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Non-amenability and visual Gromov hyperbolic spaces

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Abstract. We prove that a uniformly coarsely proper hyperbolic cone over a bounded metric space consisting of a finite union of uniformly coarsely connected components each containing at least two points is non-amenable and apply this to visual Gromov hyperbolic spaces.


Key words: Hyperbolic spaces, isoperimetry, amenability.

1 Introduction

A metric space \((X, d)\) is uniformly coarsely proper if there exist \(N: (0, \infty) \times (0, \infty) \to \mathbb{N}\) and a constant \(r_b > 0\) such that for all \(R > r > r_b\) every open ball of radius \(R\) in \(X\) can be covered by \(N(R, r)\) open balls of radius \(r\) in \(X\). A subset \(\Gamma \subseteq X\) is \((\mu, \beta)\)-cobounded if there exists a constant \(\mu > 0\) such that \(d(x, \Gamma) < \mu\) for all \(x \in X\) and uniformly locally finite if there exists \(N: (0, \infty) \to \mathbb{N}\) such that the cardinality \(#(\Gamma \cap B(x, r)) \leq N(r)\) for all \(0 < r < \infty\) and all \(x \in X\). As usual \(B(x, r) = \{y \in X: d(x, y) < r\}\). A quasi-lattice in \((X, d)\) is a cobounded uniformly locally finite subset \(\Gamma \subseteq X\), and \((X, d)\) is uniformly coarsely proper if and only if it has a quasi lattice [6, Proposition 3.D.16]. A uniformly coarsely proper space \((X, d)\) is now said to be non-amenable if there exist a quasi-lattice \(\Gamma \subseteq X\) and constants \(C > 0\) and \(r > 0\) such that for any finite subset \(F \subseteq \Gamma\)

\[
\#F \leq C \#\partial_r F
\]

where \(\partial_r F = \{x \in \Gamma: d(x, F) < r\) and \(d(x, \Gamma \setminus F) < r\}\).

A complete geodesic Gromov hyperbolic Riemannian manifold (or metric graph) with bounded local geometry and quasi-pole is non-amenable if its Gromov boundary consists of finitely many connected components of strictly positive diameter; see [3]. We show more generally that a uniformly coarsely proper hyperbolic cone over any bounded metric space with finitely many uniformly coarsely connected components each containing at least two points is non-amenable; and hence that any uniformly coarsely proper visual Gromov hyperbolic space is non-amenable if its Gromov boundary consists of finitely many uniformly coarsely connected com-
ponents of strictly positive diameter. The terminology and results are in detail as follows.

A space \((X, d)\) is \textit{Gromov hyperbolic} if it satisfies for some \(\delta \in [0, \infty)\) the Gromov product inequality
\[
(x|z)_w \geq \min\{ (x|y)_w, (y|z)_w \} - \delta
\]
for all \(x, y, z, w \in X\). The \textit{hyperbolic cone} over a bounded metric space \((Z, d)\) containing at least two points is the metric space \((H(Z), \rho)\) where \(H(Z) = Z \times [0, \infty)\),
\[
\rho((x, t), (y, s)) = 2 \log \left( \frac{d(x, y) + \max\{e^{-t}, e^{-s}\} D}{e^{-(s+t)/2} D} \right),
\]
and \(D = \text{diam}(Z)\). A space \((X, d)\) is \(\varepsilon\)-coarsely connected for \(\varepsilon > 0\) if for every \(x, y \in X\) there exists an \(\varepsilon\)-sequence from \(x\) to \(y\) in \(X\), by which we mean a finite sequence of points \(x = x_0, \ldots, x_n = y\) in \(X\) such that \(d(x_i, x_{i+1}) \leq \varepsilon\) for all \(0 \leq i \leq n - 1\). If \((X, d)\) is \(\varepsilon\)-coarsely connected for all \(\varepsilon > 0\) we say that \((X, d)\) is \textit{uniformly coarsely connected}; a uniformly coarsely connected component of \((X, d)\) is any subset of the form \(C(x, X) = \bigcup\{A: x \in A \subseteq X, A \text{ uniformly coarsely connected}\}\). If \((X, d)\) is compact its uniformly coarsely connected components are its connected components.

Our main result is the following coarse generalisation of \cite{3} Theorem 3.2.

\textbf{Theorem A.} Let \((H(Z), \rho)\) be the hyperbolic cone over a bounded space \((Z, d)\). If \((H(Z), \rho)\) is uniformly coarsely proper and \((Z, d)\) consists of a finite union of uniformly coarsely connected components each containing at least two points then \((H(Z), \rho)\) is non-amenable.

A space is visual if there exists a basepoint so that every point in the space is contained in the image of some roughly geodesic ray issuing from it; see Section 2. This gives the following generalisation of \cite{3} Main Theorem 1.1.

\textbf{Theorem B.} If \((X, d)\) is a uniformly coarsely proper visual Gromov hyperbolic space whose Gromov boundary consists of a finite union of uniformly coarsely connected components each containing at least two points then \((X, d)\) is non-amenable.

\textbf{Proof.} As \(X\) is visual Gromov hyperbolic its boundary \(\partial X\) is a bounded metric space and there exists a rough-similarity \(f: X \rightarrow H(\partial X)\); see \cite{2} Proposition 6.2, Theorem 8.2.

Since \(\partial X\) consists of finitely many uniformly coarsely connected components each containing at least two points \(H(\partial X)\) is non-amenable by Theorem A since uniformly coarsely proper is a quasi-isometry invariant by \cite{6} Corollary 3.1.D.17. The claim now follows as non-amenability is a quasi-isometry invariant by \cite{4} Corollary 2.2.

\ The Gromov boundary of a locally compact compactly generated hyperbolic group is compact so all of its uniformly coarsely connected components are connected; and if it consists of finitely many connected components containing at least two points, it consists of exactly one connected component containing these points; see for example \cite{5} Section 2.C.\]
Corollary C. Let $G$ be a locally compact compactly generated hyperbolic group whose boundary is connected and contains at least two points. Then $G$ is not geometrically amenable.

Proof. Suppose $G$ is compactly generated by $S \subseteq G$ and write $(G, d_S)$ for the corresponding word metric space noting that it is uniformly coarsely proper; see Lemma [4] By the characterisation of hyperbolic groups [4, Corollary 2.6] and the Švarc-Milnor Lemma [6, Theorem 4.C.5] there exists a quasi-isometry $f : (G, d_S) \to (X, d)$ where $(X, d)$ is some proper geodesic Gromov hyperbolic space. This induces a power-quasisymmetry $\partial f : \partial G \to \partial X$; see [2, Theorem 6.5]. Since $\partial f$ is a homeomorphism $\partial X$ is connected and contains at least two points and $(X, d)$ is non-amenable by Theorem B. In particular $(G, d_S)$ is non-amenable. The claim now follows from [7, Corollary 11.14].

1.1 Organisation of the paper

In section 2, we recall the terminology used for metric spaces not covered in the introduction and prove some folklore results claiming no originality whatsoever. Section 3 contains the gist of the paper: here we cover the hyperbolic cone construction; Cao’s graph approximation; and prove Theorem A adapting techniques from Cao [3] and Väisälä [9].

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2 Basic notions and folklore

A subset $N \subseteq X$ in $(X, d)$ is $(\mu)$-separated if there exists a constant $\mu > 0$ such that $d(x, y) \geq \mu$ whenever $x, y \in N$ are distinct. A maximal $\mu$-net in $(X, d)$ is a $\mu$-separated $\mu$-cobounded subset $N \subseteq X$. Note that a maximal $\mu$-net $N \subseteq X \neq \emptyset$ always exists for any $\mu > 0$ by Zorn’s lemma.

A function $f : X \to X'$ between $(X, d)$ and $(X, d')$ is a $(\lambda, \mu)$-quasi-isometric embedding if there exist constants $\lambda \geq 1$ and $\mu \geq 0$ such that

$$\lambda^{-1}d(x, y) - \mu \leq d'(f(x), f(y)) \leq \lambda d(x, y) + \mu$$
for all $x, y \in X$, and $\mu$-essentially surjective if $d(x', f(X)) \leq \mu$ for all $x' \in X'$. A $\mu$-essentially surjective $(\lambda, \mu)$-quasi-isometric embedding $f: X \to X'$ is a $(\lambda, \mu)$-quasi-isometry and $(X, d)$ and $(X', d')$ are said to be quasi-isometric. A $(\lambda, \mu)$-quasi-isometry $f: X \to X'$ is a $(\lambda, \mu)$-rough similarity if

$$\lambda d(x, y) - \mu \leq d'(f(x), f(y)) \leq \lambda d(x, y) + \mu$$

for all $x, y \in X$.

Abbreviating ”from $x$ to $y$” by $x \overset{\gamma}{\to} y$, we say that a $(1, \mu)$-quasi-isometric embedding $\gamma: [a, b] \to X$ from a compact interval $[a, b] \subseteq \mathbb{R}$ is a $\mu$-rough geodesic $x \overset{\gamma}{\to} y$ where $x = \gamma(a)$ and $y = \gamma(b)$. A $(1, \mu)$-quasi-isometric embedding $\gamma: [0, \infty) \to X$ is called a $\mu$-roughly geodesic ray issuing from $\gamma(0)$. A $\mu$-rough geodesic $\gamma: x \overset{\gamma}{\to} y$ can always be parametrised by $d(x, y)$.

**Lemma 1.** Given a $\mu$-rough geodesic $\gamma: [a, b] \to X$ there exists a $2\mu$-rough geodesic $\beta: [0, d(x,y)] \to X$.

**Proof.** Write $R = d(x, y)$ and assume without loss of generality that $[a, b] = [0, b]$. First assume $b < R$. Extend $\gamma$ to $\beta: [0, R] \to X$ by $\beta(t) = \gamma(t)$ for $0 \leq t \leq b$ and $\beta(t) = \gamma(b) = y$ for $b \leq t \leq R$. Restricted to $0 \leq t \leq b$ the function $\beta$ is trivially a $2\mu$-rough geodesic $x \overset{\beta}{\to} y$. Next, consider the case when $0 \leq s \leq b < t \leq R$. Now,

$$d(\beta(s), \beta(t)) = d(\gamma(s), \gamma(b)) \leq (b - s) + \mu \leq (t - s) + \mu.$$  

On the other hand, since $\gamma$ is a $\mu$-rough geodesic $|R - b| \leq \mu$, in particular since $t \leq R$ it follows from $R - b \leq \mu$ that $t - b \leq \mu$. As $t - b > 0$, $|t - b| \leq \mu$, and so also

$$d(\beta(s), \beta(t)) \geq |s - b| - \mu = |s - t| - |t - b| - \mu \geq |s - t| - 2\mu.$$  

Finally, if $b \leq s, t \leq R$, again since $R - b \leq \mu$ it follows that $0 \leq s - b \leq \mu$ and $0 \leq t - b \leq \mu$. In particular, $|t - s| \leq |t - b| + |b - s| \leq 2\mu$ and we conclude that $\beta$ is a $2\mu$-rough geodesic.

Next, assume $R < b$. This time define $\beta: [0, R] \to X$ by $\beta(t) = \gamma(t)$ for $0 \leq t \leq R$, and $\beta(R) = \gamma(b) = y$. We claim that $\beta$ is a $2\mu$-rough geodesic $x \overset{\beta}{\to} y$. Clearly $\beta: x \overset{\beta}{\to} y$, and since $|R - b| \leq \mu$ whenever $t < R$,

$$d(\beta(t), \beta(R)) \leq |t - b| + \mu \leq |t - R| + |R - b| + \mu \leq |t - R| + 2\mu,$$

and similarly,

$$d(\beta(t), \beta(R)) \geq |t - b| - \mu \geq |t - R| - |R - b| - \mu \geq |t - R| - 2\mu,$$

so $\beta$ is a $2\mu$-rough geodesic as claimed. 

A space $(X, d)$ is $(\mu)$-roughly geodesic if for every $x, y \in X$ there exists a $\mu$-rough geodesic $\gamma: [0, d(x,y)] \to X$ $x \overset{\gamma}{\to} y$, and $(\mu)$-visual if there exists $o \in X$ such that every point in $X$ is contained in the image of a $\mu$-roughly geodesic ray issuing from $o$.

We end this section with a few clarifying remarks. A space $(X, d)$ has bounded growth at some scale if there exist constants $R > r > 0$ and $N \in \mathbb{N}$ such that any
open ball of radius $R$ in $X$ can be covered by $N$ open balls of radius $r$ in $X$; see [2]. This is used by Cao in the context of geodesic spaces in [3] and we note the following.

**Lemma 2.** If $(X, d)$ is a length space then it is uniformly coarsely proper if and only if it has bounded growth at some scale.

**Proof.** If $(X, d)$ is uniformly coarsely proper it has bounded growth at some scale. So suppose $(X, d)$ has bounded growth at some scale $R > r > 0$ and cover $B(x, R)$ by $N$ open balls $B(x_1, r), \ldots, B(x_N, r)$. Since $(X, d)$ is a length space, for each $y \in B(x, 2R - r)$ there exists $y' \in B(x, R)$ such that $d(y, y') \leq R - r$. Thus, for any $y \in B(x, 2R - r)$ we can find $y' \in B(x, R)$ and $x_i$ as above such that

$$d(x_i, y) \leq d(x_i, y') + d(y', y) \leq r + R - r = R.$$

In other words, $B(x_1, R), \ldots, B(x_N, R)$ cover $B(x, 2R - r)$, and it follows that $B(x, 2R - r)$ can be covered by $N^2$ balls of radius $r$. By induction, for any $n \in \mathbb{N}$, the ball $B(x, (n + 1)R - nr)$ can be covered by $N^{n+1}$ open balls of radius $r$. \qed

Being uniformly coarsely proper is an invariant under metric coarse equivalence by [6 Corollary 3.D.17]. For the readers convenience, we give a short proof for quasi-isometries proving an explicit estimate for the scale as well.

**Lemma 3.** Suppose $f : X \to X'$ is a $(\lambda, \mu)$-quasi-isometry between $(X, d)$ and $(X', d')$. If $(X, d)$ is uniformly coarsely proper for $R > r > r_0$ then $(X', d')$ is uniformly coarsely proper for $R' > r' > \lambda \mu + \mu + r_0 \lambda$.

**Proof.** Since $f : X \to X'$ is a $(\lambda, \mu)$-quasi-isometry, it is has a $(\lambda, 3\lambda \mu)$-quasi-isometric coarse inverse $g : X' \to X$ where $d'(f(g(y)), y) \leq \lambda \mu$ for all $y \in X'$; see [5]. Let $R' > \lambda \mu + \mu + r_0 \lambda$. We claim that any $B(y, R') \subseteq X'$ can be covered by $N'(R', r')$ open balls or radius $R' > r' > \lambda \mu + \mu + r_0 \lambda$. To begin

$$g(B(y, R')) \subseteq B(g(y), \lambda R' + 3\lambda \mu),$$

and the latter can be covered by $N = N(\lambda R' + 3\lambda \mu, s)$ balls $B(x_1, s), \ldots, B(x_N, s)$ of radius $s > r_0$ as $X$ is uniformly coarsely proper. Choose $s = \lambda^{-1}r' - \lambda^{-1}\mu - \mu$. Now $E = f(B(g(y), \lambda R' + 3\lambda \mu))$ is covered by the sets $f(B(x_i, s))$ and as

$$f(B(x_i, s)) \subseteq B(f(x_i), \lambda s + \mu) = B(f(x_i), r' - \mu)$$

the balls $B(f(x_i), r' - \mu)$ cover $E$. Now since $d'(f(g(y)), y) \leq \lambda \mu$ for all $y \in X'$

$$B(y, R') \subseteq \{x \in X' : d'(x, f(g(B(y, R')))) \leq \lambda \mu\} \subseteq \{x \in X' : d'(x, E) \leq \lambda \mu\},$$

and as $E$ is covered by the balls $B(f(x_i), r' - \mu)$, the set $\{x \in X' : d'(x, E) \leq \lambda \mu\}$ is covered by the balls $B(f(x_i), r' - \mu)$ covering $B(y, R)$ as well. Letting $N'(R', r') = N$ it follows that $(X', d')$ is uniformly coarsely proper for $R' > r' > \lambda \mu + \mu + r_0 \lambda$. \qed

The following appears in the proof of Corollary C.
Lemma 4. If $G$ is locally compact and compactly generated by $S$ then $(G, d_S)$ is uniformly coarsely proper.

Proof. By [7, Proposition 6.6] the word metric space $(G, d_S)$ is quasi-isometric to a connected metric graph $(X, d)$ of bounded valency implying it has bounded growth at some scale. Since $(X, d)$ is geodesic this implies that $(X, d)$ is uniformly coarsely proper by Lemma 2. The claim now follows since being uniformly coarsely proper is a quasi-isometry invariant.

3 The hyperbolic cone

The original construction of the hyperbolic cone is due to Bonk and Schramm who introduced in [2] the metric space $(\text{Con}(Z), \rho_{BS})$ over a bounded metric space $(Z, d)$ where $\text{Con}(Z) = Z \times [0, D]$ for $D = \text{diam}(Z)$ assuming that $D > 0$, and

$$\rho_{BS}((x,t),(y,s)) = 2 \log \left( \frac{d(x,y) + \max\{t,s\}}{\sqrt{ts}} \right).$$

We note that $(\text{Con}(Z), \rho_{BS})$ and $(\mathcal{H}(Z), \rho)$ are isometric where the isometry from $(\text{Con}(Z), \rho_{BS})$ to $(\mathcal{H}(Z), \rho)$ is given by $(x,t) \mapsto (x, \log D - \log t)$. We use this observation implicitly when making use of the results in [2].

3.1 Elementary structure of the hyperbolic cone

For every $0 \leq r < \infty$, single out the following subsets of $\mathcal{H}(Z)$:

$$B_r = Z \times [0,r), \quad \overline{B}_r = Z \times [0,r], \quad \text{and} \quad S_r = Z \times \{r\}.$$  

Lemma 5. Let $(Z, d)$ be a bounded space containing at least two points. Then

(i) the hyperbolic cone $(\mathcal{H}(Z), \rho)$ is $2\mu$-roughly geodesic for some $\mu \geq 0$,

(ii) for every $x \in Z$ the map $\sigma_x : [0, \infty) \to \mathcal{H}(Z)$ given by $\sigma_x(r) \mapsto (x,r)$ is a geodesic ray in $(\mathcal{H}(Z), \rho)$,

(iii) if $(Z, d)$ is uniformly coarsely connected then $\mathcal{H}(Z) \setminus B_r$ is uniformly coarsely connected.

Proof. (i) The claim follows by Lemma 1 observing that for every $x, y \in \mathcal{H}(Z)$ there exists a $\mu$-rough geodesic $\gamma : [a, b] \to \mathcal{H}(Z)$ $x \sim y$ by [2, Theorem 7.2].

(ii) Fix $x \in Z$ and let $0 \leq r \leq s$. The claim follows from observing that now

$$\rho(\sigma_x(r), \sigma_x(s)) = 2 \log \left( \frac{e^{-s}}{e^{-s} + e^{-t}} \right) = s - r.$$  

(iii) By (ii) we can assume that $t = s = r$. As $(Z, d)$ is $(D(e^{\varepsilon/2} - 1)/e^{\varepsilon})$-coarsely connected for every $\varepsilon > 0$ the space $(S_r, \rho|_{S_r})$ is $\varepsilon$-coarsely connected for every $\varepsilon > 0$ from which the claim follows. □
Let \( t \geq 0 \) and define the projections

\[
\pi_t : \mathcal{H}(Z) \to S_t \quad \text{by} \quad \pi_t(p, s) = (p, t), \quad \text{and} \quad h : \mathcal{H}(Z) \to [0, \infty) \quad \text{by} \quad h(p, s) = s.
\]

**Lemma 6.** \( \pi_t : \mathcal{H}(Z) \to S_t \) restricted to \( \mathcal{H}(Z) \setminus B_t \) is 1-Lipschitz.

**Proof.** Let \((p, r), (q, s) \in \mathcal{H}(Z) \setminus B_t \) and \( t \leq s \leq r \). The claim follows observing that

\[
\rho(\pi_t(p, r), \pi_t(q, s)) = 2 \log \left( \frac{d(p, q)}{e^{-tD}} + 1 \right) \leq 2 \log \left( \frac{d(p, q)}{e^{-sD}} + 1 \right)
\]

\[
= 2 \log \left( \frac{d(p, q) + \max \{ e^{-s}, e^{-r} \} D}{e^{-sD}} \right)
\]

\[
\leq 2 \log \left( \frac{d(p, q) + \max \{ e^{-s}, e^{-r} \} D}{e^{-(s+r)/2} D} \right) = \rho((p, r), (q, s)).
\]

\[ \square \]

**Lemma 7.** If \((p, r) \in \mathcal{H}(Z) \) and \( \delta > 0 \) then \( B((p, r), \delta) \subseteq Z \times (r - \delta, r + \delta) \). In particular if \( x, y \in B((p, r), \delta) \) then \( |h(x) - h(y)| < 2\delta \).

**Proof.** Let \((q, s) \in B((p, r), \delta) \). The claim follows observing that

\[
|r - s| = \rho((p, r), (p, s)) = 2 \log \left( \frac{\max \{ e^{-r}, e^{-s} \} D}{e^{-(r+s)/2} D} \right)
\]

\[
\leq 2 \log \left( \frac{d(p, q) + \max \{ e^{-r}, e^{-s} \} D}{e^{-(r+s)/2} D} \right) = \rho((p, r), (q, s)) < \delta.
\]

\[ \square \]

### 3.2 Intrinsic structure of the hyperbolic cone

By Lemma 5 the hyperbolic cone \((\mathcal{H}(Z), \rho)\) is \(2\mu\)-roughly geodesic for some \( \mu \geq 0 \) and we fix \( L(\mu) = 1 + 2\mu \geq 1 \). Define \( \rho_r : \mathcal{H}(Z) \setminus B_r \times \mathcal{H}(Z) \setminus B_r \to [0, \infty] \) for all \( r \geq 0 \) by

\[
\rho_r(x, y) = \inf \left\{ \sum_{i=0}^{n-1} \rho(y_i, y_{i+1}) : x = y_0, \ldots, y_n = y \text{ } L(\mu)-\text{sequence in } \mathcal{H}(Z) \setminus B_r \right\}.
\]

This replaces \( d_r \) in [3, Section 3]. An \( L(\mu) \)-sequence \( x \rightsquigarrow y \) in \( \mathcal{H}(Z) \setminus B_r \) is called an admissible sequence for \( \rho_r(x, y) \).

**Lemma 8.** If \((Z, d)\) is a bounded uniformly coarsely connected space containing at least two points then \( \rho_r \) is a metric on \( \mathcal{H}(Z) \setminus B_r \).

**Proof.** By Lemma 5 there exists an admissible sequence \( x \rightsquigarrow y \) for any \( x, y \in \mathcal{H}(Z) \setminus B_r \) so \( \rho_r(x, y) < \infty \). That \( \rho_r(x, y) = 0 \) if and only if \( x = y \) holds as \( \rho_r(x, y) = \rho(x, y) \) if \( \rho(x, y) \leq L(\mu) \). The rest is clear. \[ \square \]
The following is left as an elementary exercise in analysis.

**Lemma 9.** For any \( \varepsilon > 0 \) there exists a constant \( \kappa(\varepsilon) > 1 \) such that
\[
1 + e^{-\varepsilon t} \leq (1 + t)^{\kappa(\varepsilon) t}
\]
for all \( s \geq 0 \) and all \( t \in [0, e^s] \).

The following now generalises [3, Lemma 3.1].

**Proposition 10.** Suppose \((Z, d)\) is a bounded uniformly coarsely connected space containing at least two points \( x, y \in Z \) and \( \sigma_x : [0, \infty) \to \mathcal{H}(Z) \), \( \sigma_x(t) = (x, t) \), and \( \sigma_y : [0, \infty) \to \mathcal{H}(Z) \), \( \sigma_y(t) = (y, t) \). Then
\[
\rho_{r+t}(\sigma_x(r + t), \sigma_y(r + t)) \geq \kappa(L(\mu)) \rho_r(\sigma_x(r), \sigma_y(r)),
\]
for all \( r \geq 0 \) and all \( t \geq 0 \).

**Proof.** Without loss of generality suppose \( t > 0 \) and let \((p_i, t_i)\), be an admissible sequence for \( \rho_{r+t}(\sigma_x(r + t), \sigma_y(r + t)) \). Since \( \sigma_r \) is 1-Lipschitz by Lemma 6, the sequence \((p_i, r_i)\), is an admissible sequence for \( \rho_r(\sigma_x(r), \sigma_y(r)) \) and
\[
\rho((p_i, r), (p_{i+1}, r)) = 2 \log \left( \frac{d(p_i, p_{i+1}) + e^{-r}D}{e^{-r}D} \right) = 2 \log \left( 1 + e^{-r} \frac{d(p_i, p_{i+1})}{e^{-r}D} \right)
\]
\[
\leq 2 \log \left( 1 + \frac{d(p_i, p_{i+1})}{e^{-r}D} \right)^{\kappa(L(\mu))^{-1}} \leq \kappa(L(\mu))^{-1} \rho((p_i, t_i), (p_{i+1}, t_{i+1}))
\]
by Lemma 9, since \( \frac{d(p_i, p_{i+1})}{e^{-r}D} \leq e^{L(\mu)} \) observing that
\[
\log \left( 1 + \frac{d(p_i, p_{i+1})}{e^{-r}D} \right) \leq \log \left( 1 + \frac{d(p_i, p_{i+1})}{e^{-(t_i + t_{i+1})/2}D} \right)
\]
\[
\leq \log \left( \frac{\max\{e^{-t_i}, e^{-t_{i+1}}\}D + d(p_i, p_{i+1})}{e^{-(t_i + t_{i+1})/2}D} \right)
\]
\[
\leq \rho((p_i, t_i)(p_{i+1}, t_{i+1})) \leq L(\mu).
\]

The claim now follows observing that
\[
\rho_r(\sigma_x(r), \sigma_y(r)) \leq \sum_{i=0}^{n-1} \rho((p_i, r), (p_{i+1}, r)) \leq \kappa(L(\mu))^{-t} \sum_{i=0}^{n-1} \rho((p_i, t_i), (p_{i+1}, t_{i+1})),
\]
and taking the infimum over all admissible sequences for \( \rho_{r+t}(\sigma_x(r + t), \sigma_y(r + t)) \).

The following lemma now replaces [3, Assertion 3.1].

**Lemma 11.** Suppose \((Z, d)\) is a bounded uniformly coarsely connected space containing at least two points. If \( y = (p, r) \in \mathcal{H}(Z) \) and \( t \geq 2L(\mu) \) then
\[
B((p, r + t), t/(2L(\mu))) \subseteq A_y^{t/2} \times [r, r + 2t]
\]
where \( A_y^{t/2} = \{x \in S_r : \rho_r(y, x) < t/2\} \).
Proof. Towards a contradiction, suppose there exists a point

\[ z \in B((p, r + t), t/(2L(\mu))) \setminus \mathcal{A}^{t/2}_y \times [r, r + 2t]. \]  

(1)

Since \( 2L(\mu) \geq 2 \), by Lemma 7

\[ r + t/2 \leq r + t - t/(2L(\mu)) \leq h(z) \leq r + t + t/(2L(\mu)) \leq r + 3t/2 \]

for all \( t \geq 2L(\mu) \). As \( z \notin \mathcal{A}^{t/2}_y \times [r, r + 2t] \)

\[ \rho_r(y, \pi_r(z)) \geq t/2, \]  

(2)

for otherwise \( \pi_r(z) \in \mathcal{A}^{t/2}_y \) and \( h(z) < r + 2t \) which implies that \( z \in \mathcal{A}^{t/2}_y \times [r, r + 2t] \) after all, contradicting (1). By Proposition 10 we now have

\[ \rho_{r+t/2}((p, r + t/2), \pi_{r+t/2}(z)) \geq \kappa(L(\mu))^{t/2} \rho_r(y, \pi_r(z)) \geq \kappa(L(\mu))^{t/2} t/2 \]

(3)

for all \( t \geq 2L(\mu) \). Estimating the left-hand side from above we arrive at a contradiction completing the proof. Towards this,

\[ \rho_{r+t/2}((p, r + t/2), \pi_{r+t/2}(z)) \]

\[ \leq \rho_{r+t/2}((p, r + t/2), (p, r + t)) + \rho_{r+t/2}((p, r + t), \pi_{r+t/2}(z)) \]

\[ \leq t/2 + \rho_{r+t/2}((p, r + t), z) + \rho_{r+t/2}(z, \pi_{r+t/2}(z)) \]

\[ \leq t/2 + \rho_{r+t/2}((p, r + t), z) + 3t/2 - t/2 \]

\[ = 3t/2 + \rho_{r+t/2}((p, r + t), z). \]  

(4)

To estimate \( \rho_{r+t/2}((p, r + t), z) \) from above, let \( \gamma: [0, \rho((p, r + t), z)] \to \mathcal{H}(Z) \) be a \( 2\mu \)-rough geodesic \((p, r + t) \sim z\) by Lemma 5 fix \( m \in \mathbb{N} \) such that \( m - 1 \leq \rho((p, r + t), z) \leq m \), and let \( x_k = \gamma((k \rho((p, r + t), z)/m)) \) for \( k \in \{0, \ldots, m\} \subseteq \mathbb{N} \). We claim that \((x_k)_k\) is an admissible sequence for \( \rho_{r+t/2}((p, r + t), z) \). To begin, \((x_k)_k\) is an \( L(\mu) \)-sequence \((p, r + t) \sim z\) of length \( m \) since \( \Lambda \)

\[ \rho(x_k, x_{k+1}) \leq \rho((p, r + t), z)/m + 2\mu \leq 1 + 2\mu = L(\mu). \]

The sequence is admissible if \( x_k \in \mathcal{H}(Z) \setminus B_{r+t/2} \) for all \( k \in \{0, \ldots, m\} \). To see that this is the case, note that if \( h(x_k) < r + t/2 \) then \( \rho(x_0, x_k) > t/2 \) and

\[ t/2 < \rho(x_0, x_k) \leq k \rho((p, r + t), z)/m + 2\mu \leq \rho((p, r + t), z) + 2\mu \leq t/(2L(\mu)) + 2\mu \]

for all \( t \geq 2L(\mu) \) which is not possible. Thus,

\[ \rho_{r+t/2}((p, r + t), z) \leq mL(\mu) \leq (t/(2L(\mu)) + 1)L(\mu), \]

which together with (4) gives that

\[ \rho_{r+t/2}((p, r + t/2), \pi_{r+t/2}(z)) \leq 3t/2 + (t/(2L(\mu)) + 1)L(\mu) \leq 5L(\mu)t/2 \]

for all \( t \geq 2L(\mu) \). Together with (3) this implies that \( 5L(\mu) \geq \kappa(L(\mu))^{t/2} \) for all \( t \geq 2L(\mu) \) which is impossible. Thus, \( z \) as in (1) cannot exist and the claim follows. \( \square \)
3.3 Cao’s graph structure

In this section we approximate the hyperbolic cone by a graph structure due to Cao in [3]. Here by a graph we mean a 1-dimensional abstract simplicial complex $\Gamma$ whose 0-simplexes are its vertices and its 1-simplexes its edges. We write $\Gamma^{(0)}$ for the set of vertices and $\Gamma^{(1)}$ for the set of edges, and whenever $\{u, v\} \in \Gamma^{(1)}$ we say that $u$ and $v$ are neighbours and write $u \sim v$. Let $N(v) = \{u: u \sim v\}$. If for some constant $c \in \mathbb{N}$ it holds that $\#N(v) \leq c$ for all $v \in \Gamma^{(0)}$ we say that $\Gamma$ has bounded valency (by $c$).

A graph structure $(\Gamma X, d_\Gamma)$ on $(X, d)$ is a pair where $\Gamma X$ is a graph with vertex set $\Gamma X^{(0)} = X$ and $d_\Gamma: X \times X \to [0, \infty]$ is given by

1. $d_\Gamma(x, y) = 0$ if and only if $x = y$,  
2. $d_\Gamma(x, y) = n$ if the shortest edge path $x \lhd y$ is of length $n$,  
3. $d_\Gamma(x, y) = \infty$ if there is no edge path $x \lhd y$,

where an edge path $x \lhd y$ (of length $n \in \mathbb{N}$) is any finite sequence $x = x_0, \ldots, x_n = y$ of points in $X$ such that $x_i \sim x_{i+1}$ for all $0 \leq i \leq n - 1$.

Cao’s graph structure

Suppose $(\mathcal{H}(Z), \rho)$ is $2\mu$-roughly geodesic and uniformly coarsely proper for $R > r > r_b$ and fix $\delta > 0$ and $r_0 > 0$ such that

$$r_0/3 > \delta > c(\mu)(r_b + 1) \quad \text{and} \quad \kappa(L(\mu))r_0 > 8\delta N(10\delta, \delta/c(\mu)) \quad \text{($\theta_0$)}$$

where $c(\mu) = 2L(\mu) \geq 2$ hold. For $i \in \mathbb{N}$, let $N_{i r_0} = \{(p_{i, \alpha}, i r_0): \alpha \in \mathcal{I}_i\}$ be a maximal $\delta$-net in $(S_{i r_0}, \rho_{i r_0})$ indexed by $\mathcal{I}_i$ and write

$$q_{i, \alpha} = (p_{i, \alpha}, i r_0), \quad v_{i, \alpha} = \pi_{i r_0 + \delta}(q_{i, \alpha}), \quad A(q_{i, \alpha}) = B_{\rho_{i r_0}}(q_{i, \alpha}, 3\delta) \cap S_{i r_0}, \quad V(v_{i, \alpha}) = A(q_{i, \alpha}) \times [i r_0, (i + 1)r_0].$$

The graph structure $(\Gamma \mathcal{H}(Z), d_\Gamma)$ where

$$\Gamma \mathcal{H}(Z)^{(0)} = \bigcup_{i \in \mathbb{N}} \pi_{i r_0 + \delta}(N_{i r_0}) \quad \text{and} \quad \Gamma \mathcal{H}(Z)^{(1)} = \{\{u, v\}: V(u) \cap V(v) \neq \emptyset\}$$

is called Cao’s graph structure and $\Gamma \mathcal{H}(Z)$ the Cao graph.

3.4 Basic properties of Cao’s graph structure

We now prove that Cao’s graph structure approximates the hyperbolic cone.

Proposition 12. Let $(Z, d)$ be a bounded uniformly coarsely connected space containing at least two points with uniformly coarsely proper hyperbolic cone $(\mathcal{H}(Z), \rho)$. Then
(i) $\Gamma H(Z)^{(0)}$ is $\delta/c(\mu)$-separated in $(H(Z), \rho)$.

(ii) $\Gamma H(Z)^{(0)}$ is $2r_0$-cobounded in $(H(Z), \rho)$.

(iii) $(\Gamma H(Z), d_{\Gamma})$ is quasi-isometric to $(H(Z), \rho)$.

(iv) $\Gamma H(Z)^{(0)}$ is countable and $\Gamma H(Z)$ has bounded valency by $N(10r_0, \delta/c(\mu))$.

Proof. (i) Suppose $v \in \Gamma H(Z)^{(0)}$ where $v = \pi_{ir_0+\delta}(q)$ for $q \in N_{ir_0}$. By Lemma 11

$$B(v, \delta/c(\mu)) \subseteq A_{q}^{\delta/2} \times [h(q), h(q) + 2\delta],$$

and

$$(A_{q}^{\delta/2} \times [h(q), h(q) + 2\delta]) \cap (A_{p}^{\delta/2} \times [h(p), h(p) + 2\delta]) = \emptyset$$

if $q \in N_{ir_0}$ and $p \in N_{ir_0}$ are distinct points since $N_{ir_0}$ is a maximal $\delta$-net in $(S_{ir_0}, \rho_{ir_0})$ and $r_0 > 3\delta$ by 7. Hence $\rho(u, v) \geq \delta/c(\mu)$ if $u$ and $v$ are distinct vertices in the Cao graph. The claim now follows.

(ii) Let $z \in H(Z)$, $i \in \mathbb{N}$ such that $ir_0 \leq h(z) < (i + 1)r_0$, and $q \in N_{ir_0}$ such that $\rho_{ir_0}(\pi_{ir_0}(z), q) \leq \delta$. Now

$$\rho(z, \Gamma H(Z)^{(0)}) \leq \rho(z, \pi_{ir_0+\delta}(q)) \leq \rho(z, \pi_{ir_0}(z)) + \rho(\pi_{ir_0}(z), \pi_{ir_0+\delta}(q))$$

$$= r_0 + \rho_{ir_0}(\pi_{ir_0}(z), q) + \rho(q, \pi_{ir_0+\delta}(q)) \leq r_0 + \delta < 2r_0$$

since $r_0 > 3\delta$ by 7. The claim now follows.

(iii) By (ii) it suffices to show that the inclusion $(\Gamma H(Z)^{(0)}, d_{\Gamma}) \hookrightarrow (H(Z), \rho)$ is a quasi-isometric embedding. Explicitly, we prove that

$$\frac{1}{8r_0} \rho(u, v) \leq d_{\Gamma}(u, v) < 3r_0 \rho(u, v) \quad (5)$$

for all $u, v \in \Gamma H(Z)^{(0)}$. We begin by proving the right-hand side of (5). Let $u, v \in \Gamma H(Z)^{(0)}$ be distinct vertices, $\gamma: [0, r] \to H(Z)$ a $2\mu$-rough geodesic $u \searrow v$ where $r = \rho(u, v)$ which exists by Lemma 5 and $m \in \mathbb{N}$ such that $m - 1 < r \leq m$.

Now,

$$\rho(\gamma(kr/m), \gamma((k + 1)r/m)) \leq r/m + 2\mu \leq 1 + 2\mu = L(\mu)$$

for every $k \in \{0, \ldots, m - 1\} \subseteq \mathbb{N}$. For each $k \in \{0, \ldots, m\} \subseteq \mathbb{N}$ choose $q_{i(k), \alpha(k)} \in N_{i(k)r_0}$ such that $\pi_{i(k)+\delta}(q_{i(k), \alpha(k)}) = u$, $\pi_{i(m)+\delta}(q_{i(m), \alpha(m)}) = v$, and

$$i(k)r_0 \leq h(\gamma(kr/m)) < (i(k) + 1)r_0,$$

$$\rho_{i(k)r_0}(q_{i(k), \alpha(k)}, \pi_{i(k)r_0}\gamma(kr/m)) < \delta,$$

and write $v_{i(k), \alpha(k)} = \pi_{i(k)r_0+\delta}(q_{i(k), \alpha(k)})$ as usual. Let $i_0 = \min\{i(k), i(k + 1)\}$. Since the restriction of $\pi_{i_0r_0}$ to $H(Z) \setminus B_{ir_0}$ is 1-Lipschitz by Lemma 6

$$\rho(\pi_{i_0r_0}\gamma(i(k)r/m), \pi_{i_0r_0}\gamma(i(k + 1)r/m)) \leq L(\mu)$$

so $\rho_{i_0r_0}(\pi_{i_0r_0}\gamma(i(k)r/m), \pi_{i_0r_0}\gamma(i(k + 1)r/m)) \leq L(\mu)$. Choose $p, q \in N_{i_0r_0}$ such that

$$\rho_{i_0r_0}(p, \pi_{i_0r_0}\gamma(i(k)r/m)) < \delta,$$

$$\rho_{i_0r_0}(q, \pi_{i_0r_0}\gamma(i(k + 1)r/m)) < \delta,$$

11.
and note that by Lemma 6 and 7

\[ \rho_{i_0} (p, q) \leq \rho_{i_0} (p, \pi_{i_0} \gamma (i(k)r/m)) + \rho_{i_0} (\pi_{i_0} \gamma (i(k)r/m), \pi_{i_0} \gamma (i(k+1)r/m)) + \rho_{i_0} (\pi_{i_0} \gamma (i(k+1)r/m), q) < \delta + L(\mu) + \delta < 3\delta. \]

Thus \( p \in A(q) \) so \( \pi_{i_0} + \delta (p) \in V(\pi_{i_0} + \delta (q)) \) and \( d(\pi_{i_0} + \delta (p), \pi_{i_0} + \delta (q)) \leq 1 \) giving

\[ d_\Gamma (v_{i(k), \alpha(k)}, v_{i(k+1), \alpha(k+1)}) \leq 1 + d_\Gamma (v_{i(k), \alpha(k)}, \pi_{i_0} + \delta (p)) + d_\Gamma (\pi_{i_0} + \delta (q), v_{i(k+1), \alpha(k+1)}). \]

However, since \( |i(k) - i(k+1)| \leq 1 \)

\[ \pi_{i(k)r_0} (\gamma (kr/m)) \in V(v_{i(k), \alpha(k)}) \cap V(\pi_{i_0} + \delta (p)), \]

\[ \pi_{i(k+1)r_0} (\gamma (i(k+1)r/m)) \in V(v_{i(k+1), \alpha(k+1)}) \cap V(\pi_{i_0} + \delta (q)). \]

so \( d_\Gamma (v_{i(k), \alpha(k)}, \pi_{i_0} + \delta (p)) \leq 1 \) and \( d_\Gamma (v_{i(k+1), \alpha(k+1)}, \pi_{i_0} + \delta (q)) \leq 1 \), and altogether

\[ d_\Gamma (v_{i(k), \alpha(k)}, v_{i(k+1), \alpha(k+1)}) \leq 3. \]

Finally

\[ d_\Gamma (u, v) \leq \sum_{k=0}^{m-1} d_\Gamma (v_{i(k), \alpha(k)}, v_{i(k+1), \alpha(k+1)}) \leq 3m \leq 3(r+1) \]

\[ = 3 \rho (u, v) + 3 \leq 3(1 + c(\mu)/\delta) \rho (u, v) < 3r_0 \rho (u, v) \]

as \( \rho (u, v) \geq \delta/c(\mu) \) by (i) which gives the right-hand side of 59. To prove the left-hand side of 59 let \( u, v \in \Gamma H (Z)^{(0)} \) be two vertices. Without loss of generality, assume that \( d_\Gamma (u, v) = n \in \mathbb{N} \setminus \{0\} \) realised by the edge path \( u = x_0, \ldots, x_n = v \).

Since \( x_i \sim x_{i+1} \) it follows that \( V(x_i) \cap V(x_{i+1}) \neq \emptyset \) where \( \text{diam}(V(x_i)) < 4r_0 \) for all \( 0 \leq i \leq n-1 \).

Thus \( \rho (x_i, x_{i+1}) < 8r_0 \) for all \( 0 \leq i \leq n-1 \) and

\[ \rho (u, v) \leq \sum_{k=0}^{n-1} \rho (x_i, x_{i+1}) < 8r_0 n = 8r_0 d_\Gamma (u, v), \]

which gives the left-hand side of 59 and the claim follows.

(iv) For \( n \in \mathbb{N} \) let

\[ C_n = \left\{ B(v, \delta/c(\mu)) : v \in \Gamma H (Z)^{(0)} \text{ and } h(v) \leq nr_0 + \delta \right\} \text{ so } \Gamma H (Z)^{(0)} \subseteq \bigcup_{n \in \mathbb{N}} \bigcup C_n. \]

We claim that \( \# C_n < \infty \) for every \( n \in \mathbb{N} \) from which the claim then follows. By Lemma 7 for any \( z \in S_0 \) and \( n \in \mathbb{N} \)

\[ \bigcup C_n \subseteq B \left( z, nr_0 + \delta/c(\mu) + \delta + \text{diam} \left( S_{nr_0 + \delta/c(\mu) + \delta} \right) \right) \subseteq B(z, 5(n+2)r_0) \]
using (\[8\]) together with
\[
\text{diam}(S_{nr_0 + \delta/c(\mu) + \delta}) \leq 2 \log \left( \frac{D + e^{-(nr_0 + \delta/c(\mu) + \delta)} D}{e^{-(nr_0 + \delta/c(\mu) + \delta)} D} \right) = 2 \log(c^{nr_0 + \delta/c(\mu) + \delta} + 1).
\]

Let $R(n) = 5(n + 2)r_0$. Since $R(n) > r_\theta$ by (\[8\]) and $(H(Z), \rho)$ is uniformly coarsely proper $B(z, R(n))$ is covered by $N(R(n), \delta/c(\mu))$ balls of radius $\delta/c(\mu)$ and $\#C_n \leq N(R(n), \delta/c(\mu))$ by part (i) and it follows that $\Gamma H(Z)^{(0)}$ is countable.

To see that $\Gamma H(Z)$ has bounded valency note that if $v \sim u$ then $d_{\Gamma}(v, u) \leq 1$ and by inequality (\[5\]) above $\rho(v, u) \leq 8r_0$. In particular

\[
B(u, \delta/c(\mu)) \subseteq B(v, 9r_0 + \delta/c(\mu)),
\]

and $B(v, \delta/c(\mu)) \cap B(v, \delta/c(\mu)) = \emptyset$ by part (i) if $v$ and $u$ are distinct vertices in the Cao graph. Once again, since $(H(Z), \rho)$ is uniformly coarse proper

\[
\#N(v) \leq N(9r_0 + \delta/c(\mu), \delta/c(\mu)) \leq N(10r_0, \delta/c(\mu))
\]

from which the claim follows.

The following lemma now replaces \[3\] Assertion 3.2.

**Lemma 13.** Let $(Z, d)$ be a bounded uniformly coarsely connected space containing at least two points with uniformly coarsely proper hyperbolic cone $(H(Z), \rho)$. Then for any $i \in \mathbb{N}$ and any $q \in S_{ir_0}$

\[
\#V(i, q) \leq N(10\delta, \delta/c(\mu))
\]

where $V(i, q) = \{v_{i, \alpha} \in \Gamma H(Z)^{(0)}: \rho_{ir_0}(q_{i, \alpha}, q) < 4\delta\}$.

**Proof.** Suppose $v_{i, \beta} \in V(i, q)$. As $A_{q_{i, \beta}}^{1/2} \subseteq A(q_{i, \beta})$ and $\rho \leq \rho_{ir_0}$

\[
B(v_{i, \beta}, \delta/c(\mu)) \subseteq A(q_{i, \beta}) \times [ir_0, ir_0 + 2\delta] \subseteq B(q, 10\delta)
\]

by Lemma (11). By Proposition (12) the balls $B(v_{i, \alpha}, \delta/c(\mu))$ and $B(v_{i, \beta}, \delta/c(\mu))$ are disjoint if $v_{i, \alpha} \neq v_{i, \beta}$. Thus $\#V(i, q) \leq N(10\delta, \delta/c(\mu))$ since $(H(Z), \rho)$ is uniformly coarsely proper and $\delta/c(\mu) > r_\theta$.

We use this to find a uniform upper bound for the downward flow in the Cao graph.

**Proposition 14.** Let $(Z, d)$ be a bounded uniformly coarsely connected space containing at least two points with uniformly coarsely proper hyperbolic cone $(H(Z), \rho)$. Let $v \in \Gamma H(Z)^{(0)}$ and $N^-(v) = \{w \in \Gamma H(Z)^{(0)}: w \sim v \text{ and } h(w) = h(v) - r_0\}$. Then

\[
\#N^-(v) \leq N(10\delta, \delta/c(\mu))
\]

for all $v \in \Gamma H(Z)^{(0)}$. 

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Proof. Fix $v_{i,\alpha} \in \Gamma \mathcal{H}(Z)^{(0)}$. If $i = 0$ then $N^-(v_{i,\alpha}) = \emptyset$ so assume $i \geq 1$ and let $v_{j,\beta} \in N^-(v_{i,\alpha})$. Then $j = i - 1$ and $V(v_{i,\alpha}) \cap V(v_{i-1,\beta}) \neq \emptyset$. In particular, there exists $y \in A(q_{i,\alpha})$ such that $\rho_{r_0}(q_{i,\alpha}, y) < 3\delta$ and
\[
\rho_{r_0}(\pi_{(i-1)r_0}(y), \pi_{(i-1)r_0}(q_{i,\alpha})) \leq \kappa(L(\mu))^{-r_0} \rho_{r_0}(y, q_{i,\alpha}) < 3\delta \kappa(L(\mu))^{-r_0} < \delta
\]
by Proposition 10. As $y \in V(v_{i-1,\beta})$
\[
\rho_{r_0}(\pi_{(i-1)r_0}(q_{i-1,\beta}), \pi_{(i-1)r_0}(q_{i,\alpha})) \leq \rho_{r_0}(\pi_{(i-1)r_0}(q_{i-1,\beta}), \pi_{(i-1)r_0}(y)) + \rho_{r_0}(\pi_{(i-1)r_0}(y), \pi_{(i-1)r_0}(q_{i,\alpha})) < 3\delta + \delta = 4\delta
\]
so $q_{i-1,\beta} \in V(i - 1, \pi_{(i-1)r_0}(q_{i,\alpha}))$ and so $\# N^-(v_{i,\alpha}) \leq N(10\delta, \delta/c(\mu))$ by Lemma 13.

The following gives a uniform lower bound for the upward flow in the Cao graph.

**Proposition 15.** Let $(Z, d)$ be a bounded uniformly coarsely connected space containing at least two points with uniformly coarsely proper hyperbolic cone $(\mathcal{H}(Z), \rho)$. Let $v \in \Gamma \mathcal{H}(Z)^{(0)}$ and $N^+(v) = \{ w \in \Gamma \mathcal{H}(Z)^{(0)} : w \sim v \text{ and } h(w) = h(v) + r_0 \}$. Then
\[
2N(10\delta, \delta/c(\mu)) \leq \# N^+(v)
\]
for all $v \in \Gamma \mathcal{H}(Z)^{(0)}$ with $h(v) > \delta$.

**Proof.** Let $v_{i,\alpha} \in \Gamma \mathcal{H}(Z)^{(0)}$ such that $\delta < h(v_{i,\alpha}) = ir_0 + \delta$. Now $i \geq 1$ and
\[
\text{diam}_{\rho_{r_0}}(S_{ir_0}) \geq \text{diam}_{\rho}(S_{ir_0}) \geq 2\log \left( \frac{D + e^{-ir_0}D}{e^{-ir_0}D} \right) = 2\log \left( e^{ir_0} + 1 \right) \geq 2ir_0 \geq 2r_0 > 6\delta
\]
by (10). Fix $m = 2N(10\delta, \delta/c(\mu))$ and let $k \in \{0, \ldots, m\} \subseteq \mathbb{N}$. Now, for all $0 \leq k/m \leq 1$ there exist $x_{k/m} \in S_{ir_0}$ such that
\[
k/m - \varepsilon \leq \rho(q_{i,\alpha}, x_{k/m}) \leq k/m + \varepsilon
\]
for any $0 < \varepsilon < 1/(4m)$ since $(S_{ir_0}, \rho|_{S_{ir_0}})$ is uniformly coarsely connected. In particular if $k_1/m \neq k_2/m$, say $k_1 > k_2$, then
\[
\rho_{r_0}(x_{k_1/m}, x_{k_2/m}) \geq \rho(x_{k_1/m}, x_{k_2/m}) \geq \rho(q_{i,\alpha}, x_{k_2/m}) \geq k_1/m - \varepsilon - (k_2/m + \varepsilon) \geq (k_1 - k_2)/m - 2\varepsilon \geq 1/m - 2\varepsilon > 1/(2m)
\]
as $0 < \varepsilon < 1/(4m)$. For each $k \in \{0, \ldots, m\}$ let $y_{k} = \pi_{(i+1)r_0}(x_{k/m})$. As previously for $k_1 > k_2$
\[
\rho_{(i+1)r_0}(y_{k_1}, y_{k_2}) = \rho_{(i+1)r_0}(\pi_{(i+1)r_0}(x_{k_1/m}), \pi_{(i+1)r_0}(x_{k_2/m})) \geq \kappa(L(\mu))^{r_0} \rho_{r_0}(x_{k_1/m}, x_{k_2/m}) \geq \frac{\kappa(L(\mu))^{r_0}}{2m} \geq \frac{\kappa(L(\mu))^{r_0}}{4N(10\delta, \delta/c(\mu))} > 2\delta
\]
by Lemma\ref{lem:10} and \ref{lem:11}. Now for each \(y_k\) choose \(q_k \in N_{(i+1)r_0}\) such that
\[
\rho_{(i+1)r_0}(y_k, q_k) < \delta
\]
noting that \(\pi_{(i+1)r_0+\delta}(q_k) \sim v_{i,\alpha}\) since \(y_k \in V(\pi_{(i+1)r_0+\delta}(q_k)) \cap V(v_{i,\alpha})\). Moreover, \(\pi_{(i+1)r_0+\delta}(q_k) \neq \pi_{(i+1)r_0+\delta}(q_{k_2})\) whenever \(k_1 > k_2\) since
\[
\rho_{(i+1)r_0}(q_k, q_{k_2}) \geq \rho_{(i+1)r_0}(y_k, y_{k_2}) - \rho_{(i+1)r_0}(y_k, q_k) - \rho_{(i+1)r_0}(y_{k_2}, q_{k_2}) > 2\delta - 2\delta = 0.
\]
Thus \(\{0, \ldots, m\} \to N^+(v_{i,\alpha})\) for \(k \mapsto \pi_{(i+1)r_0+\delta}(q_k)\) is an injection and \(#N^+(v_{i,\alpha}) > 2N(10\delta, \delta/c(\mu))\) for \(i \geq 1\) proving the claim. \(\square\)

### 3.5 Non-amenability of the hyperbolic cone

Let \((\mathcal{H}(Z), \rho)\) be uniformly coarsely proper, let \(\mathbb{R}^{\Gamma\mathcal{H}(Z)^{(0)}} = \{f : \Gamma\mathcal{H}(Z)^{(0)} \to \mathbb{R}\}\), and let \(\Delta : \mathbb{R}^{\Gamma\mathcal{H}(Z)^{(0)}} \to \mathbb{R}^{\Gamma\mathcal{H}(Z)^{(0)}}\) be the graph Laplacian given by
\[
\Delta f(v) = \frac{1}{\#N(v)} \left( \sum_{w \sim v} f(w) \right) - f(v).
\]

**Lemma 16.** Let \((Z, d)\) be a bounded uniformly coarsely connected space containing at least two points with uniformly coarsely proper hyperbolic cone \((\mathcal{H}(Z), \rho)\). If there exist a Lipschitz function \(f : \Gamma\mathcal{H}(Z)^{(0)} \to \mathbb{R}\) and \(C > 0\) such that \(\Delta f(v) > C\) for every \(v \in \Gamma\mathcal{H}(Z)^{(0)}\) then \((\mathcal{H}(Z), \rho)\) is non-amenable.

**Proof.** By Proposition\ref{prop:12} the assumptions in \[3, Proposition 2.3\] hold so the Cheeger constant of \(\Gamma\mathcal{H}(Z)\) is strictly positive, equivalently, \((\Gamma\mathcal{H}(Z), d)\) is non-amenable. The claim follows as \((\mathcal{H}(Z), \rho)\) and \((\Gamma\mathcal{H}(Z), d)\) are quasi-isometric. \(\square\)

**Theorem A.** Let \((\mathcal{H}(Z), \rho)\) be the hyperbolic cone over a bounded space \((Z, d)\). If \((\mathcal{H}(Z), \rho)\) is uniformly coarsely proper and \((Z, d)\) consists of a finite union of uniformly coarsely connected components each containing at least two points then \((\mathcal{H}(Z), \rho)\) is non-amenable.

**Proof.** First suppose \((Z, d)\) is uniformly coarsely connected and contains at least two points. Since \(|h(v_{i,\alpha}) - h(v_{j,b})| = |ir_0 + \delta - jr_0 - \delta| \leq |i - j|r_0 \leq r_0\) whenever \(v_{i,\alpha} \sim v_{j,b}\) it follows that \(h = r_0\)-Lipschitz on \(\Gamma\mathcal{H}(Z)^{(0)}\). Moreover, if \(i \geq 1\)
\[
\frac{\Delta h(v_{i,\alpha})}{r_0} = \frac{1}{\#N(v_{i,\alpha})r_0} \sum_{v_{j,b} \sim v_{i,\alpha}} (h(v_{j,b}) - h(v_{i,\alpha})) = \frac{\#N^+(v_{i,\alpha}) - \#N^-(v_{i,\alpha})}{\#N(v_{i,\alpha})} \geq \frac{2N(10\delta, \delta/c(\mu)) - N(10\delta, \delta/c(\mu))}{N(10r_0, \delta/c(\mu))} \geq \frac{1}{N(10r_0, \delta/c(\mu))} > 0
\]
by Lemma\ref{lem:14} and Lemma\ref{lem:15} where \(#N(v_{i,\alpha}) \leq N(10r_0, \delta/c(\mu))\) for all \(v_{i,\alpha} \in \Gamma\mathcal{H}(Z)^{(0)}\) by Lemma\ref{lem:12}. If \(i = 0\) we have \(N^-(v_{0,\alpha}) = \emptyset\) and the same lower bound holds for \(\Delta h\). Thus \((\mathcal{H}(Z), \rho)\) is non-amenable by Lemma\ref{lem:16}.
Now suppose that $(Z, d)$ is a finite union of uniformly coarsely connected components $Z = Z_1 \sqcup \cdots \sqcup Z_n$ where each component $Z_i$ contains at least two points. To see that $(H(Z), \rho)$ is non-amenable let $\Gamma = \Gamma_1 \sqcup \cdots \sqcup \Gamma_n \subseteq H(Z)$ be a quasi-lattice in $(H(Z), \rho)$ such that $\Gamma_i \subseteq H(Z_i)$ is a quasi-lattice in $(H(Z_i), \rho|_{H(Z_i)})$, and let $F \subseteq \Gamma$ be any finite set and write $F_i = F \cap H(Z_i)$ so that $F = F_1 \sqcup \cdots \sqcup F_n$. By the first part of the proof each $(H(Z_i), \rho|_{H(Z_i)})$ is non-amenable, so for some constants $C_i > 0$ and $r_i > 0$ the isoperimetric inequality $\# F_i \leq C_i \# \partial_r F_i$ holds and hence

$$\# F = \# F_1 + \cdots + \# F_n \leq C_1 \# \partial_r F_1 + \cdots + C_n \# \partial_r F_n \leq C \# \partial_r F$$

for $C = \max\{C_1, \ldots, C_n\}$ and $r = \max\{r_1, \ldots, r_n\}$. The claim now follows.

References


