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Tools for discovery?

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The role of diagrams in contemporary mathematics: Tools for discovery?

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1 Abstract

My presentation draws upon and combines results from some of my articles on the role of diagrams in contemporary mathematics. Referring to [1] and [2] I will present examples of how diagrams function as tools for discovery in contemporary analysis. The purpose in this talk is to analyse why these diagrams are fruitful. In most of the talk I use 'diagram' in its ordinary sense referring to certain (2-dimensional) visual representations composed of lines and sometimes letters standing for mathematical objects.

Carter [1] considers a case study from free probability theory (a field combining analysis with probability theory) where certain diagrams are used to represent permutations (and constructions on permutations). The diagrams give rise to the concepts of a 'crossing permutation' (Fig. 2) and a 'neighbouring pair' (Fig. 1). That is, new concepts are found by representing permutations by diagrams. Furthermore these diagrams can be "manipulated" and by doing this new results have been discovered. One of these results establishes a connection between the two concepts, that is, of a crossing permutation and a permutation having no neighbouring pairs. Note that these properties can be *shown* in a diagram – in a crossing permutation lines *cross* and neighbouring pairs are *neighbouring numbers.* That is, in addition to giving rise to new concepts, the diagrams visualise these concepts as well as relations holding between them.

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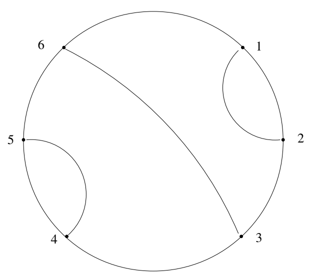

Figure 1. A representation of a permutation with neighbouring pairs, (1,2) and (4,5).

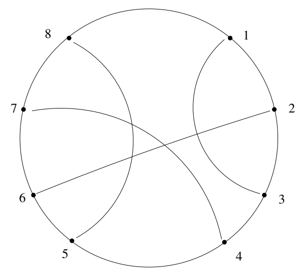

Figure 2. A representation of a crossing permutation.

Carter [2] discusses an example where manipulations with diagrammatic presentations of directed graphs have contributed to establish that a certain type of C*-algebras exists. In the theory of C*-algebras an important question concerns their classification, that is, determining which different types of algebras exist up to isomorphism. An important tool in order to do this is to compute their K-groups, denoted $K_0$and K_1. Unfortunately these K-groups are difficult to define. Recently an easier way has been found by instead generating C*-algebras and their corresponding K-groups from so-called directed graphs. A directed graph consists of a collection of vertices and directed edges between these vertices. See figure 3 for an example.

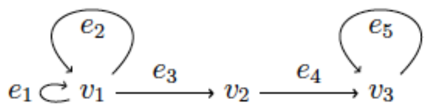

Figure 3. A picture of a directed graph with vertices, v_1, v_2, v_3 and edges $e_1, e_2 \dots e_5$. Arrows show the direction of the edges.

A particular graph gives rise to a collection of generators and relations from which a C*-algebra may be defined. Read in a different way the graph gives rise to a linear map. From this map one can easily define the two groups, K_0 and K_1. One natural question that arises in this context is how many different C*-algebras can be obtained from directed graphs. It is a response to this question that is considered in [2]. The result is found by *manipulating* the directed graphs while calculating their associated K-groups.

In both of the above cases diagrams function as objects – or signs – that can be manipulated and so experimented on. Furthermore I noted in the first case study that certain properties and relations are directly observable in diagrams. These two features are central to Peirce's description of 'diagrammatic reasoning' and so in the second part of my talk I will give a few relevant details of what he means by this (referring to [3], [4] and [5]). First it should be noted that according to Peirce a 'diagram' includes representations that one would normally not consider as diagrams. Moreover it should be stressed that it is not entirely clear what Peirce took a diagram to be. Peirce states that a diagram is an icon and in some places a representation of *relations* or *rationally related objects* see ([6], pp. 316-317). According to Peirce diagrammatic reasoning is a description of (mathematical) necessary reasoning. Thus an important role of a diagram is to allow you to see the necessary relation holding between the antecedent and conclusion of a proposition. Reasoning proceeds by constructing "*a diagram, or visual array of characters or lines. Such a construction is formed according to a precept furnished by the hypothesis. Being formed, the construction is submitted to the scrutiny of observation, and new relations are discovered among its parts, not stated in the precept by which it was formed, and are found, by a little mental experimentation, to be such that they will always be present in such a construction*"([7], CP 3.560). This seems to fit the procedure in Euclid's Elements. A diagram is constructed so that it represents the hypothesis of the stated proposition. Sometimes further constructions have to be made (e.g. lines drawn) until the conclusion can be seen to follow from the diagram. What is remarkable is that Peirce holds that this characterises mathematical reasoning in general (and so accordingly he extends the notion of 'a diagram'). To mention a simple example, a proof consisting of a calculation to prove the proposition *the product of the sum and difference of two numbers is equal to the difference between their squares* would also be a diagrammatic proof. It is important for Peirce that a proof – or the drawn diagram – is a concrete representation, in other words a token, so that it is possible to *observe*. At the same time the diagram is the representation of a symbolic statement (or the interpretant of a symbolic statement) and so general. The combination of the two gives the necessity of the conclusion.

Peirce's idea of diagrams as concrete objects that can be *experimented on* and *observed* fits well with the observations made in the two case studies above. Firstly, in both cases are certain representations that were manipulated with when discovering new results. The idea of a sign that can be manipulated gives rise to the notion of a 'faithful representation', which I propose may be used to explain the fruitfulness of certain

representations [2,8]. Note that a faithful representation can be any type of representation, including a formal expression, in case it fulfils the characterisation given below.
A faithful representation fulfils that i) it resembles or shares certain relations with the represented object (i.e., represents iconically) and ii) manipulations can be performed on the representations, respecting relevant relations between the represented objects, so that new relations may become visible.

Secondly I noted in the first case study that certain concepts as well as relations between them are shown in the diagrams. (Now taking 'diagram' in its usual meaning.) In both case studies it is possible to compare diagrammatic representations with formal representations of the same concept. Here the diagrams *show* the relations holding whereas the formal expressions *describe* them. Furthermore in both cases it can be argued that the diagrammatic presentations offer a cognitive advantage over formal expressions. In the case study on graph algebras, for example, a case can be made that it is easier to read off relevant information from the diagrammatic presentation of a graph. I measure 'cognitive advantage' in terms of the number of cognitive resources drawn upon in order to perform a given task. In conclusion diagrams seem to offer two advantages that I propose are relevant components in a characterisation of understanding. First is their capacity in some cases to *show* relations in contrast to describing them. Second is the fact that they sometimes offer cognitive advantages.

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