive uncertainties in demand function. In the existing literature, it has been proved that both optimal order quantity and optimal price are decreasing as inventory increases when the retailer faces all-or-nothing supplier and demand uncertainties[8,10]. It also has been shown that the optimal order quantity and optimal order price may not be monotonic in the inventory level when both supplier yield uncertainty and multiplicative demand noise are involved (see for example [32,8,7]). Being different from the existing literature, the supplier’s risk is captured by capacity uncertainty instead of yield uncertainty in our paper. The all-or-nothing supplier is a special case of both capacity uncertainty and yield uncertainty. Feng and Shi [23] study a model with both constrained supply capacity and general demand noise. When supplier uncertainty is eliminated, they verified the monotonicity of optimal price and optimal order quantity. The optimal policy is a threshold policy when the retailer faces multiple general random capacity suppliers with demand decision. The relative inventory models can also be found at Katehakis, et al. [29] and Shi, et al. [35].

In contrast to the existing literature, we consider an inventory model with delivery flexibility and one uncertain capacity-limited supplier. Our result on optimal price is consistent with the findings in the existing literature which shows that optimal price is monotonic in inventory level under some conditions. However, unlike these findings in the literature, where the optimal price and optimal order quantity are either both monotonic or non-monotonic in
inventory level, our results show that the optimal order quantity could be non-monotonic while the optimal price is monotonic in inventory level, which is a phenomenon that has not been discussed in the existing literature. When the retailer faces a capacity-constrained supplier and general demand noises, the optimal average demand is increasing in the inventory level, which implies the optimal average price is decreasing in the inventory level. Moreover, in general, optimal order quantity follows a threshold policy; and the change of the optimal order quantity is smaller than the change of inventory level, which is due to the trade-off between the demand uncertainty and the supplier uncertainty. On the one hand, the retailer has an intention to order less to mitigate the supplier uncertainty. On the other hand, she also has an incentive to keep a high inventory level so that no demand is missed to increase profit. The interaction of these two factors contributes to the characteristics of optimal decisions in our system, namely the optimal average demand is increasing in inventory level and the change of optimal order quantity is smaller than the change of the inventory level.

In a special case when demand uncertainty is additive, we also explore the change of optimal order quantity, optimal average demand, as well as marginal profits of the current inventory over time horizon. Federgruen and Zipkin[21] showed that the marginal profits and optimal order quantity are decreasing with time when the inventory system has one fixed capacity limited supplier. We extend their results to the inventory system with one uncertain supplier and price-dependent demand to show that both the marginal profits and the optimal order quantity are decreasing and the average demand is increasing with time.

The major contribution of the paper is introducing the delivery flexibility as well as exploring its influence on the optimal policies and profits. To study the benefit of this delivery flexibility, we compare it with two systems. One is called Immediate Shipping inventory system and the other is called One-period Shipping&Handling system. Our numerical studies show that the delivery flexibility increases the profit in our inventory system by 10%-40% compared with
these two systems. The impact of holding cost, unit cost, uncertainties on the benefits from the delivery flexibility are also investigated by numerical studies in the paper.

The remainder of the paper is organized as follows. In § 2, we review the relevant literature. In § 3, we present our multi-period inventory model with delivery flexibility under supplier and demand uncertainties. In § 4, the structure of the optimal policy is studied and discussed. In § 5, the change of the optimal policy across time horizon is investigated for a special case with additive demand uncertainty. In § 6, we study the benefits of delivery flexibility in different scenarios. § 7 concludes the paper and discusses the limitation of the study.

2 Literature Review

Our paper relates to a few streams of research. The first stream studies inventory problem with uncertain supplier. When the price is exogenous, the optimal order policy is a reorder-point policy (see for example [2,15,20,9]). There are three types of supplier uncertainty. The first well-known supplier uncertainty is all-or-nothing such as [2,10]. In more general case, one way to incorporate the supplier uncertainty into the model is to use a random yield rate [27,8,7], i.e. the actual arrival amount is the multiplication of a random percentage and the order quantity. Another way to model supplier uncertainty is to use random capacity [14,44,22], i.e. the actual arrival amount is the minimum of the order quantity and the random production capacity. All-or-nothing is a special case of both random yield uncertainty and random capacity uncertainty. In our paper, we discuss the supplier with random capacity uncertainty. All of our results can be applied to all-or-nothing case.

The second stream of research related to our paper studies price dependent demand. Among these studies, the models closely related to our research are finite periodic-review inventory models, where a retailer uses dynamic pricing to either intensify or reduce stochastic demand in response to current and
future supply availability. Federgruen and Heching [19] demonstrated how to characterize and compute simultaneous pricing and inventory policies. Chen and Simchi-Levi [11] extended the work of Federgruen and Heching to model an additional fixed cost component of ordering costs as well as more general demand processes. More recent extensions to this vein of research include modeling of substitute products [18], incorporating lost sales as opposed to backlogging [40], and analyzing demand learning with finite capacity [3]. The demand model we adopt in this paper is a linear function of price which includes both additive and multiplicative stochastic terms. This model is used in both [1] and [11]. For the additive type of demand uncertainty, Chen and Simchi-Levi [11] proved the optimality of (s, S) policies, but they also showed that those results do not hold for more general demand functions, i.e. multiplicative plus additive. Feng [22] proved the monotonicity of the optimal order quantity and the optimal price for the inventory system with additive demand uncertainty and the capacity-constrained supplier. In other similar studies, optimal prices are shown to be decreasing in inventory level, which includes [12, 13, 30, 34]. Alternative modeling considerations regarding the interplay of demand, price, and inventory include inventory-dependent demand, e.g., [38, 16] as well as perishable inventory, e.g., [6, 24, 47, 33, 31].

The third stream of research related to our paper considers a system with one-period lead time to study optimal pricing and inventory replenishing decisions. The one period lead time has been used widely to simplify the inventory system in the existing literature, e.g., [28, 4, 25, 41, 43, 17]. To simplify the model, we also consider one period delivery flexibility in this paper. However, our study is different from the models studied in the literature reviewed above in that we consider both supplier capacity uncertainty and the general demand noise in an inventory system with delivery flexibility. Wang and Yan [42] studied a multi-period inventory model in which a supplier provides two alternative lead-time choices to customers, either a short or a long lead time. We adopt their modeling technology and consider one period delivery flexibility on the retailer side. The retailer can either ship the order immediately or defer the
shipping to the end of the period when the ordered products arrive, depending on whether there is enough inventory to satisfy the demand.

The last stream of research that is also relevant is on the structure of optimal policy in periodic review inventory system, which has been extensively studied in the last decade. For example, both Federgruen and Heching[19] showed that the optimal order quantity is decreasing in inventory level when demand uncertainty is of general form. Li and Zheng[32] showed that both optimal order quantity and optimal price are decreasing in inventory level when supply uncertainty is modeled by production ratio and demand uncertainty is additive. Chen etc.[8] provided a summary of periodic review inventory models with demand uncertainty and supply uncertainty. They showed that the order quantity is monotonic in inventory when only one multiplicative uncertainty is involved in profit function in each period. However, when more than one multiplicative uncertainties are involved, they illustrated by using examples that monotonicity of order quantity may no longer hold. More recently, Chen and Tan[10] proved that the optimal order quantity is still monotonic in a more general case where more than one suppliers with capacity uncertainties are involved. But, they didn’t consider the pricing decision. Being different from these studies, we consider a case involving one supplier with capacity uncertainty and multiplicative demand. In the literature, monotonic price and order quantity are always present together. In contrast, we find that the optimal price is monotonic in inventory but the optimal order quantity may not be monotonic. In a special case when demand uncertainty is only additive, both optimal price and order quantity become monotonic. The result in this special case is consistent with the findings in Chen etc.[8].

To our best knowledge, the delivery flexibility in periodic-preview inventory system with ordering and pricing decisions has not been studied in the literature, making our paper a pioneer in this direction.
3 The Basic Model

We consider a retailer who sells a single product over a finite time horizon. The retailer aims to maximize the total expected discounted profit by adjusting the selling price $p_t$ and the order quantity $q_t$ in each period. The periods are indexed forward, namely the first period is indexed by 1 and the last period is indexed by $T$ ($T \geq 2$) for a finite time horizon. To improve the efficiency of the inventory management, the retailer would like to satisfy the demand immediately if they have enough products on hand. Specifically, when demand arrives, partial demand may be satisfied immediately by the current available inventory. The remaining amount of unmet demand can be satisfied by the incoming order quantity at the end of the period. For the retailer, there is no additional cost induced by postponing the delivery of products to the end of the period. For tractability, we assume that excess demand is fully backlogged at a specified per unit per period penalty cost. The model resembles scenarios where more and more retailers begin to use a combination of technology and instant customer service to manage the current inventory, order quantity, and demand.

In each period, the retailer can order from its upstream supplier. The order will be realized at the end of the period subject to the production capacity of the upstream. Specifically, if an order of $q_t$ is placed, the actual delivered amount at the end of the period is $\min(q_t, k_t)$, where $k_t$ is the realization of the supplier's random production capacity. The probability distribution function of the suppliers' random production capacity is $G(\cdot)$. Both the price $p_t$ and the order quantity $q_t$ must be determined before the actual demand and the supply uncertainties are observed. The manager pays $c$ dollars for each delivered unit. In each period, there are also inventory holding/backorder costs. Holding each item incurs a holding cost $h$ and backlogging one item incurs a backlog penalty $p$. We refer to $h$ as unit holding cost and $p$ as unit shortage cost.

Following [11], we consider two types of demand uncertainty, i.e., additive and multiplicative. Specifically, we assume that the demand in each period $t$
is stochastic, denoted by $D_t$, which can be represented as

$$D_t = \epsilon_t d_t + \omega_t$$

where, $\epsilon_t$ is a random variable representing multiplicative demand noise with $E(\epsilon_t) = 1$ and $\epsilon_t \in [\epsilon_l, \epsilon_r] \subset [0, \frac{3}{2}]$; and $\omega_t$ is a random variable representing the additive demand noise with $E(\omega_t) = 0$, which is independent of $\epsilon_t$. We also assume that random perturbations $\{\epsilon_t, \omega_t\}$ are independent across time. By using this expression, two types of customers will be captured. Price-sensitive customers are captured by the multiplicative uncertainty in the model, whereas non-price-sensitive customers are captured by the additive uncertainty in the model. Such an assumption on demand model is consistent with Smith[36], which pointed out that loyal customers are less sensitive to the price and regular customers are more sensitive to the price. For example, the launch of Target’s Lilly Pulitzer product lines attracted a large amount of customers in 2015 [37], making Target’s website crashed many times. Such a surge in demand is more appropriate to be captured by the multiplicative noise rather than an additive noise in demand model.

The deterministic part of the demand $d_t$ in each period is given by a function $d_t = d(p^*_t)$, which can be interpreted as the mean demand, i.e., $E(D_t) = d(p^*_t)$, under the assumptions about $\epsilon_t$ and $\omega_t$. A special case of this mean demand function is $d(p^r) = a(p^r)^{-b}$ (for $a > 0$, $b > 1$), which is a well-known isoelastic demand function widely adopted in the literature, e.g., [39], [5], and [46].

We also assume that the mean demand function $d(p^r_t)$ has a strict decreasing inverse function $p^r(d_t)$. Under this assumption, $p^*_t$ is uniquely determined by $p^*_t = p^r(d_t)$. Following [32], we treat $d_t$ as our decision variable in this paper. Since the demand cannot be negative, there exists a lower bound for the demand in each period, i.e., $d_t \geq d$ for any $t$. According to the inverse demand function $p^r(d)$, the upper bound for price is given by $\overline{p} = p^r(d)$.

Throughout the paper, we often simplify notation by omitting the argument in demand function to denote demand by $d$. Revenue is expressed as
$R(d) = dp'(d)$. It is assumed to be a strictly increasing, concave, and twice-differentiable function in $d$. We also assume that $\lim_{d \to +\infty} R'(d) = 0$, which implies that the marginal profit is zero when the mean of demand $d$ goes to infinity, i.e., the price goes to zero. To avoid the trivial case, we also assume that $R'(d) \geq c + h$. This assumption says that the margin profit is greater than the sum of purchasing cost and unit holding cost when the highest price is charged, which may be true in reality when marginal revenue is always greater than the total cost of producing and holding a unit product.

The unmet demands are fully backlogged. The sequence of events in each period $t$, $1 \leq t \leq T$, is described below.

1. The inventory level, $I_t$, is reviewed.
2. The retailer decides the order quantity $q_t$ and the average demand $d_t$, i.e. the selling price $p_t^* = p'(d_t)$, based on the current inventory level.
3. The actual demand $D_t = \epsilon_t d(p_t^*) + \omega_t$ arrives. The partial demand $\min\{D_t, I_t\}$ is satisfied immediately.
4. At the end of each period, the order quantity $\min(q_t, k_t)$ arrives where $k_t$ is the realization of the supplier’s random production capacity. When the order quantity arrives, the remaining amount $\max\{D_t - I_t, 0\}$ is satisfied immediately by the incoming order quantity.
5. At the end of the period, the remaining inventory is carried over to the next period and unmet demand is fully backlogged, which induces penalty cost. Hence,

$$I_{t+1} = I_t + q_t \wedge k_t - \epsilon_t d - \omega_t$$

where $q_t \wedge k_t = \min\{q_t, k_t\}$.

Let $V_t(I)$ be the optimal expected profit from period $t$ to the end of the horizon. Without loss of generality, we assume $V_{T+1}(I) = c_s I^+ - c_u I^-$ where $I^+ = \max(I, 0)$ and $I^- = \max(-I, 0)$. This assumption implies that the retailer has salvage value $c_s$ for each item of leftover inventory; she can also obtain items from other places as an emergency delivery at cost $c_u$ per unit for
unmet demand. Naturally, we assume \( c_s < c < c_u \), namely the salvage value is less than the purchasing cost from the supplier, which is also less than the cost from emergency delivery. To simplify the notation, we call the inventory system with delivery flexibility as DF system. Let \( \alpha \in (0, 1) \) be a discount factor. The dynamic programming formulation of our inventory system is written as

\[
\text{DF System: } V_t(I) = \max_{d \geq d^*, q \geq 0} J_t(I, q, d),
\]

where

\[
J_t(I, q, d) = R(d) - cE(q \land k_t) - he[(I - \epsilon_t d - \omega_t)^+] - pE[(I + q \land k_t - \epsilon_t d - \omega_t)^-] \\
+ \alpha E[V_{t+1}(I + q \land k_t - \epsilon_t d - \omega_t)].
\]  
(1)

where \( q \land k_t = \min(q, k_t) \). \( x^+ = \max(x, 0) \) and \( x^- = \max(-x, 0) \). In each period, the retailer starts with a given inventory level \( I_t \). The order quantity \( q \) and the average demand \( d \) are predetermined before the realization of the demand and the supply uncertainties. The expected revenue is denoted \( R(d) \), which is expressed as \( R(d) = dp(d) = E[(\epsilon_t d + \omega_t)p(d)] \). We use \((q_t^*(I), d_t^*(I))\) to denote the corresponding maximizer in (1), which are the optimal pricing and inventory replenishing decisions respectively\(^1\).

4 The Structure of The Optimal Solution

First, we show that the expected optimal profit function \( V_t(I) \) is concave in inventory level \( I \) in Lemma 1, i.e., the marginal profit of the current available inventory is decreasing.

**Lemma 1** \( V_t(I) \) is concave in \( I \).

The concavity of the optimal profit function allows us to further characterize the structure of the optimal policy. To simplify the discussion, we assume that the additive demand noise \( \omega_t \) has a continuous density so that the optimal

\(^1\) we always pick up the smallest, if there are multiple solutions, namely the retailer always prefers the solution with the least ordering unit and the highest selling price.
cost function $V_t$ is differentiable, which is an assumption commonly adopted in the literature, e.g., [25, 2, 14, 15]. The objective function can be rewritten as

$$J_t(I, q, d) = R(d) - cE(q \land k_t) + E[L(I - \epsilon_t d)] + E[L(I + q \land k_t - \epsilon_t d)]$$

$$+ \alpha E[V_{t+1}(I + q \land k_t - \epsilon_t d - \omega_t)].$$

where

$$L(x) = -hE[(x - \omega_t)^+] \text{ and } \hat{L}(x) = -pE[(x - \omega_t)^-].$$

From the continuity of $\omega_t$ and the structure of functions, we have $L$, $\hat{L}$, and $V_{t+1}$ are all concave and differentiable. The structure of the optimal policy are described in the following theorem.

**Theorem 1** The following statements are true.

(i) $d_t^*(I)$ increases in $I$.

(ii) $I + q_t^*(I)$ increases in $I$.

(iii) If $\epsilon_t \equiv 1$, then $q_t^*(I)$ decreases in $I$ and $I - d_t^*(I)$ increases in $I$.

Theorem 1(i) shows that the optimal average demand is increasing in inventory, hence the optimal price is decreasing. In the literature, e.g., [32, 8], it has been shown that neither average demand nor the order quantity is monotonic in inventory level when the system involves both multiplicative demand uncertainty and supplier uncertainty. Being different from these studies, the yield uncertainty is captured by random capacity instead of random yield in our paper. Feng [22] shows the monotonicity of optimal price when an inventory system with additive demand uncertainty and supplier’s capacity uncertainty is considered. Theorem 1(i) extends Feng [22]’s results into the inventory system with multiplicative demand uncertainty.

Theorem 1(ii) shows that, when inventory level decreases, the increase in optimal order quantity is greater than the decrease in inventory level. Intuitively, at a lower inventory level, the retailer prefers to order more from the supplier. The extra ordered quantity induced by a lower inventory level comprises two parts. One part is used to compensate the decrease in the inventory
level. The other part is used to hedge a higher supplier’s risk from the larger order quantity. In total, the increased order quantity outweighs the decrease in the inventory level. When the retailer carries more inventory, the decrease in the optimal order quantity should be less than the increase in inventory, because in this scenario the retailer still wants to keep some order to buffer the impact of supplier uncertainty.

Theorem 1(iii) further discusses results with additive demand uncertainty. Those results are consistent with the finding in [22]. Theorem 1(iii) extends them to the inventory system with delivery flexibility. It is interesting to notice that in the previous literature, the monotonicity of optimal price and optimal order quantity are always present together, either both monotonic or both non-monotonic, e.g., [22, 32, 8, 22]. Theorem 1 shows that our previous understanding cannot be extended to an inventory system with a random capacity-constrained supplier and an uncertain demand. The optimal price is decreasing as the inventory increases in the general demand uncertainty case (multiplicative and additive). However, for the optimal order quantity, it can only be shown that the optimal order quantity is decreasing in the inventory level when the retailer faces additive demand uncertainty. Further, when demand uncertainty is additive, the increase in optimal average demand is less than the increase in the inventory level. This is probably due to the fact that the retailer want to keep some space to hedge the demand uncertainty.

The next theorem shows that the optimal order quantity is a threshold policy.

**Theorem 2** There exists $I_{q,t}$ satisfying that

(i) $q_t^* > 0$ for $I < I_{q,t}$
(ii) $q_t^* = 0$ for $I \geq I_{q,t}$.

**Remark 1** From Theorems 1 and 2, we can conclude that there are two thresholds $I_{d,t}$ and $I_{q,t}$ such that: (1) $d_t^*(I) > d$ if $I > I_{d,t}$; and $d_t^*(I) = d$ if $I \leq I_{d,t}$, (2) $q_t^*(I) = 0$ if $I \geq I_{q,t}$; and $q_t^*(I) > 0$ if $I < I_{q,t}$. 
We cannot obtain the monotonicity of the optimal order quantity in the general case in Theorem 1. However, we can show that the optimal policy is a threshold policy (Theorem 2 and Remark 1). Whether to place an order is an important decision for the retailer. Theorem 2 above gives a guideline to such a decision in the inventory system with delivery flexibility, supply capacity uncertainty, and general demand uncertainty. If it is optimal for the retailer to place a positive order at a certain inventory level, it is also optimal for the retailer to place a positive order for all inventory levels that are lower than this level. If it is optimal for the retailer not to place an order at a certain inventory level, it is also optimal for the retailer not to place any order at an inventory level that are higher than this level.

**Remark 2** In view of the proof of Theorem 2, we can also show that the threshold of whether to place a positive order under random capacity assumption is equal to the threshold of this decision in an inventory system without supplier uncertainty in the current period.

It is hard for the retailer to make the decision by considering uncertainty from both supply side and demand side. Remark 2 provides a guideline to the retailer. When the retailer needs to decide whether to place an order, he can assume the supplier is reliable, i.e. without capacity uncertainty $k_t \equiv +\infty$. If facing a reliable supplier, the retailer needs to place an order, then she also need to place an order when facing an unreliable supplier, i.e., with general capacity uncertainty. If facing a reliable supplier, the retailer needs not to place an order, then she does not need to place an order from an unreliable supplier either. Combining Remarks 1 and 2, the retailer places a positive order from an unreliable supplier if and only if the inventory level is less than the threshold at which she needs to place an order from a reliable supplier.

Another interesting question is how the optimal policy changes over time at the same inventory level. In § 5, we provide theoretical analysis of the change of marginal profit, optimal order quantity, and optimal average demand over time when demand uncertainty is only additive. We show that the marginal
profit is decreasing in time; the optimal order quantity is also decreasing in
time; and the optimal average demand is increasing in time.

5 Additive Demand Noise

In the section, we discuss some results under the assumption of additive de-
mand noise. In this case, the retailer still faces the supplier uncertainty (the
random capacity risk) as well as the demand uncertainty (the additive noise).
It is different from the general case, where changes of average demand can
influence the variance of the inventory level through multiplicative demand
uncertainty. In contrast, when there is only additive demand uncertainty, the
change of average demand only affects the mean but not variance of the ending
inventory level in each period.

In the following discussion, we first explore the structure of the optimal
policy under the additive demand uncertainty. Then, we construct an uncon-
strained profit optimization problem where both order quantity \( q_t \) and average
demand \( d_t \) are unbounded. The relation between the optimal decisions of this
unconstrained problem and the original problem is established and discussed.
With such a relation, we can show how the marginal profit, optimal order
quantity, and optimal average demand change over finite time horizon at some
fixed inventory level.

**Theorem 3** If \( \epsilon_t \equiv 1 \), then \( I_{d,t} < I_{q,t} \) and
\[
I + q_t^*(I) - d_t^*(I) = I_{q,t} - d_t^*(I_{q,t}), \quad \forall \ I \leq I_{q,t}.
\]

In previous study [45], the base stock policy is suboptimal in inventory
system with supplier uncertainty. The reorder point policy has also been dis-
cussed under different supplier uncertainty environments, e.g., [19,32]. The
results obtained in the Theorems 1 and 2 imply that the retailer will charge
a unit price lower than the highest price if and only if the inventory level is
higher than \( I_{d,t} \) and the retailer will place a positive order if and only if the
inventory level is less than \( I_{q,t} \).
Theorem 3 shows that there exists a non-empty interval \([I_{d,t}, I_{q,t}]\), in which the retailer will reach the optimality by making decisions on pricing and order quantity jointly in each period. Further, Theorem 3 shows that the retailer would keep a base as \(I_{q,t} - d_t^*(I_{q,t})\) by coordinating the order quantity and the mean demand, which reveals that the optimal policy is not only a reorder point policy but also a base stock policy when the demand uncertainty is additive. Specifically, the mean of the optimal demand increases in inventory level; and the optimal order quantity decreases in inventory level. But, the sum of the current inventory and the difference between the optimal order quantity and the optimal mean demand is a constant \(I_{q,t} - d_t^*(I_{q,t})\), when the inventory level is less than the threshold, i.e. \(I \leq I_{q,t}\).

This type of base stock policy may be very interesting especially in situations when the inventory \(I\) is very small or negative. As \(I_{d,t} < I_{q,t}\) and \(d_t^*(I) = d\) when \(I \leq I_{d,t}\), Theorem 3 implies that \(q_t^*(I) = I_{q,t} - d_t^*(I_{q,t}) + d - I\) when \(I \leq I_{d,t}\). Thus, when inventory is very low in some period, i.e. \(I \leq I_{d,t}\), the reorder decision in that period reduces to a simply decision by subtracting the current inventory level \(I\) form a constant \(I_{q,t} - d_t^*(I_{q,t}) + d\).

**Lemma 2** When \(\epsilon_t \equiv 1\),

(i) \(V_t'(I) \geq R'(d)\) if \(I \leq I_{d,t}\).
(ii) \(V_t'(I) = R'(d_t^*(I))\) if \(I \geq I_{d,t}\).

Lemma 2 explains how marginal profit changes with inventory. Li and Zheng[32] shows that the marginal profit equals to the unit purchasing cost when the optimal quantity is positive in a system where the retailer faces both additive demand uncertainty and supplier uncertainty. However, when delivery flexibility involves, under additive demand uncertainty, the marginal profit is not equal to the unit purchasing cost but the marginal revenue. \(R\) is the revenue at the optimal average demand when the inventory level is greater than the threshold. In other word, the delivery flexibility increases marginal profit from the unit purchasing cost to marginal profit of the optimal mean of the demand.
The next theorem studies how marginal profit and optimal decisions change over time.

**Theorem 4** When $\epsilon_t \equiv 1$, the following statements are true. For all $t \leq T$,

(i) $V'_t(I) \geq V'_{t+1}(I)$

(ii) $q^*_t(I) \geq q^*_{t+1}(I)$

(iii) $d^*_t(I) \leq d^*_{t+1}(I)$

Federgruen and Zipkin [21] showed that the marginal profits and optimal order quantity are decreasing in time when the inventory system has one fixed capacity limited supplier. The inventory system becomes more complex when the retailer faces an uncertain supplier and price decisions. We extend results of Federgruen and Zipkin [21] to the inventory system with one uncertain supplier and price-dependent demand. Theorem 4 explains how the optimal marginal profit and the optimal policy change over time.

First, the marginal profit of the inventory level decreases in time, namely the value of a unit inventory decreases as the remaining time horizon decreases. Intuitively, the retailer has the opportunity to sell products in the current period or the following periods. With the passage of time, the number of periods over which the retailer can sell products decreases. Therefore, the marginal profit of inventory level decreases, as the chance of selling the inventory decreases when there are less periods left to sell the inventory. Second, the optimal order quantity decreases in time. The intuition and explanation is consistent with how the marginal profit changes with the inventory level. It is optimal to order less when the horizon is approaching to the end, as the number of periods left to sell products is decreasing. Third, the optimal average demand increases in time. That is to say, the optimal price decreases over the entire horizon under the fixed inventory level. To hold less inventory on hand when the time horizon is approaching to the end, it is optimal for the retailer to sell more by charging a lower price at the same inventory level, resulting in an increasing average demand over time horizon.
Due to the technical issue, the above analytical results cannot be extended to general demand uncertainty model, i.e., demand with both multiplicative and additive noises. In next section, we discuss this issue further by conduct numerical studies for the general demand model.

6 The Benefit of Delivery Flexibility

Our paper adopts a periodic-review inventory model to study the joint pricing and inventory replenishing decisions with delivery flexibility under both demand and supply uncertainties. In this section, we discuss the advantage of delivery flexibility considered in our DF system by comparing it with two other systems, i.e., the Immediately Shipping (IS) system and the One-period Handling and Shipping (1H&S) system. IS system has been considered in the previous literature, e.g., [27, 26]. One-period Handling and Shipping system are commonly employed by a lot of online retailers, such as Kmart, Apple, Drugstore, etc., where a period for preparing the order for shipping is usually required. Through the numerical studies, we show how the delivery flexibility modeled in this paper can improve the profit in an inventory system.

IS System.

\[
V^o_t(I) = \max_{q \geq 0} J^o_t(I, q, d)
\]

\[
J^o_t(I, q, d) = R(d) - cE(q \land k_t) - hE[(I - \epsilon_t d - \omega_t)^+] - pE[(I - \epsilon_t d - \omega_t)^-] + \alpha E[V^o_{t+1}(I + q \land k_t - \epsilon_t d - \omega_t)].
\]

where \((q^o_t(I), d^o_t(I))\) is the corresponding maximizer.

1H&S System.

\[
V^l_t(I) = \max_{q \geq 0} J^l_t(I, q, d)
\]

\[
J^l_t(I, q, d) = R(d) - cE(q \land k_t) - hE[(I + q \land k_t - \epsilon_t d - \omega_t)^+] - pE[(I + q \land k_t - \epsilon_t d - \omega_t)^-] + \alpha E[V^l_{t+1}(I + q \land k_t - \epsilon_t d - \omega_t)].
\]

where \((q^l_t(I), d^l_t(I))\) is the corresponding maximizer.
In IS system, the demand is required to be satisfied immediately. Otherwise, a penalty cost \( p \) for the delay of each item will be incurred. This type of system can be commonly observed in the past decades, when customers usually assume that the commodity should be shipped immediately upon placing the order. In 1H&S system, handling and preparing order for shipping usually take a period of time. For example, the orders placed at “www.tonbon.com” and “www.kmart.com” usually cannot be processed until the end of the day and it may take even a few days to prepare an order. Intuitively, handing orders placed online in a batch at the same time can save some managerial cost, which is usually implemented by retailers. However, Bloomingdales disagrees such a process. Bloomingdales claims that they want to satisfy their customers as fast as they can. In the last five years, the duration of handling and preparing for shipping decreases from 24 hours to 1 hour. Apparently, Bloomingdales has been focusing on moving from 1H&S System to our “flexible delivery system.”

In the following discussion, we examine the long-term optimal ordering and sales strategies to study the advantage of delivery flexibility as well as the impact of different parameters on this advantage.

We consider a base model for all three systems, i.e, DF, IS, and 1H&S. In the base model, the time horizon \( T = 10 \) and the discount factor \( \alpha = 0.9 \). The cost parameters are assumed as: unit ordering cost \( c = 2 \), holding cost \( h = 0.5 \), and penalty cost \( p = 5 \). In all the three systems, we assume the terminal condition given below

\[
V(T + 1) = 0.5 \times \max(I, 0) - 3 \times \max(-I, 0)
\]

The supplier’s capacity \( k_t \) is assumed to be i.i.d with \( \text{Prob}(k_t = +\infty) = \theta = 0.9 \) and \( \text{Prob}(k_t = 0) = 1 - \theta = 0.1 \). The multiplicative demand noises \( \epsilon_t \) are i.i.d following a truncated normal distribution with mean \( \mu_\epsilon = 1 \) and standard deviation \( \sigma_\epsilon = 0.16 \). The additive demand noises \( \omega_t \) are also i.i.d, following a truncated normal distribution with mean \( \mu_\omega = 0 \) and standard deviation \( \sigma_\omega = 0.65 \).
Besides solving for both the optimal demand and order quantity decisions numerically in each period for each system for the base case, we also change the costs $c$, $p$, and $h$, the horizon $T$, the uncertainty parameters $k_t$, $\epsilon_t$, and $\omega_t$ in the model of each system one by one to study the effects from different parameters on the optimal decisions as well as optimal profits. The optimal policies and some performance measurements are compared across three systems. All these results are listed in three separate tables, Table 1, Table 2, and Table 3, where the bolded parameters represents the values used in our base case.

6.1 General comparison and some measurements

To measure the optimal average order quantity and demand level over the entire time horizon, we define the following average quantity and demand for the DF system. Let

$$ \bar{q}^* = \frac{1}{T} \sum_{t=1}^{T} E q_t^*(I_t) / T $$

and

$$ \bar{d}^* = \frac{1}{T} \sum_{t=1}^{T} E d_t^*(I_t) / T $$

be the average order per period and the average demand per period for the DF system. Similarly, for IS and 1H&S systems, we define $\bar{q}^* = \sum_{t=1}^{T} E q_t^*(I_t) / T$ and $\bar{d}^* = \sum_{t=1}^{T} E d_t^*(I_t) / T$ as optimal average order per period and optimal average demand per period, where $j = o, l$ correspond to the IS and 1H&S system respectively.

As one of the benefits from the delivery flexibility is the ability to smooth the mismatch between order quantity and demand in an inventory system, we define a ratio of realized expected average order quantity $\theta \bar{q}^*$ to the average demand $\bar{d}^*$, which measures how well the mismatch is handled in each of the three inventory systems and is referred as ordering-sale ratio in the paper.

$$ R_{OS}^j = \frac{\theta \bar{q}^*}{\bar{d}^*}, j = f, o, l $$

Moreover, we also define measurements to capture how average sales and total profit in DF system are improved relative to the IS and 1H&S systems
respectively.

\[ \delta_{d_o} = \frac{\bar{d}^* - \bar{d}_1^*}{\bar{d}_1^*} \times 100\% \quad \text{and} \quad \delta_{V_o} = \frac{V_1^*(0) - V_{1^*}(0)}{V_{1^*}(0)} \times 100\%, \quad \forall j = o, l. \]

The percentage \( \delta_{d_o} \) (\( \delta_{d_l} \)) measures the increase in average sales in DF system relative to that in IS (1H&S) system; and \( \delta_{V_o} \) (\( \delta_{V_l} \)) measures the increase in total average profit of DF system relative to that in IS (1H&S) system. From Tables 1-3, we can observe that the average \( \delta_{d_o} \) is around 20% and the highest \( \delta_{d_l} \) is 47.65%. So, in general, DF system will sell 20% more products than IS system and in the extreme case, delivery flexibility can increase over 40% on sales. The average \( \delta_{d_o} \) is around 17% and the highest \( \delta_{d_l} \) is 37.24%. These numbers indicate that the delivery flexibility can bring 17% more sales compared to the 1H&S system in general; and this increase may reach up to 35% more sales in certain cases. From the profit aspect, Tables 1-3 show that \( \delta_{V_o} \) is around 20% and \( \delta_{V_l} \) is around 8%. These numbers imply that, on average, delivery flexibility can bring 20% more profit compared to IS system and 8% more profit compared to 1H&S system.

Therefore, in term of both sales and profits, the delivery flexibility captured in DF system can produce greater improvements compared to IS than that compared to 1H&S systems. This is because IS system is the least flexible one among the three systems. Compared with 1H&S system, IS system will behave more conservative when making decision on average demand, as the extra demand cannot be covered by using the order quantity. This more conservative decision makes the overall sales and profits in IS become less than these in 1H&S system. In an extreme case, when there are only two periods, the order quantity in the first period in IS system can only be used to satisfy the demand that has not been realized in the second period. In contrast, the order quantity in both periods in 1H&S system can be used to cover the high demand in each period, which gives a “back up” for demand decision making. Moreover, when horizon is finite, the effective order quantity in IS system is one period less than that in 1H&S system. In the two period case, only the first period order
quantity can be used in the IS system, but the order quantity in both periods in 1H&S are useful.

In the following subsections, we discuss how changes in three sets of parameters, i.e., the cost structure, time horizon parameters, and uncertainties, influence the optimal policies in the three inventory systems. By comparing the measurements defined above across three systems under different values of parameters, we obtain better ideas about the benefit from the delivery flexibility in an inventory system.

6.2 Effect of Cost Structure

In this section, we study the impact of cost structure on the advantage of delivery flexibility. Three cost parameters are considered, the unit production cost $c$, the unit holding cost $h$, and the unit penalty cost $p$. Results are summarized in Table 1.

<table>
<thead>
<tr>
<th>$c$</th>
<th>$h$</th>
<th>Order-Sale Ratio</th>
<th>IS System</th>
<th>1H&amp;S System</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>0.50</td>
<td>0.90 8.94 8.82 110.35</td>
<td>8.34 110.92% 80.40 22.47% 21.48%</td>
<td></td>
</tr>
<tr>
<td>0.50</td>
<td>0.50</td>
<td>0.90 8.94 8.82 110.35</td>
<td>8.34 110.92% 80.40 22.47% 21.48%</td>
<td></td>
</tr>
<tr>
<td>0.75</td>
<td>0.50</td>
<td>0.90 8.94 8.82 110.35</td>
<td>8.34 110.92% 80.40 22.47% 21.48%</td>
<td></td>
</tr>
<tr>
<td>1.00</td>
<td>0.50</td>
<td>0.90 8.94 8.82 110.35</td>
<td>8.34 110.92% 80.40 22.47% 21.48%</td>
<td></td>
</tr>
</tbody>
</table>

First, we notice that the ordering-sale ratio $R_{OS}^{j}$, $j = f, o, l$ is relatively stable and close to 1 in all the three systems in different scenarios. Therefore, we may conclude that all the three systems can achieve a good match between the average order quantity and average demand. This is probably a result of the terminal condition used in the model, which assumes that leftover inventory can be salvaged and the unmet demand can be satisfied. Under this condition,
both understocking and overstocking in the last period can be easily handled. Therefore, all the three systems may easily achieve a good match over the entire horizon without being too conservative or bold when making order decision.

Although the ordering-sale ratio does not reveal too much difference among the three systems, the numerical results in Table 1 does reveal a significant improvement on both sales and profits when DF system is compared with IS and 1H&S system, which are measured by \( \delta_{d_0}, \delta_{V_0}, \delta_{d^*}, \) and \( \delta_{V^*} \) respectively. By checking the columns in Table 1 corresponding to these four measurements, we can find that they are all positive in each scenarios, which indicates that both sales and profits in the DF system are improved relative to the other two systems. In addition, the improvements on sales and profits of DF system relative to IS system are much greater than the improvements of DF system relative to 1H&S system. This observation matches the intuition that DF system is more flexible than 1H&S system, which is more flexible than the IS system in term of managing both inventory and delivery. The DF system is the most flexible one among the three systems. When penalty cost is very high, DF system can use the incoming order quantity that arrives at the end of a period to satisfy the demand if the current inventory is not enough at the beginning of the period. When holding cost is very high, DF system can reduce the inventory level immediately by using current inventory to satisfy the demand. These two types of flexibility make DF system have a better ability to balance holding cost and penalty cost. Compared with DF system, 1H&S system does not have the flexibility of using current inventory to satisfy the demand immediately, since there is a one period delay for handling the order. However, 1H&S system does have the option to use the incoming order quantity to satisfy the demand, which helps to reduce the penalty cost. In contrast, the IS system is the least flexible one, as it must satisfy the demand immediately.

Finally, we can also observe that both \( \delta_{V_0} \) and \( \delta_{V^*} \) are increasing in \( h \) and \( p \) respectively in Table 1. This reveals that the improvements on profits of DF system relative to IS and 1H&S system become more significant when
either holding cost or penalty cost is higher. However, when the unit ordering cost \( c \) increases, the higher cost produces opposite effects. When DF system is compared with IS system, higher \( c \) makes profit improvement \( V_o \) larger. In contrast, when DF system is compared with 1H&S system, higher \( c \) makes the improvement \( V_l \) smaller. This is probability because larger \( c \) tend to make retailer order less, which makes the flexibility of satisfying the demand by the incoming order quantity at the end of a period become more important in optimizing the profit. While IS system does not have this flexibility, 1H&S system does have it. Therefore, when DF system is compared with the other two systems at a higher level of ordering cost \( c \), the advantage of DF system relative to IS system is enhanced, but this advantage relative to the 1H&S is diminished.

6.3 Effect of Time Horizon

In this subsection, we explore the impact of time horizon on the advantage of delivery flexibility. There are two factors considered here. One is the length of the time horizon \( T \) and the other is the discount factor \( \alpha \). The numerical results are summarized in Table 2.

<table>
<thead>
<tr>
<th>Table 2</th>
<th>The effect of Time Horizon</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>DF System</td>
</tr>
<tr>
<td></td>
<td>( \delta_{V_o} )</td>
</tr>
<tr>
<td>( T = 10 )</td>
<td>7.26</td>
</tr>
<tr>
<td>( T = 20 )</td>
<td>7.42</td>
</tr>
<tr>
<td>( T = 30 )</td>
<td>7.45</td>
</tr>
<tr>
<td>( T = 40 )</td>
<td>7.46</td>
</tr>
</tbody>
</table>

First, we observe that both \( \delta_{V_o} \) and \( \delta_{V_l} \) decrease in \( T \); but both \( \delta_{V_o} \) and \( \delta_{V_l} \) increase in \( T \). The opposite effects from a longer horizon on IS and 1H&S systems can be explained as follows. The results in Table 2 reveal that the ordering-sales ratio in IS system becomes closer to 1 as \( T \) increases. This
implies that the retailer in the IS system can make supply and demand match better over a longer time horizon. In IS system, retailer tends to overstock products, which can be observed by the fact that the ordering-sale ratio is greater than 1. In this case, a longer time horizon means that there are more periods for the retailer to sell the overstocked commodities, which gives a good chance to match supply and demand. This better match makes the advantage of DF system over the IS system become less significant when the time horizon increases. On the contrary, on a longer horizon, 1H&S system can suffer more from holding current inventory to the end of period. Therefore, the advantage of DF system relative to 1H&S system tends to be enhanced as time horizon increases. Besides the results shown in Table 2, our numerical study also shows that the profit benefit of DF vs IS converges to 17% and the benefit of DF vs 1H&S converges 8.5% when time horizon becomes very long.

In Table 2, we can also observe that when $\alpha$ varies, it also has opposite effects on the advantages of DF systems relative to IS system and 1H&S system, measured by $\delta_{V^o}$ and $\delta_{V^l}$ respectively. Specifically, when $\alpha$ increases, $\delta_{V^o}$ has a decreasing tendency, but $\delta_{V^l}$ has an increasing tendency. When $\alpha$ decreases, values in future periods become less important. The advantage of DF over IS is enhanced at smaller values of $\alpha$, which indicates that $\alpha$ has greater impact on the optimal policies in IS system than that in 1H&S system. When $\alpha$ decreases, the impact from remaining periods decreases, which makes the retailer focus more on the current period. In IS system, the order quantity in a period cannot be used to hedge the demand uncertainty in that period. Therefore, the retailer suffers more in IS system when $\alpha$ is small. For example when $\alpha = 0.84$, the average sales of DF system is 29.78% higher than that of IS system. This kind of dependence on the current period further leads to a decreasing profit. Therefore, the improvements on sales and profit of DF system relative to IS system increase as $\alpha$ decreases. In contrast, the demands in both DF system and 1H&S system can be satisfied by both the current on-hand inventory and the incoming order quantity. Therefore, the improvement of profit of DF system over 1H&S system decrease as $\alpha$ decreases.
Finally, as increasing time horizon can increase the number of periods that generate profits, it has a similar effect to increasing the discount factor $\alpha$. Therefore, we can observe that the measurements of relative benefits measurements behave in the same way when either $T$ is increased or $\alpha$ is increased.

6.4 Effect of Supply and Demand Uncertainty

In this subsection, we discuss the impact of supply and demand uncertainty. The supply uncertainty is captured by the arrival rate $\theta$. Demand uncertainty is captured by the standard deviations of multiplicative noise $\epsilon$ and additive noise $\eta$. Results are summarized in Table 3.

**Table 3** The effect of Supply and Demand Uncertainty

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$\epsilon = 0.16$</th>
<th>$\epsilon = 0.24$</th>
<th>$\epsilon = 0.33$</th>
<th>$\epsilon = 0.41$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta = 0.95$</td>
<td>$q^*_i$</td>
<td>$q^*_j$</td>
<td>$q^*_k$</td>
<td>$q^*_l$</td>
</tr>
<tr>
<td>$q^*_i$</td>
<td>$q^*_j$</td>
<td>$q^*_k$</td>
<td>$q^*_l$</td>
<td>$q^*_m$</td>
</tr>
<tr>
<td>$q^*_i$</td>
<td>$q^*_j$</td>
<td>$q^*_k$</td>
<td>$q^*_l$</td>
<td>$q^*_m$</td>
</tr>
<tr>
<td>$q^*_i$</td>
<td>$q^*_j$</td>
<td>$q^*_k$</td>
<td>$q^*_l$</td>
<td>$q^*_m$</td>
</tr>
<tr>
<td>$q^*_i$</td>
<td>$q^*_j$</td>
<td>$q^*_k$</td>
<td>$q^*_l$</td>
<td>$q^*_m$</td>
</tr>
<tr>
<td>$q^*_i$</td>
<td>$q^*_j$</td>
<td>$q^*_k$</td>
<td>$q^*_l$</td>
<td>$q^*_m$</td>
</tr>
<tr>
<td>$q^*_i$</td>
<td>$q^*_j$</td>
<td>$q^*_k$</td>
<td>$q^*_l$</td>
<td>$q^*_m$</td>
</tr>
<tr>
<td>$q^*_i$</td>
<td>$q^*_j$</td>
<td>$q^*_k$</td>
<td>$q^*_l$</td>
<td>$q^*_m$</td>
</tr>
</tbody>
</table>

First, we discuss the impact of supply uncertainty $\theta$ on the optimal policies. From Table 3, we can observe that when $\theta$ increases, average order quantities $q^*_j$ decrease and average demand $d^*_j$ increases in all three systems ($j = f, o, l$). These opposite effects from $\theta$ can be explain as follows. When $\theta$ becomes bigger, the retailer tends to order less because a larger fraction of the order quantity will be realized, which makes the over all average order quantity $q^*_j$ become smaller. In the same time, given the same order quantity, a bigger $\theta$ also means more products will be received by the system at the end of each period, which makes the retailer tend to raise the mean demand at the beginning of each period, resulting a higher level of average demand $d^*_j$. When it comes to the relative advantages on both sales and profits measured by $\delta_d$ and $\delta_{V'}$, for
Table 3 reveals that increasing $\theta$ can enhance the advantage of DF over IS but reduce the advantage of DF over 1H&S. In IS system, commodities must be shipped immediately to satisfy the demand, which lacks the flexibility to satisfy the demand by using incoming order quantity at the end of the period. Therefore, as a larger part of order quantity can be received at a higher level of $\theta$, the advantage of using incoming ordered quantity to satisfy demand in DF system over IS is enhanced. Compared with 1H&S system, the advantage of the DF resides in the flexibility to satisfy the demand immediately by using on hand inventory to reduce holding cost. At a higher level of $\theta$, the advantage from using incoming order quantity increases, which decreases the advantages from using on hand inventory to satisfy demand relatively. Therefore, at a higher level of $\theta$, the advantage of DF over 1H&S becomes less significant.

The demand uncertainty comes from both the additive noise and the multiplicative noise. In terms of additive noise, we notice that the impacts from the standard deviation of additive noise (std) $\mu_\omega$ on the optimal policies are less than the impacts from $\theta$ and $\epsilon$ in all three systems. The optimal average demand and order quantity are almost unchanged when additive uncertainty varies. This is probably because the impact of $\theta$ and $\epsilon$ will be enlarged at a higher level of the optimal order quantity and optimal average demand respectively. In contrast, additive noise is appended to the mean demand, which has less impact on the optimal policy as the variance will not be enlarged by multiplying a higher order quantity or demand.

Finally, we discuss the impact of multiplicative demand uncertainty on the optimal policies as well as the relative advantages. In terms of the optimal policies, from Table 3, it can be observed that both $\tilde{q}^*$ and $\tilde{d}^*$ decrease when the standard deviation of the multiplicative noise $\sigma_\epsilon$ increases. As multiplicative noise can be enlarged at higher level of average demand, when this uncertainty is increased, the retailer tends to reduce the average demand so as to reduce the overall uncertainty in the system. Corresponding to the reduced average demand, the order quantity is also reduced. Besides the optimal policies, the multiplicative uncertainty also has a more significant impact on the relative
advantage of DF system over IS system than that over 1H&S system. This can be observed from that fact that the ranges of both $\delta_{V_o}$ and $\delta_{d_o}$ are bigger than the ranges of both $\delta_{V_1}$ and $\delta_{d_1}$ when $\sigma_\epsilon$ is varying from 0.16 to 0.41.

7 Conclusion

We formulate an inventory system with joint inventory and pricing decisions to study the advantage of delivery flexibility. The structure of the optimal policy is described first. Our results about the optimal policies can also be extended to an inventory system without delivery flexibility. Previous literature suggests that the optimal policy will become very complex when both multiplicative demand uncertainty and supply uncertainty are involved. In our paper, we consider a case where supply uncertainty is captured by uncertain production capacity and demand uncertainty is modeled by both multiplicative and additive demand noises. In Table 4, we summarize the optimal policies of inventory system with multiplicative demand in the literature and compare them with our results obtained from delivery flexibility.

Table 4 Periodic inventory system with pricing and inventory decisions (multiplicative demand noise)

<table>
<thead>
<tr>
<th>Supplier</th>
<th>Optimal Policy</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>a reliable supplier</td>
<td>◦ the optimal order quantity is a base stock policy</td>
<td>[20]</td>
</tr>
<tr>
<td></td>
<td>◦ the optimal average demand increases in the inventory level</td>
<td></td>
</tr>
<tr>
<td>an all-or-nothing supplier</td>
<td>◦ the optimal order quantity is a threshold policy</td>
<td>[2]</td>
</tr>
<tr>
<td></td>
<td>◦ the optimal average demand increases in the inventory level</td>
<td></td>
</tr>
<tr>
<td>a supplier with capacity</td>
<td>◦ the change of the optimal order quantity is less than the change of inventory level</td>
<td></td>
</tr>
<tr>
<td>uncertainty</td>
<td>◦ the optimal policy is a threshold policy</td>
<td>our paper</td>
</tr>
<tr>
<td></td>
<td>◦ the optimal average demand increases in the inventory level</td>
<td></td>
</tr>
<tr>
<td>a supplier with yield uncertainty</td>
<td>◦ the optimal policy is a near-reorder point policy</td>
<td>[8]</td>
</tr>
<tr>
<td></td>
<td>◦ optimal order quantity may increase or decrease in the inventory level</td>
<td></td>
</tr>
<tr>
<td></td>
<td>◦ optimal average demand may increase or decrease in the inventory level</td>
<td></td>
</tr>
</tbody>
</table>

Besides exploring the optimal policy in our system, we also investigate how the optimal policy changes over the entire time horizon, which provides some insights into how to adjust the pricing and ordering decision over time.
Specifically, we show that the marginal profit from higher inventory level is decreasing in time; the order quantity is decreasing in time; and the average demand is increasing in time under the same inventory level. These results imply that the retailer prefers to order less and sell more when approaching to the end of the horizon, which is consistent with the intuition that less periods left means less flexibility for the retailer. The results can also be extended to a case without delivery flexibility. The managerial insight for the retailer is that she should be more aggressive to place order and sell product at the beginning of the horizon and become mild when approaching to the end of the horizon.

Finally, we explore the advantage of delivery flexibility over two inventory systems without this flexibility by numerical study. First, our results show that delivery flexibility can dramatically increase sales. Especially, it can increase sales by 20% compared to immediately shipping system and 15% compared to one-period handling&shipping system. Second, delivery flexibility can increase profit. Our results show that it can increase profit by 16% compared to immediately shipping system and 9% compared to one-period handling&shipping system. Third, additive demand noise and multiplicative demand noise can have different impacts on the optimal order quantity and optimal average demand. The benefits of this delivery flexibility we shown in this paper stems from the fact that it can not only instantly use the current inventory but also use the incoming order quantity to satisfy demand. The former reduces the supply uncertainty and the latter alleviates the demand uncertainty. With the help of the delivery flexibility, the retailer can obtain good perfect match between uncertain supply and uncertain demand to maximize her profit.

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A Appendix: Proofs

**Proof of Lemma 1.** We have $V_{T+1}(I) = c_s I^+ - c_u I^-$, where $c_s < c < c_u$. It is concave in $I$. We assume $V_{T+1}(I)$ is concave in $I$. Denote

$$
\pi(I, q, d) = R(d) - c_q - \mathbb{E}[h(I - \epsilon d - \omega_t)^+ - p(\epsilon d + \omega_t - I - q)^+] + \alpha \mathbb{E}[V_{T+1}(I + q - \epsilon d - \omega_t)].
$$

$$
\bar{q}(I, d) = \arg \max_{q \geq 0} \pi(I, q, d).
$$

Clearly, $\pi(I, q, d)$ is jointly concave in $(I, q, d)$. We also have that $J_t(I, q, d) = \mathbb{E}[\pi(I, q \wedge k_t, d)]$. 

For every \( \lambda \in (0,1) \), \( I_1 \), \( I_2 \), denote \( \lambda = \lambda I_1 + (1 - \lambda)I_2 \), \( d = \lambda d_1^* (I_1) + (1 - \lambda)d_2^* (I_2) \). We have

\[
\lambda V_1 (I_1) + (1 - \lambda)V_1 (I_2) = \lambda J_1 (I_1, q^*_1 (I_1), d^*_1 (I)) + (1 - \lambda)J_1 (I_1, q^*_1 (I_1), d^*_1 (I))
\]

\[
= \lambda E[\pi (I_1, q^*_1 (I_1) \land k_1, d^*_1 (I))] + (1 - \lambda)E[\pi (I_2, q^*_2 (I_2) \land k_2, d^*_2 (I))]
\]

\[
= E[\lambda \pi (I_1, q^*_1 (I_1) \land k_1, d^*_1 (I)) + (1 - \lambda)\pi (I_2, q^*_2 (I_2) \land k_2, d^*_2 (I))]
\]

\[
\leq E[\pi (\lambda q^*_1 (I_1) \land k_1) + (1 - \lambda)q^*_2 (I_2) \land k_2) \land d] \quad (2)
\]

We next show that consider every realization of \( \hat{k}_1 \). If \( \hat{k}_1 < \hat{q}(\hat{I}, \hat{d}) \), then

\[
\lambda (q^*_1 (I_1) \land k_1) + (1 - \lambda)(q^*_1 (I_2) \land k_2) \leq \lambda k_1 + (1 - \lambda)k_2 = k < \hat{q}(\hat{I}, \hat{d}).
\]

Combining with the concavity of \( \pi \) and the optimality of \( \hat{q} \),

\[
\pi (\hat{I}, \lambda (q^*_1 (I_1) \land k_1) + (1 - \lambda)(q^*_1 (I_2) \land k_2), \hat{d}) \leq \pi (\hat{I}, \hat{q}(\hat{I}, \hat{d}), \hat{k}_1).
\]

If \( \hat{k}_1 \geq \hat{q}(\hat{I}, \hat{d}) \), then

\[
\pi (\hat{I}, \lambda (q^*_1 (I_1) \land \hat{k}_1) + (1 - \lambda)(q^*_1 (I_2) \land \hat{k}_1), \hat{d}) \leq \pi (\hat{I}, \hat{q}(\hat{I}, \hat{d}), \hat{d}) = \pi (\hat{I}, \hat{q}(\hat{I}, \hat{d}), \hat{k}_1, \hat{d}).
\]

Hence, we have for every realization of \( \hat{k}_1 \),

\[
\pi (\hat{I}, \lambda (q^*_1 (I_1) \land \hat{k}_1) + (1 - \lambda)(q^*_1 (I_2) \land \hat{k}_1), \hat{d}) \leq \pi (\hat{I}, \hat{q}(\hat{I}, \hat{d}), \hat{k}_1, \hat{d}).
\]

Combining with (2), we have

\[
\lambda V_1 (I_1) + (1 - \lambda)V_1 (I_2) \leq E[\pi (\hat{I}, \hat{q}(\hat{I}, \hat{d}), \hat{k}_1, \hat{d})]
\]

\[
\leq E[\pi (\hat{I}, \hat{q}(\hat{I}, \hat{d}^*_1 (\hat{I}) \land k_1, d^*_1 (\hat{I}))]
\]

\[
= E[\pi (\hat{I}, \hat{q}(\hat{I}, \hat{d}^*_1 (\hat{I}) \land k_1, d^*_1 (\hat{I})))] = V_1 (\hat{I})
\]

Therefore \( V_1 (I) \) is concave in \( I \). We conclude our proof. 

\[ \blacksquare \]

**Proof of Lemma 1.** The proof is based on the concavity of \( V_{t+1}(\cdot) \), \( R(\cdot) \) and \( -()^+ \). The first order conditions with respect to \( q_t (I) \) and \( d_t (I) \) can be written as follows.

\[
\frac{\partial J_t}{\partial q_t} = G(q) \phi_{q,t} (I, q, d)
\]

\[
\frac{\partial J_t}{\partial d_t} = \phi_{d,t} (I, q, d)
\]

where

\[
\phi_{q,t} (I, q, d) = -c + E[L(I + q - \varepsilon d)] + \alpha E[V_{t+1}^t (I + q - \varepsilon d - \omega_t)]
\]

\[
\phi_{d,t} (I, q, d) = R'(d) - E[\varepsilon L(I - \varepsilon d)] - E[\varepsilon L(I + q - \varepsilon d)] - \alpha E[\epsilon_t V_{t+1}^t (I + q - \varepsilon d - \omega_t)]
\]

Notice that \( \phi_{q,t} (I, q, d) \) decreases in \( q \) and increases in \( d \). \( \phi_{d,t} (I, q, d) \) decreases in \( d \) and increases in \( q \).
To simplify notation let \( v_{t, \omega} = -L''(I + q - \epsilon d) - V''_{t+1}(I + q - \epsilon d - \omega l) \), \( u_{t} = -L''(I - \epsilon d) \), and \( \hat{v}_{t+1, \omega} = -L''(I + q \wedge k_0 - \epsilon d) - V''_{t+1}(I + q \wedge k_0 - \epsilon d - \omega l) \). From the definition of above variables, we have that they are all positive and

\[
E_{k} [\hat{v}_{t+1, \omega}] \geq 0 (1 - G(q)) + G(q) v_{t, \omega} = G(q) v_{t, \omega}
\]  

(3)

From the envelop theorem, we have

\[
- \frac{dq^{*}(I)}{dI} \{E[v_{t, \omega}]\} + \frac{dd^{*}(I)}{dI} \{E[v_{t+1, \omega}]\} = E[v_{t, \omega}]
\]

\[
- \frac{dq^{*}(I)}{dI} \{E[t v_{t, \omega}]\} - \frac{dd^{*}(I)}{dI} \{ - R''(d) + E[c_{t}^{2} u_{t}] + E[c_{t}^{2}\hat{v}_{t+1, \omega}]\} = -E[u_{t}] - E[\hat{v}_{t+1, \omega}]
\]

First, consider a trivial case \( E[v_{t, \omega}] = 0 \), then \( v_{t, \omega} = 0 \) for every \( \epsilon \) and \( \omega \). It is easy to see that \( \frac{dd^{*}(I)}{dI} \geq 0 \). Since we always pick up the lowest \( q \). Under such scenario, \( q \) will not increase as \( I \) increase.

In the following discussion, we focus on the case \( E[v_{t, \omega}] > 0 \).

\[
\frac{dd^{*}(I)}{dI} = \frac{E(v_{t, \omega})[E(\epsilon t u_{t}) + E(\epsilon t \hat{v}_{t+1, \omega}) - G_{t}(q)E(v_{t+1, \omega})]}{- R''(d) + E[c_{t}^{2} u_{t}] + E[c_{t}^{2}\hat{v}_{t+1, \omega}]E(v_{t+1, \omega}) - G_{t}(q)E^{2}(v_{t+1, \omega})}
\]

From Cauchy-Schwarz inequality, we have

\[
E[(\epsilon t \sqrt{v_{t, \omega}})^{2}]E[\sqrt{v_{t, \omega}}, t^{2}] - E^{2}(\epsilon t \sqrt{v_{t, \omega}}) \geq 0
\]  

(4)

Combining (3) and (4), we have

\[
E[\epsilon t \hat{v}_{t+1, \omega}] - G_{t}(q)E(v_{t+1, \omega}) = E\{\epsilon t[E(\hat{v}_{t+1, \omega}) - G_{t}(q)v_{t+1, \omega}]\} \geq 0
\]

and

\[
E(c_{t}^{2}\hat{v}_{t+1, \omega})E(v_{t+1, \omega}) - G_{t}(q)E^{2}(v_{t+1, \omega}) \geq 0
\]

Therefore, \( \frac{dd^{*}(I)}{dI} \geq 0 \)

To see part (ii), we also notice that

\[
1 + \frac{dq^{*}(I)}{dI} = \frac{E(v_{t+1, \omega})[E(u_{t}) + E(\hat{v}_{t+1, \omega}) - G_{t}(q)E(v_{t+1, \omega})]}{- R''(d) + E(u_{t}) + E(\hat{v}_{t+1, \omega})E(v_{t+1, \omega}) - G_{t}(q)E^{2}(v_{t+1, \omega})}
\]

\[
= \frac{E(u_{t}) + E(\hat{v}_{t+1, \omega}) - G_{t}(q)E(v_{t+1, \omega})}{- R''(d) + E(u_{t}) + E(\hat{v}_{t+1, \omega}) - G_{t}(q)E(v_{t+1, \omega})} < 1
\]

Hence, \( 1 + \frac{dq^{*}(I)}{dI} \geq 0 \).

To see part (iii), we first show that \( 1 + \frac{dq^{*}(I)}{dI} \leq 1 \) when \( \epsilon \equiv 1 \).

\[
1 + \frac{dq^{*}(I)}{dI} = \frac{E(v_{t+1, \omega})[E(u_{t}) + E(\hat{v}_{t+1, \omega}) - G_{t}(q)E(v_{t+1, \omega})]}{- R''(d) + E(u_{t}) + E(\hat{v}_{t+1, \omega})E(v_{t+1, \omega}) - G_{t}(q)E^{2}(v_{t+1, \omega})}
\]

\[
= \frac{E(u_{t}) + E(\hat{v}_{t+1, \omega}) - G_{t}(q)E(v_{t+1, \omega})}{- R''(d) + E(u_{t}) + E(\hat{v}_{t+1, \omega}) - G_{t}(q)E(v_{t+1, \omega})} < 1
\]
Follow the similar argument, we have $1 - \frac{\alpha h_t(I) - q}{\alpha h_t(I) - d} \geq 1$ when $\epsilon \equiv 1$. We conclude the proof of part (iii).

**Proof of Theorem 2.** We first define a system with $k_I \equiv +\infty$ as follows.

$$V_t^{++}(I) = \max_{d \geq 0, q \geq 0} J_t(I, q, d),$$

where

$$J_t^{++}(I, q, d) = R(d) - cq - h_t E[(I - \epsilon_t d - \omega_t)^+] - p_t E[(I + q - \epsilon_t d - \omega_t)\]
+ \alpha_t E[V_{t+1}^{++}(I + q - \epsilon_t d - \omega_t)].$$

Denote $(q_t^{++}(I), d_t^{++}(I))$ as the corresponding maximizer. Follow the similar argument as Theorem 1, we will have both $q_t^{++}(I)$ decreases in the inventory level and $d_t^{++}(I)$ increases in the inventory level. That is to say, the optimal ordering policy for the $+$ system follows a threshold policy. The FOC for the $+$ system can be written as

$$\frac{\partial J_t^{++}}{\partial q} = \phi_{q,t}(I, q, d) = -c + E[L(I + q - \epsilon_t d)] + \alpha_t E[V_{t+1}^{++}(I + q - \epsilon_t d - \omega_t)]$$

$$\frac{\partial J_t^{++}}{\partial d} = \phi_{d,t}(I, q, d) = R'(d) - c I + L'(I + q - \epsilon_t d) - E[L'(I + q - \epsilon_t d)] - \alpha_t E[V_{t+1}^{++}(I + q - \epsilon_t d - \omega_t)].$$

Next we will show that $q_t^*(I) = 0$ if and only if $q_t^{++}(I) = 0$.

First, if $q_t^{++}(I) = 0$, then we have $\phi_{q,t}(I, 0, d_t^{++}(I)) = \phi_{d,t}(I, 0, d_t^{++}(I)) = \phi_{q,t}(I, 0, d_t^{++}(I)) = \phi_{d,t}(I, 0, d_t^{++}(I))$. Hence, $q_t^*(I) = q_t^{++}(I) = 0$ and $d_t^*(I) = d_t^{++}(I)$.

Second, if $q_t^{++}(I) > 0$, then we assume $q_t^*(I) = 0$ and will reach a contradiction. If $q_t^*(I) = 0$, then $\phi_{q,t}(I, 0, d_t^*(I)) = \phi_{d,t}(I, 0, d_t^*(I))$ and $\phi_{q,t}(I, q, d_t^{++}(I)) = \phi_{d,t}(I, q, d_t^{++}(I))$. Hence, $q_t^{++}(I) = q_t^*(I) = 0$ and $d_t^{++}(I) = d_t^*(I)$. It contradicts with $q_t^{++}(I) > 0$.

From above discussion, we have $q_t^*(I) = 0$ if and only if $q_t^{++}(I) = 0$. Combining with the monotonicity of $q_t^{++}(I)$, we conclude the proof.

**Proof of Theorem 3.** When $\epsilon_t \equiv 1$, we have

$$\phi_{q,t}(I, q, d) = -c + L(I + q - d) + \alpha_t E[V_{t+1}^{++}(I + q - d - \omega_t)].$$

$$\phi_{d,t}(I, q, d) = R'(d) - L'(I + q - d) - E[L'(I + q - \epsilon_t d - \omega_t)] - \alpha_t E[V_{t+1}^{++}(I + q - \epsilon_t d - \omega_t)].$$

From the definition of $I_{q,t}$, we have $\phi_{q,t}(I_{q,t}, 0, d_t^{*}(I_{q,t})) = 0$. Hence, Notice that

$$\phi_{d,t}(I_{q,t}, 0, d_t^{*}(I_{q,t})) = R'(d_t^{*}(I_{q,t})) - L'(I_{q,t} - d_t^{*}(I_{q,t})) - E[L'(I_{q,t} - d_t^{*}(I_{q,t} - \omega_t))] - \alpha_t E[V_{t+1}^{++}(I_{q,t} - d_t^{*}(I_{q,t} - \omega_t))]
= R'(d_t^{*}(I_{q,t})) - L'(I_{q,t} - d_t^{*}(I_{q,t})) - E[L'(I_{q,t} - d_t^{*}(I_{q,t} - \omega_t))]
= R'(d_t^{*}(I_{q,t})) - L'(I_{q,t} - d_t^{*}(I_{q,t}) - c$$

Notice that $R'(d) - L'(I - d) - c \geq R'(d) - h - c > 0$. Hence, $d_t^{*}(I_{q,t}) > d_t$. That is to say, $I_{d,t} < I_{q,t}$.
For every $I \leq I_{q,t}$, let $q = q_{q,t} - d_t^*(I_{q,t}) - I - d$, we have

$$\phi_{q,t}(I, q, d) = -c + \mathbb{E}[L'(I_{q,t} - d_t^*(I_{q,t}))] + \alpha \mathbb{E}[V_{t+1}^\prime(I_{q,t} - d_t^*(I_{q,t}) - \omega_t)] = \phi_{q,t}(I_{q,t}, 0, d_t^*(I_{q,t})) = 0.$$ 

$$\phi_{d,t}(I, q, d) = R'(d) - L'(I - d) - \mathbb{E}_{k_t} \max \left\{ L'(I + q - d) - \alpha \mathbb{E}_{\omega_t}[V_{t+1}^\prime(I + q - d - \omega_t)], \right.$$

$$L'(I + k_t - d) + \alpha \mathbb{E}_{\omega_t}[V_{t+1}^\prime(I + k_t - d - \omega_t)] \bigg\} \bigg\}$$

That is to say $q^*_t(I) = I_{q,t} - d_t^*(I_{q,t}) - I + d_t^*(I)$.

**Proof of Lemma 2.** When $I \geq I_{d,t}$, we have that

$$V'(I) = L'(I - d) + \mathbb{E}[L'(I + q^*_t(I) - d_t^*(I))] + \alpha \mathbb{E}[V_{t+1}^\prime(I + q^*_t(I) - d_t^*(I) - \omega_t)] \bigg\} \bigg\}$$

When $I \leq I_{d,t}$, $d_t^*(I) = \bar{d}$ and $\phi_{d,t}(I, q^*_t(I), d_t^*(I)) \leq 0$. When $I \geq I_{d,t}$, $\phi_{d,t}(I, q^*_t(I), d_t^*(I)) = 0$. From lemma 3, we have $I_{d,t} < I_{q,t}$. Hence, $0 = \phi_{q,t}(I, q^*_t(I), \bar{d})$ for all $I \leq I_{d,t}$. Therefore,

$$(V'(I) = L'(I - d) + \mathbb{E}_{k_t} \max \left\{ L'(I + q^*_t(I) - d_t^*(I)) + \alpha \mathbb{E}[V_{t+1}^\prime(I + q^*_t(I) - d_t^*(I) - \omega_t)], \right.$$

$$L'(I + k_t - d_t^*(I)) + \alpha \mathbb{E}[V_{t+1}^\prime(I + k_t - d_t^*(I) - \omega_t)] \bigg\} \bigg\}$$

We conclude the proof.

To investigate how the marginal profit changes over the time horizon. To obtain the results, we construct an unconstraint profit problem first and then build up the relation between the optimal solutions of the unconstraint profit problem and the original problem. This relation is used to obtain result about how marginal profit changes over time in Theorem 4. By relaxing the boundary assumptions on the decision variables $q_t$ and $d_t$ in the DF inventory problem, we obtain an unconstraint problem below.

$$(\hat{q}_t(I), \hat{d}_t(I)) = \arg \max_{q, d} J_t(I, q, d)$$

where $(\hat{q}_t(I), \hat{d}_t(I))$ is the optimal solution for the profit function. Next Lemma reveals the relation between optimal solutions of the unconstrained optimal solution and the original optimal problem.

**Lemma 3** When $\epsilon_t \equiv 1$, the relation of $(\hat{q}_t(I), \hat{d}_t(I))$ and $(q^*_t(I), d_t^*(I)(I))$ is as follows.

(i) If $I \leq I_{d,t}$, then $\hat{q}_t(I) \leq q^*_t(I)$ and $\hat{d}_t(I) \leq \bar{d} = d_t^*(I)(I)$.

(ii) If $I \in [I_{d,t}, I_{q,t}]$, then $q^*_t(I) = \hat{q}_t(I)$ and $d_t^*(I) = \hat{d}_t(I)$.

(iii) If $I \geq I_{q,t}$, then $\hat{q}_t(I) \leq 0 = q^*_t(I)$ and $\hat{d}_t(I) \leq d_t^*(I)$.
Proof of Lemma 3. Notice that \((\tilde{q}_t(I), \tilde{d}_t(I))\) are the optimal solution without constraints \(q \geq 0\) and \(d \geq d^\text{d}\). Hence, (5) and (6) are also FOC of the optimal problem \(\max_{d, q} J_t(I, q, d)\).

We also have \(\tilde{d}_t(I, q, d) > d\) from Lemma 3

when \((q, d) = (\tilde{q}_t, \tilde{d}_t)\) we have

\[
0 = \phi_{q,t}(I, q, d) = -c + \mathbb{E}[L'(I + q - d)] + \alpha \mathbb{E}[V'_{t+1}(I + q - d - \omega_t)](7)
\]

\[
0 = \phi_{d,t}(I, q, d) = R'(d) - L'(I - d) - \mathbb{E}_k \max \{e, L'(I + k - d + \alpha \mathbb{E}_w[I_{t+1}(I + k - d - \omega_t)]\}(8)
\]

If is clear that if \(I \in [I_{d,t}, I_{q,t}]\), then constrains are satisfied automatically. Hence, \(q_t^*(I) = \tilde{q}_t(I)\) and \(d_t^*(I) = \tilde{d}_t(I)\).

If \(I \leq I_{d,t}\), then when \((q, d) = (q_t^*(I), d)\)

\[
0 \leq \phi_{q,t}(I, q, d) = -c + \mathbb{E}[L'(I + q - d)] + \alpha \mathbb{E}[V'_{t+1}(I + q - d - \omega_t)].
\]

\[
0 \leq \phi_{d,t}(I, q, d) = R'(d) - L'(I - d) - \mathbb{E}_k \max \{e, L'(I + k - d + \alpha \mathbb{E}_w[I_{t+1}(I + k - d - \omega_t)]\}.
\]

Comparing \(\phi_{d,t}(I, q^*_t(I), d)\) and \(\phi_{d,t}(I, \tilde{q}_t, \tilde{d}_t)\), we have \(\tilde{d}_t(I) \leq d = d_t^*(I)\). Comparing \(\phi_{q,t}(I, q^*_t(I), d)\) and \(\phi_{q,t}(I, \tilde{q}_t, \tilde{d}_t)\), we have \(\tilde{q}_t - \tilde{d}_t = q^*_t(I) - d\). Hence, \(\tilde{q}_t \leq q^*_t(I)\).

If \(I \geq I_{q,t}\), then if \(q^*_t(I) > 0\), then we can construct a solution with \(q_t^*(I) = q^*_t(I)\) and \(d_t^*(I) = d_t^*(I)\) to satisfy the FOC. It contradicts with \(I \geq I_{q,t}\). If \(q^*_t(I) = 0\), then \(q_t^*(I) = \tilde{q}_t(I)\) and \(d_t^*(I) = \tilde{d}_t(I)\). Next, we consider \(q^*_t(I) < 0\). Notice that when \((q, d) = (0, d_t^*(I))\)

\[
0 \leq \phi_{q,t}(I, q, d) = -c + \mathbb{E}[L'(I - d)] + \alpha \mathbb{E}[V'_{t+1}(I - d - \omega_t)].
\]

Hence, \(\tilde{q}_t(I) - \tilde{d}_t(I) \geq -d_t^*(I)\). That is to say, \(\tilde{d}_t(I) \leq d_t^*(I) + \tilde{q}_t(I) \leq d_t^*(I)\). We conclude the proof.

Proof of Theorem 4. In the following discussion, the concavity of \(R(\cdot), L(\cdot), L'(\cdot)\) and \(V_I(\cdot)\) is used. From their concavity, we have \(R'(\cdot), L'(\cdot), L''(\cdot)\) and \(V''_I(\cdot)\) are all decreasing functions.

We will use mathematic proof to obtain results. First, from Lemma 2 and the assumption \(V_{T+1} = c_s I^+ - c_s I^-\), we have

\[
V_{T+1}^L(I) = \lim_{t \to +\infty} R'(d) = 0 \geq V_{T+1}^\text{prime}(I) \text{almost everywhere}.
\]

Next, we will show that if \(V_{T+1}'(I) \geq V_{T+1}'(I)\) for all \(\tau \geq t\) and \(I\) almost everywhere, then \(V_I'(I) \geq \max \{e, L'(I + k - d + \alpha \mathbb{E}_w[I_{t+1}(I + k - d - \omega_t)]\}\) almost everywhere.

From the FOC of \((\tilde{q}_t(I), \tilde{d}_t(I))\) and \((\tilde{q}_{t+1}(I), \tilde{d}_{t+1}(I))\) and \(V_I'(I) \geq V_{I+1}'(I)\), for all \(I\), we have

\[
I + \tilde{q}_t(I) - \tilde{d}_t(I) \geq I + \tilde{q}_{t+1}(I) - \tilde{d}_{t+1}(I)
\]

\[
\tilde{d}_t(I) \leq \tilde{d}_{t+1}(I)
\]

We have

\[
\tilde{d}_t(I) \leq \tilde{d}_{t+1}(I) \text{ and } \tilde{q}_t(I) \geq \tilde{q}_{t+1}(I) \tag{9}
\]
Combing with Lemma 3, we have \( I_{q,t} \geq I_{q,t+1} \) and \( I_{d,t} \geq I_{d,t+1} \).

Next we discuss three cases to see that \( q'_t(I) \geq q'_{t+1}(I) \) and \( d'_t(I) \leq d'_{t+1}(I) \) for all \( I \).

Case (1) \( I \geq I_{q,t} \). We have \( q'_t(I) = q'_{t+1}(I) = 0 \). Then we have \( d'_t(I) \) (\( i = t, t + 1 \)) are solutions of

\[
0 = \phi_{d,i}(I, 0, d'_t(I))
= R'(d'_t(I)) - L'(I - d'_t(I)) - L'(I - d'_t(I)) - \alpha E_{\omega_t}[V'_{t+1}(I - d'_t(I) - \omega_t)]
\]

From \( V'_{t+1}(I) \geq V'_{t+2}(I) \), we have \( d'_t(I) \leq d'_{t+1}(I) \).

Case (2) \( I \in [I_{d,t+1}, I_{q,t}] \).

To see \( d'_t(I) \leq d'_{t+1}(I) \) for all \( I \). If \( d'_t(I) = \underline{d} \) then it is clear that \( d'_t(I) \leq d'_{t+1}(I) \). If \( d'_t(I) > \underline{d} \). Then from Lemma 3 and (9), we have

\[
d'_{t+1}(I) \geq \hat{d}_{t+1}(I) \geq \hat{d}_t(I) = d'_t(I)
\]

To see \( q'_t(I) \geq q'_{t+1}(I) \) for all \( I \). If \( q'_{t+1}(I) = 0 \), then it is clear that \( q'_t(I) \geq q'_{t+1}(I) \). If \( q'_t(I) > 0 \). Then from Lemma 3 and (9), we have

\[
q'_{t+1}(I) = \hat{q}_{t+1}(I) \leq \hat{q}_t(I) \leq q'_t(I)
\]

Case (3) \( I \leq I_{d,t+1} \). We have \( d'_t(I) = d'_{t+1}(I) = \underline{d} \). Notice that

\[
0 = \phi_{d,i}(I, q'_t(I), \underline{d}) = \phi_{d,t+1}(I, q'_t(I), \underline{d})
\]

Hence, \( q'_t(I) \geq q'_{t+1}(I) \).

From above discussion, we have \( q'_t(I) \geq q'_{t+1}(I) \) and \( d'_t(I) \leq d'_{t+1}(I) \) for all \( I \).

We further show that \( V'_t(I) \geq V'_{t+1}(I) \).

If \( I \geq I_{d,t} \), then from Lemma 2, we have

\[
V'_t(I) = R'(d'_t(I)) \geq R'(d'_{t+1}(I)) = V'_{t+1}(I)
\]

If \( I \in [I_{d,t+1}, I_{d,t}] \), we have

\[
V'_t(I) \geq R'(\underline{d}) \geq R'(d'_{t+1}(I)) = V'_{t+1}(I)
\]

If \( I \leq I_{d,t+1} \), from Lemma 2, we have

\[
V'_t(I) = L'(I - d') + E_{\omega_t} \max \{ c, L'(I + k_t - \underline{d}) + \alpha E[V'_{t+1}(I + k_t - d - \omega_t)] \}
\]

\[
\geq L'(I - \underline{d}) + E_{\omega_t} \max \{ c, L'(I + k_t - \underline{d}) + \alpha E[V'_{t+2}(I + k_t - d - \omega_t)] \} = V'_{t+1}(I)
\]

Therefore, if \( V'_{t+1}(I) \geq V'_{t+2}(I) \) for all \( \tau \geq t \) and \( I \), then \( V'_t(I) \geq V'_{t+1}(I) \), \( q'_t(I) \geq q'_{t+1}(I) \) and \( d'_t(I) \leq d'_{t+1}(I) \). We conclude the proof by mathematic induction. \( \blacksquare \).