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ON ENDOMORPHISMS OF THE CUNTZ ALGEBRA WHICH
PREERVE THE CANONICAL UHF-SUBALGEBRA, II

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ABSTRACT. It was shown recently by Conti, Rørdam and Szymański that there exist endomorphisms \( \lambda_u \) of the Cuntz algebra \( \mathcal{O}_n \) such that \( \lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n \) but \( u \notin \mathcal{F}_n \), and a question was raised if for such a \( u \) there must always exist a unitary \( v \in \mathcal{F}_n \) with \( \lambda_u|_{\mathcal{F}_n} = \lambda_v|_{\mathcal{F}_n} \). In the present paper, we answer this question to the negative. To this end, we analyze the structure of such endomorphisms \( \lambda_u \) for which the relative commutant \( \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n \) is finite dimensional.

1. INTRODUCTION AND PRELIMINARIES

This paper is devoted to continuation of the line of investigation of exotic endomorphisms of the Cuntz algebras initiated in [4]. Our main result is solution of a question raised therein, see below for details. Our strategy is based on a detailed analysis of such endomorphisms \( \lambda_u \) of \( \mathcal{O}_n \) that globally preserve the core UHF subalgebra \( \mathcal{F}_n \) and have finite dimensional relative commutant \( \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n \), and builds on the earlier results in this direction obtained in [10].

The Cuntz algebra \( \mathcal{O}_n \), \( n \geq 2 \), is the \( C^* \)-algebra generated by isometries \( S_1, \ldots, S_n \) satisfying \( \sum_{i=1}^n S_i S_i^* = 1 \). It is a purely infinite, simple \( C^* \)-algebra, independent of the choice of generating isometries, [7]. We denote by \( W_n^k \) the set of \( k \)-tuples \( \mu = (\mu_1, \ldots, \mu_k) \) with \( \mu_m \in \{1, \ldots, n\} \), and by \( W_n \) the union \( \bigcup_{k=0}^\infty W_n^k \), where \( W_n^0 = \{0\} \). If \( \mu \in W_n^k \) then \( |\mu| = k \) is the length of \( \mu \). If \( \mu = (\mu_1, \ldots, \mu_k) \in W_n \) then \( S_\mu = S_{\mu_k} \cdots S_{\mu_1} \) (\( S_0 = 1 \) by convention) is an isometry in \( \mathcal{O}_n \). Every word in \( \{S_i, S_i^* \mid i = 1, \ldots, n\} \) can be uniquely expressed as \( S_\mu S_\nu^* \), for \( \mu, \nu \in W_n \) [7, Lemma 1.3].

The gauge action \( \gamma \) of the circle group \( \mathbb{T} \) on \( \mathcal{O}_n \) is defined by \( \gamma_z(S_i) = zS_i, \ z \in \mathbb{T} \). Let \( \mathcal{F}_n \) be the fixed point algebra of \( \gamma \). Denote \( \mathcal{F}_n^{(k)} := \text{span}\{S_\mu S_\nu^* \mid \mu, \nu \in W_n^k\} \). Then \( \mathcal{F}_n \) is generated by \( \mathcal{F}_n^{(k)}, \ k = 1, 2, \ldots, \) and each \( \mathcal{F}_n^{(k)} \) is isomorphic to the matrix algebra \( M_{n^k}(\mathbb{C}) \). Thus \( \mathcal{F}_n \) is isomorphic to the UHF-algebra of type \( n^{\infty} \), and hence it has a

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unique tracial state $\tau$. There exists a faithful conditional expectation $E : \mathcal{O}_n \to \mathcal{F}_n$, defined by integration with respect to the Haar measure on $\mathbb{T}$ as

$$E(x) = \int_\mathbb{T} \gamma_z(x) dz.$$ 

For each $k \in \mathbb{Z}$ we denote by $\mathcal{O}_n^{(k)}$ the corresponding spectral subspace for $\gamma$ in $\mathcal{O}_n$,

$$\mathcal{O}_n^{(k)} := \{ x \in \mathcal{O}_n | \gamma_z(x) = z^k, \, \forall z \in \mathbb{T} \}.$$ 

Thus, in particular, $\mathcal{O}_n^{(0)} = \mathcal{F}_n$.

The $C^*$-subalgebra of $\mathcal{O}_n$ generated by projections $P_\mu := S_\mu S_\mu^*$, $\mu \in W_n$, is a MASA (maximal abelian subalgebra) in $\mathcal{O}_n$. We call it the diagonal and denote $\mathcal{D}_n$, also writing $\mathcal{D}_n^k$ for $\mathcal{D}_n \cap \mathcal{F}_n^{(k)}$.

The canonical shift endomorphism $\varphi : \mathcal{O}_n \to \mathcal{O}_n$ is defined by

$$\varphi(x) = \sum_{i=1}^n S_i x S_i^*.$$ 

It is easy to see that $S_i x = \varphi(x) S_i$ and $x S_i^* = S_i^* \varphi(x)$ for all $x \in \mathcal{O}_n$.

As shown by Cuntz in [8], there exists a bijective correspondence between unitaries in $\mathcal{O}_n$ (whose collection is denoted $\mathcal{U}(\mathcal{O}_n)$) and unital $*$-endomorphisms of $\mathcal{O}_n$, determined by

$$\lambda_u(S_i) = u S_i, \quad i = 1, \ldots, n.$$ 

We have $\text{Ad}(u) = \lambda_u(\varphi(u^*))$ for all $u \in \mathcal{U}(\mathcal{O}_n)$. If $u \in \mathcal{U}(\mathcal{O}_n)$ then for each positive integer $k$ we denote

$$u_k = u \varphi(u) \cdots \varphi^{k-1}(u).$$ 

Here $\varphi^0 = \text{id}$, and we agree that $u_k^*$ stands for $(u_k)^*$. If $\alpha$ and $\beta$ are multi-indices of length $k$ and $m$, respectively, then $\lambda_u(S_{\alpha} S_{\beta}^*) = u_k S_{\alpha} S_{\beta}^* u_m^*$.

The Cuntz correspondence between unitaries and endomorphisms of $\mathcal{O}_n$ provides a very efficient tool for investigations of the latter. In this note, we continue the study (by several authors) of those unital endomorphisms which globally preserve the UHF-subalgebra $\mathcal{F}_n$. For example, such endomorphisms were analyzed from the point of view of the Jones-Kosaki-Watatani index theory in [12] and [3], and in connection with Hopf algebra actions in [9] and [13]. More recently, interesting combinatorial approaches to the study of permutative endomorphisms of this type have been found (e.g. see [6], [2], and a survey article [1]).

It was observed by Cuntz in his groundbreaking paper [8] that an automorphism $\lambda_u$ globally preserves $\mathcal{F}_n$ if and only if $u \in \mathcal{F}_n$. The situation is more complex with proper endomorphisms. Clearly, $u \in \mathcal{F}_n$ implies $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$, [8], but the question if the converse is true remained open until very recently. Indeed, it was shown in [4] that
there exist unitaries $u$ in $\mathcal{O}_n \setminus \mathcal{F}_n$ such that $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$. All such examples found therein were of the form $u = wv$ with $w \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ and $v \in \mathcal{F}_n$. In such a case, we also have $\lambda_u(x) = \lambda_v(x)$ for all $x \in \mathcal{F}_n$. Thus a natural question arises if such a factorization of $u$ is always possible whenever $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ (cf. [11, Problem 5.3]).

Some progress towards answering this question has been made recently in [10] and [11]. The main purpose of the present paper is to develop definite methods for analyzing endomorphisms $\lambda_u$ of $\mathcal{O}_n$ satifying $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ and an additional condition that the relative commutant $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$ be finite dimensional. In particular, we give a verifiable criterion for determining if the aforementioned decomposition is possible, Corollary 3.4. Based on this criterion, in Section 3 we give an explicit example of a unitary $u \in \mathcal{O}_2$ such that $\lambda_u(\mathcal{F}_2) \subseteq \mathcal{F}_2$ and $\text{dim } \lambda_u(\mathcal{F}_2)' \cap \mathcal{F}_2 < \infty$ but there is no unitary $v \in \mathcal{F}_2$ such that $\lambda_u|_{\mathcal{F}_2} = \lambda_v|_{\mathcal{F}_2}$, see Example 3.6. In this way, we answer to the negative the question raised in [4] and [1].

2. THE RELATIVE COMMUTANTS

We begin by recording for future references a few simple facts, essentially contained in [4] and [10].

Proposition 2.1. Let $u \in \mathcal{U}(\mathcal{O}_n)$. Then the following conditions are equivalent.

1. $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$,
2. $\lambda_{\gamma_z(u)}|_{\mathcal{F}_n} = \lambda_u|_{\mathcal{F}_n}$ for all $z \in \mathbb{T}$,
3. $u\gamma_z(u^*) \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ for all $z \in \mathbb{T}$.

Proof. Clearly, $\gamma_z \lambda_u \gamma_z^{-1} = \lambda_{\gamma_z(u)}$ for all $z \in \mathbb{T}$. Thus condition (2) above is equivalent to $\gamma_z \lambda_u|_{\mathcal{F}_n} = \lambda_u|_{\mathcal{F}_n}$ for all $z \in \mathbb{T}$. Obviously, this holds if and only if $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$. That is, (1) is equivalent to (2).

It is an immediate consequence of Proposition 2.1 and Proposition 4.7 from [11] that $\lambda_u|_{\mathcal{F}_n} = \lambda_u|_{\mathcal{F}_n}$ if and only if $vu^* \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$. This gives (2) is equivalent to (3). □

Proposition 2.2. If $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ and $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n = \mathbb{C}1$, then $u \in \mathcal{F}_n$.

Proof. If $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ and $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n = \mathbb{C}1$, then $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n = \mathbb{C}1$ as well, [10, Theorem 1.1]. As shown in [4], this implies that $u \in \mathcal{F}_n$. □

Proposition 2.3. Let $u$ be a unitary in $\mathcal{O}_n$. Then $u = wv$ for some $w \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ and a unitary $v \in \mathcal{F}_n$ if and only if there exists a unitary $y \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ such that $u\gamma_z(u^*) = y\gamma_z(y^*)$ for all $z \in \mathbb{T}$.

Proof. If $u = wv$ for some $w \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ and $v \in \mathcal{U}(\mathcal{F}_n)$, then $u\gamma_z(u^*) = w\gamma_z(w^*)$, and it suffices to put $y = w$. □
Conversely, if there exists a unitary \( y \in \lambda_u(F_n)' \cap \mathcal{O}_n \) such that \( u \gamma_z(u^*) = y \gamma_z(y^*) \) for all \( z \in \mathbb{T} \) then \( y^*u \) is fixed by all \( \gamma_z \). Thus \( y^*u \in F_n \) and it suffices to put \( w = y \) and \( v = y^*u \).

From now on, we make a standing assumption that \( u \in \mathcal{U}(\mathcal{O}_n) \) is such that
\[
(2) \quad \lambda_u(F_n) \subseteq F_n \quad \text{and} \quad \dim \lambda_u(F_n)' \cap F_n < \infty.
\]

As shown in [10], assumption (2) above entails a number of important consequences, which we summarize as follows.

- We also have \( \dim \lambda_u(F_n)' \cap \mathcal{O}_n < \infty \).
- There exists a unitary group \( \{u_z\}_{z \in \mathbb{T}} \) in the center of \( \lambda_u(F_n)' \cap F_n \) such that \( \text{Ad} u_z(x) = \gamma_z(x) \) for all \( x \in \lambda_u(F_n)' \cap \mathcal{O}_n \).
- Minimal projections in \( \lambda_u(F_n)' \cap F_n \) are minimal in \( \lambda_u(F_n)' \cap \mathcal{O}_n \) as well. Thus \( \lambda_u(F_n)' \cap \mathcal{O}_n \) contains a MASA consisting of projections in \( \lambda_u(F_n)' \cap F_n \).

The proof of the following theorem is modelled after that of [10] Lemma 1.11.

**Theorem 2.4.** Let \( u \in \mathcal{U}(\mathcal{O}_n) \) be such that \( \lambda_u(F_n) \subseteq F_n \) and \( \dim \lambda_u(F_n)' \cap F_n < \infty \). Then there exist unitaries \( w \in \lambda_u(F_n)' \cap \mathcal{O}_n \) and \( v \in \mathcal{O}_n \), and a unitary group \( \{v_z\}_{z \in \mathbb{T}} \subseteq \lambda_u(F_n)' \cap F_n \) satisfying \( u = wv \) and \( \gamma_z(v) = v_zv \) for all \( z \in \mathbb{T} \).

**Proof.** At first we note that \( u \gamma_z(u^*)u_z \) is a unitary group in \( \lambda_u(F_n)' \cap \mathcal{O}_n \). Indeed,
\[
(u \gamma_z(u^*)u_z)(u \gamma_z(u^*)u_z^*) = u \gamma_z(u^*)(\text{Ad} u_z)(u)u_z \gamma_z(u^*)u_z^* = u \gamma_z(u^*)^* u_z \gamma_z(u^*) u_z^* = u \gamma_z^z(u^*) u_z^z.
\]

Since \( \dim \lambda_u(F_n)' \cap \mathcal{O}_n < \infty \), this unitary group may be diagonalized. On the other hand, \( \lambda_u(F_n)' \cap \mathcal{O}_n \) contains a MASA composed of projections in \( \lambda_u(F_n)' \cap F_n \). Thus, there exists a unitary \( w \in \lambda_u(F_n)' \cap \mathcal{O}_n \) such that \( y_z := w^*(u \gamma_z(u^*)u_z)w \) is a unitary group in \( \lambda_u(F_n)' \cap F_n \). Since each \( u_z \) is in the center of \( \lambda_u(F_n)' \cap F_n \), the unitary groups \( \{y_z\}_{z \in \mathbb{T}} \) and \( \{u_z\}_{z \in \mathbb{T}} \) commute.

Set \( v_z := u_z y_z^*, \ z \in \mathbb{T} \), and \( v := w^*u \). Then \( v_z \) is a unitary group in \( \lambda_u(F_n)' \cap F_n \) and
\[
\gamma_z(v) = \gamma_z(w^* u) = u_z (u_z^* \gamma_z(w^* u) u_z^*) w^* u = u_z w^* u = v_z v.
\]
for all \( z \in \mathbb{T} \). This completes the proof. \( \square \)

We keep the notation from Theorem 2.4 assuming that unitaries \( w, v \) and \( v_z \) have the properties described therein. Thus, in particular, \( \lambda_u|_{F_n} = \lambda_u|_{F_n} \) by [4] Proposition 2.1. Consequently, \( \text{Ad} v \circ \varphi \) is an automorphism of \( \lambda_u(F_n)' \cap \mathcal{O}_n \), by [4] Proposition 2.3 and Lemma 2.4.

**Lemma 2.5.** With unitaries \( u, v, v_z \) and \( u_z \) as above, put
\[
X_z := (\text{Ad} v \circ \varphi)(u_z)u_z^*v_z.
\]
Then \( \{X_z\}_{z \in T} \) is a unitary group in the center of \( \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n \), and we have
\[
\gamma_z(v) = X_z u_z (\text{Ad} v \circ \varphi)(u^*_z)v.
\]

**Proof.** For each \( x \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n \), we see that
\[
\begin{align*}
    u_z (\text{Ad} v \circ \varphi)(x) u_z^* &= \gamma_z(v \varphi(x) v^*) = \gamma_z(v) \gamma_z(\varphi(x)) \gamma_z(v)^* = \gamma_z(v) \varphi(\gamma_z(x)) \gamma_z(v)^* \\
    &= v_z v \varphi(u_z x u_z^*) v_z^* v_z^* = v_z v \varphi(u_z) v^* v \varphi(x) v^* v \varphi(u_z^*) v^* v_z^* \\
    &= v_z \text{Ad}(\varphi(u_z) v^*)((\text{Ad} v \circ \varphi)(x)) v_z^*.
\end{align*}
\]
Hence, we have
\[
\text{Ad}(v_z u_z)((\text{Ad} v \circ \varphi)(x)) = \text{Ad}((\text{Ad} v \circ \varphi)(u_z))((\text{Ad} v \circ \varphi)(x)).
\]
Since \( \text{Ad} v \circ \varphi \) is an automorphism of \( \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n \), this shows that
\[
\text{Ad}(v_z u_z) = \text{Ad}((\text{Ad} v \circ \varphi)(u_z)) \quad \text{on} \quad \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n.
\]
Consequently, \( X_z \) belongs to the center of \( \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n \).

Now, \( \{u_z\}_{z \in T} \) and \( \{v_z\}_{z \in T} \) are commuting unitary groups, and both commute with \( X_z \), by the above argument. Therefore the unitary group \( (\text{Ad} v \circ \varphi)(u_z) = X_z u_z v_z^* \) commutes with both of them. Consequently, \( X_z \) being a product of three mutually commuting unitary groups itself is a unitary group.

The final claim of the lemma now follows from the fact that \( \gamma_z(v) = v_z v \). \( \square \)

Before proceeding further, we introduce the following notation. For \( x \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n \) and \( k \in \mathbb{N} \), we set
\[
(4) \quad x^{(k)} := x(\text{Ad} v \circ \varphi)(x)(\text{Ad} v \circ \varphi)^2(x) \cdots (\text{Ad} v \circ \varphi)^{k-1}(x).
\]

**Lemma 2.6.** With unitaries \( u, v, v_z \) and \( u_z \) as above, and \( x \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n \), for all \( g \in \mathcal{U}(\mathcal{O}_n) \), \( z \in T \) and \( k \in \mathbb{N} \) we have the following identities.

(i) \( \gamma_z(v_k) = v_z^{(k)} v_k \),

(ii) \( (\text{Adg} \circ \varphi)(x) = g_k \varphi^k(x) g_k^* \),

(iii) \( v_z^{(k)} = X_z^{(k)} u_z (\text{Ad} \circ \varphi)^k(u_z^*) \),

(iv) \( (gv)_k = g^{(k)} v_k \).

**Proof.** In all three cases, we proceed by induction on \( k \).

Ad (i). Case \( k = 1 \) is the identity \( \gamma_z(v_1) = \gamma_z(v) = v_z v = v_z^{(1)} v_1 \) from Theorem 2.4. For the inductive step, we calculate
\[
\gamma_z(v_{k+1}) = \gamma_z(v_k \varphi^k(v)) = \gamma_z(v_k) \varphi^k(\gamma_z(v)) = v_z^{(k)} v_k \varphi^k(v_z) v_k^* v_k \varphi^k(v) = v_z^{(k+1)} v_k v_k^* v_k \varphi^k(v) = v_z^{(k+1)} v_{k+1}.
\]
In this calculation we used identity (ii) of the present lemma, whose proof does not depend on (i).
Ad (ii). Case \( k = 1 \) is clear. For the inductive step, we have
\[
(\text{Ad } g \circ \varphi)^{k+1} = (\text{Ad } g \circ \varphi)(g_k \varphi^k(x)g_k^*) = g \varphi(g_k \varphi^k(x) \varphi(g_k^*)g_k^* = g_{k+1} \varphi^{k+1}(x)g_{k+1}^*.
\]

Ad (iii). Case \( k = 1 \) is clear. For the inductive step, we see that
\[
v_z^{(k+1)} = v_z^{(k)}(\text{Ad } v \circ \varphi)^k(v_z) = X_z^{(k)}u_z(\text{Ad } v \circ \varphi)^k(u_z^*) \text{Ad } v \circ \varphi)^k(v_z)
\]
\[
= X_z^{(k)}u_z(\text{Ad } v \circ \varphi)^k(u_z^* v_z) = X_z^{(k)}u_z(\text{Ad } v \circ \varphi)^k(X_z(\text{Ad } v \circ \varphi)(u_z^*)
\]
\[
= X_z^{(k)}u_z(\text{Ad } v \circ \varphi)^k(X_z(\text{Ad } v \circ \varphi)^{k+1}(u_z^*) = X_z^{(k+1)}u_z(\text{Ad } v \circ \varphi)^{k+1}(u_z^*).
\]

Ad (iv). Case \( k = 1 \) is clear. For the inductive step, we calculate using part (ii) above,
\[
(gv)^{k+1} = (gv)_k \varphi^{k+1}(gv) = g_k(v_k \varphi^k(g) v_k^*)v_k \varphi^k(v) = g^{(k+1)}u^{k+1},
\]
and this completes the proof.

The following lemma provides a key step in the proof of our second main result, Theorem 2.4 below. We continue keeping the notation of Theorem 2.4. Here we remark that since \( v = w^* v, w \in \lambda_n(F_n)' \cap \mathcal{O}_n \), and \( \text{Ad } v \circ \varphi \) is an automorphism of \( \lambda_n(F_n)' \cap \mathcal{O}_n \), we see that \( \text{Ad } v \circ \varphi = \text{Ad } w^* \circ (\text{Ad } u \circ \varphi) \) is an automorphism of \( \lambda_n(F_n)' \cap \mathcal{O}_n \) as well.

We also note that for each positive integer \( k \), \( \{X_z^{(k)}\}_{z \in T} \) is a unitary group in the center of \( \lambda_n(F_n)' \cap \mathcal{O}_n \).

**Lemma 2.7.** With unitaries \( u, v, v_z \) and \( u_z \) as above, there exist a positive integer \( k \) and a unitary \( U \in \lambda_n(F_n)' \cap \mathcal{O}_n \) such that
\[
(\text{Ad } v \circ \varphi)^k(x) = \text{Ad } U(x) \quad \text{for all } x \in \lambda_n(F_n)' \cap \mathcal{O}_n.
\]

Then \( X_z^{(k)} = 1 \). Furthermore, for such \( U \) and \( k \), we have \( U^* v_k \in F_n \).

**Proof.** Since \( \text{Ad } v \circ \varphi \) is an automorphism of a finite dimensional C*-algebra \( \lambda_n(F_n)' \cap \mathcal{O}_n \), its restricts to the center has finite order. Thus there exists a positive integer \( k \) and a unitary \( U \in \lambda_n(F_n)' \cap \mathcal{O}_n \) such that \( (\text{Ad } v \circ \varphi)^k = \text{Ad } U \) on \( \lambda_n(F_n)' \cap \mathcal{O}_n \). We claim that \( U^* v_k \in F_n \).

Indeed, by Lemma 2.6 for all \( z \in T \) we have
\[
\gamma_z(v_k) = v_z^{(k)}v_k = X_z^{(k)}u_z(\text{Ad } v \circ \varphi)^k(u_z^* v_k = X_z^{(k)}u_z U u_z^* U^* v_k = X_z^{(k)} \gamma_z(U) U^* v_k,
\]
and this yields
\[
\gamma_z(U^* v_k) = X_z^{(k)} U^* v_k.
\]
Since \( \{X_z^{(k)}\}_{z \in T} \) is a unitary group in the center of \( \lambda_n(F_n)' \cap \mathcal{O}_n \), there exists a partition of unity \( 1 = \sum_i p_i \) in \( Z(\lambda_n(F_n)' \cap \mathcal{O}_n) \) and integers \( k_i \) such that
\[
X_z^{(k_i)} = \sum_i z^{k_i} p_i.
\]
We have \((\text{Ad} \circ \varphi)^k(p_i) = U p_i U^* = p_i\) for all \(i\). Combining this with part (ii) of Lemma 2.6, we get

\[
(6) \quad v_k^* p_i v_k = \varphi^k(p_i).
\]

We want to show that \(k_i = 0\) for all \(i\). Suppose for a moment this is not the case and let \(k_i > 0\) for some \(i\). We set \(K := p_i U^* v_k(S_1^*)^{k_i}\). Since \(p_i\) being in \(\mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n)\) belongs to \(\mathcal{F}_n\) as well, it follows from identity (6) above that

\[
\gamma_z(K) = \gamma_z(p_i U^* v_k(S_1^*)^{k_i}) = p_i X_z^{(k)} U^* v_k \gamma_z((S_1^*)^{k_i}) = z^{k_i} p_i U^* v_k (z^{-k_i}(S_1^*)^{k_i}) = K.
\]

Hence \(K\) belongs to \(\mathcal{F}_n\). We have \(KK^* = p_i\). On the other hand, using identity (6) we get

\[
K^* K = S_1^{k_i} v_k^* p_i v_k (S_1^*)^{k_i} = S_1^{k_i} \varphi^k(p_i)(S_1^*)^{k_i} = \varphi^{k+k_i}(p_i) S_1^{k_i} (S_1^*)^{k_i}.
\]

It easily follows that \(\tau(K K^*) > \tau(K^* K)\), which is a contradiction. A similar argument applies in the case \(k_i < 0\). Hence \(k_i = 0\) for all \(i\) and thus \(X_z^{(k)} = 1\). Now, identity (5) implies that \(U^* v_k\) is fixed by the gauge action and hence belongs to \(\mathcal{F}_n\). \(\square\)

Now, we are ready to prove the second main result of this paper.

**Theorem 2.8.** Let \(u \in \mathcal{U}(\mathcal{O}_n)\) be such that \(\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n\) and \(\dim \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n < \infty\). Then there exist a positive integer \(k\) and unitaries \(W \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n\) and \(V \in \mathcal{F}_n\) such that \(u_k = WV\).

**Proof.** By Theorem 2.4 and Lemma 2.7, there exist unitaries \(w, U \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n\), a unitary group \(\{v_z\}_{z \in \mathbb{T}}\) in \(\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n\) and a positive integer \(k\) satisfying \(u = wv\), \(\gamma_z(v) = v_z v\), \(U^* v_k \in \mathcal{F}_n\). By part (iv) of Lemma 2.6 we have \(w^{(k)} v_k = u_k\). Thus to complete the proof, it suffices to put \(W := w^{(k)} U\) and \(V := U^* v_k\). \(\square\)

It was observed in [1] (just above Remark 4.4) that if \(\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n\) and \(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n = \mathbb{C}1\) then \(u \in \mathcal{F}_n\). The following corollary gives a sharp strengthening of that result.

**Corollary 2.9.** Let \(u\) be a unitary in \(\mathcal{O}_n\). If \(\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n\), \(\dim \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n < \infty\) and the automorphism \(\text{Ad} u \circ \varphi\) of \(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n\) is inner, then there exist a unitary \(w \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n\) and a unitary \(v \in \mathcal{F}_n\) such that \(u = wv\), and hence also \(\lambda_u|_{\mathcal{F}_n} = \lambda_v|_{\mathcal{F}_n}\). In particular, this is the case whenever \(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n\) is a factor.

**Remark 2.10.** The assumption in Corollary 2.9 above that the automorphism \(\text{Ad} u \circ \varphi\) of \(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n\) be inner, is equivalent to demanding existence of a unitary \(g\) in the relative commutant \(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n\) such that

\[
\lambda_{gu}(\mathcal{F}_n)' \cap \mathcal{O}_n = \lambda_{gu}(\mathcal{O}_n)' \cap \mathcal{O}_n.
\]

Indeed, if \(\text{Ad} u \circ \varphi\) is inner then \(\text{Ad} gu \circ \varphi = \text{id}\) for a suitable unitary \(g\) in \(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n\). Hence \(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n = \lambda_{gu}(\mathcal{F}_n)' \cap \mathcal{O}_n = \lambda_{gu}(\mathcal{O}_n)' \cap \mathcal{O}_n\). Conversely, if \(\lambda_{gu}(\mathcal{F}_n)' \cap \mathcal{O}_n = \lambda_{gu}(\mathcal{O}_n)' \cap \mathcal{O}_n\), then \(\text{Ad} gu \circ \varphi = \text{id}\) in \(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n\), and hence \(\text{Ad} u \circ \varphi = \text{id}\) as well.
Let
\begin{equation}
\lambda_{gu}(\mathcal{O}_n)' \cap \mathcal{O}_n \text{ then } \text{Ad } gu \circ \varphi = \text{id on } \lambda_u(F_n)' \cap \mathcal{O}_n = \lambda_{gu}(F_n)' \cap \mathcal{O}_n, \text{ and hence Ad } u \circ \varphi \text{ is inner.}
\end{equation}

Remark 2.11. We remark that the implication in Corollary 2.9 above cannot be reversed. In fact, there exist unitaries \( u \in F_n \) such that \( \lambda_u(F_n)' \cap \mathcal{O}_n \) is finite dimensional and the automorphism \( \text{Ad } u \circ \varphi \) is outer on \( \lambda_u(F_n)' \cap \mathcal{O}_n \). For example, take
\begin{equation}
u = S_{22}S_{11}^* + S_{12}S_{22}^* + S_{11}S_{12}^* + P_2,
\end{equation}
a permutative unitary in \( F_2 \). Then \( \text{Ad } u \circ \varphi \) is outer on \( \lambda_u(F_2)' \cap \mathcal{O}_2 \). For otherwise let \( h \) be a unitary in \( \lambda_u(F_2)' \cap \mathcal{O}_2 \) such that \( \text{Ad } u \circ \varphi = \text{Ad } h \) on \( \lambda_u(F_2)' \cap \mathcal{O}_2 \). Then \( \text{Ad } u \circ \varphi(h) = h \) and thus \( h \in \lambda_u(\mathcal{O}_2)' \cap \mathcal{O}_2 \). But it can be shown that \( \lambda_u \) is irreducible on \( \mathcal{O}_2 \) (e.g., see [3], where this endomorphism is denoted \( \rho_{1,2} \)), and hence \( \lambda_u(\mathcal{O}_2)' \cap \mathcal{O}_2 = \mathbb{C}1 \). Thus \( h \) is a scalar and consequently \( \text{Ad } u \circ \varphi \) is identity on \( \lambda_u(F_2)' \cap \mathcal{O}_2 \). This however is not the case, since one can calculate directly that \( \text{Ad } u \circ \varphi \) permutes \( P_1 \) and \( P_2 \), and both these projections are in \( \lambda_u(F_2)' \cap \mathcal{O}_2 \).

We want to elaborate a little bit the statement of Theorem 2.8 above. We continue keeping our standing assumption (2).

Lemma 2.12. Let \( \alpha \) be an automorphism of \( \lambda_u(F_n)' \cap \mathcal{O}_n \) and let \( k \in \mathbb{N} \) be such that \( \alpha^k \) acts trivially on \( Z(\lambda_u(F_n)' \cap \mathcal{O}_n) \). Then there exists a MASA \( D \) of \( \lambda_u(F_n)' \cap F_n \) and a unitary \( g \in \lambda_u(F_n)' \cap \mathcal{O}_n \) such that
\begin{enumerate}
\item \( (\text{Ad } g \circ \alpha)^k = \text{id, and} \\
\item \( (\text{Ad } g \circ \alpha)(D) = D. \)
\end{enumerate}

Proof. Automorphism \( \alpha \) permutes the finitely many minimal central projections of \( \lambda_u(F_n)' \cap \mathcal{O}_n \). Write this permutation as a product of disjoint cycles. Clearly, it suffices to prove the lemma for each cycle separately. Thus we may simply assume that \( \alpha \) acts transitively on minimal projections \( p_1, p_2, \ldots, p_l \) in \( Z(\lambda_u(F_n)' \cap \mathcal{O}_n) \), so that \( \alpha(p_i) = p_{i+1} \), with \( p_{l+1} = p_1 \). Let \( \{ e^{(i)}_{r,s} \} \) be matrix units of the full matrix algebra \( p_i(\lambda_u(F_n)' \cap \mathcal{O}_n) \), such that all \( e^{(i)}_{r,s} \) are in \( \lambda_u(F_n)' \cap F_n \). Then \( D := \text{span}\{ e^{(i)}_{r,s} \} \) is a MASA in \( \lambda_u(F_n)' \cap F_n \). Since \( p_i(\lambda_u(F_n)' \cap \mathcal{O}_n) \cong p_{i+1}(\lambda_u(F_n)' \cap \mathcal{O}_n) \), we can find a unitary \( g_i \in p_{i+1}(\lambda_u(F_n)' \cap \mathcal{O}_n) \) such that \( (\text{Ad } g_i \circ \alpha)(e^{(i)}_{r,s}) = e^{(i+1)}_{r,s} \). Setting \( g := \sum_{i=1}^l g_i \), we obtain the desired result.

Lemma 2.13. Let \( u \in U(\mathcal{O}_n) \) be such that \( \lambda_u(F_n) \subseteq F_n \) and \( \text{dim} \lambda_u(F_n)' \cap F_n < \infty \). Then there exist a positive integer \( k \), a unitary \( g \in \lambda_u(F_n)' \cap \mathcal{O}_n \), and a unitary group \( \{ d_z \}_{z \in \mathbb{T}} \subseteq \lambda_u(F_n)' \cap F_n \) such that \( (\text{Ad } g \circ \varphi)(d_z) = d_z g v \).

Proof. Put \( \alpha := \text{Ad } v \circ \varphi \), and let \( g \) and \( k \) be as in Lemma 2.12. Then we have
\begin{equation}
(\text{Ad } g v \circ \varphi)^k = \text{id}.
\end{equation}
and thus
\[ \text{Ad } v_k \circ \varphi^k = (\text{Ad } v \circ \varphi)^k = \text{Ad}(g^{(k)})^* \]
by parts (ii) and (iv) of Lemma 2.6. Then arguing as in the proof of Lemma 2.7 (with \( g^{(k)} \) playing the role of \( U \)), we get
\[ (gv)_k = g^{(k)}v_k \in \mathcal{F}_n. \]
Now, let \( D \) be a MASA as in Lemma 2.12. For all \( x \in D \) and \( z \in \mathbb{T} \), we see that
\[ gv\varphi(x)v^*g^* = \gamma_z(gv\varphi(x)v^*g^*) = \gamma_z(g)v_zv\varphi(x)v^*v_z^*\gamma_z(g^*) \]
\[ = (\gamma_z(g)v_zg^*)(gv\varphi(x)v^*g^*)(\gamma_z(g)v_zg)^* \]
which implies that \( \gamma_z(g)v_zg^* \) is in the commutant of MASA \( D \), and hence in \( D \) itself. Set \( d_z = \gamma_z(g)v_zg^* \), a unitary in \( D \). Now, \( d_z = u_zgu_z^*v_zg^* \) implies \( u_zd_z = g(u_z^*v_z)g^* \). Since \( \{u_z\}_{z \in \mathbb{T}} \) and \( \{v_z\}_{z \in \mathbb{T}} \) are commuting unitary groups, so is \( \{u_z^*d_z\}_{z \in \mathbb{T}} \), and consequently also is \( \{d_z\}_{z \in \mathbb{T}} \). Finally, we see that \( \gamma_z(gv) = \gamma_z(g)v_zv = d_zgv. \)

Now, we are ready to prove the following result.

**Theorem 2.14.** Let \( u \in \mathcal{U}(\mathcal{O}_n) \). If \( \lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n \) and \( \dim \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n < \infty \), then there exists a unitary \( W \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n \) satisfying the following.

(i) There exists a unitary group \( \{d_z\}_{z \in \mathbb{T}} \subseteq \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n \) such that \( \gamma_z(Wu) = d_zWu \) for all \( z \in \mathbb{T} \).

(ii) There exists a positive integer \( k \) such that \( (Wu)_k \in \mathcal{F}_n \).

**Proof.** Let \( u = wv \) be a factorization as in Theorem 2.4 and let \( k \in \mathbb{N} \) and \( g \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n \) be as in Lemma 2.13 above. Then setting \( W := gw^* \) gives the claim. \( \square \)

### 3. The criterion and examples

In this section, we give a dynamic characterization of those unitaries \( u \in \mathcal{O}_n \) satisfying our standing assumptions which either belong to \( \mathcal{F}_n \) (Theorem 3.2) or admit a unitary \( v \in \mathcal{F}_n \) such that \( \lambda_u|_{\mathcal{F}_n} = \lambda_u|_{\mathcal{F}_n} \) (Corollary 3.4). Before proving these results, we still need one technical lemma about the structure of the relative commutants. We keep our standing assumptions (2).

**Lemma 3.1.** There exist a unitary group \( \{q_z\}_{z \in \mathbb{T}} \) in \( \mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n) \) such that
\[ X_z = q_z(\text{Ad } v \circ \varphi)(q_z^*). \]

**Proof.** Since \( \text{Ad } v \circ \varphi \) restricts to an automorphism of \( \mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n) \), there exist minimal projections \( p_i^{(j)}, j = 1, \ldots, N, i = 1, \ldots, n_j \), in \( \mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n) \) such that
\[ \mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n) = \bigoplus_{j=1}^N \bigoplus_{i=1}^{n_j} p_i^{(j)} \]
and
\[(\text{Ad} v \circ \varphi)(p_i^{(j)}) = p_{i+1}^{(j)} \text{ for } i < n_j, \quad \text{and} \quad (\text{Ad} v \circ \varphi)(p_{n_i}^{(j)}) = p_1^{(j)}.
\]
Then \(X_z\) from Lemma 2.5 can be written as
\[X_z = \sum_{j=1}^{N_n} \sum_{i=1}^{n_j} z^{m_i^{(j)}} p_i^{(j)},
\]
for some \(m_i^{(j)} \in \mathbb{N}\). Now, let \(k \in \mathbb{N}\) be such that \(\text{Ad} v \circ \varphi\) is an inner automorphism of \(\lambda_u(F_n)' \cap O_n\). Then
\[X_z^{(k)} = X_z(\text{Ad} v \circ \varphi)(X_z)(\text{Ad} v \circ \varphi)^2(X_z) \ldots (\text{Ad} v \circ \varphi)^{k-1}(X_z) = 1
\]
by Lemma 2.7 Since each \(n_j\) divides \(k\), this implies that
\[\sum_{i=1}^{n_j} m_i^{(j)} = 0
\]
for each \(j = 1, \ldots, N\). Now, we want to define \(q_z\) as follows,
\[q_z = \sum_{j=1}^{N_n} \sum_{i=1}^{n_j} z^{r_i^{(j)}} p_i^{(j)},
\]
for suitable chosen integers \(r_i^{(j)}\), so that \(X_z = q_z(\text{Ad} v \circ \varphi)(q_z^*)\). To this end, it suffices to put
\[r_1^{(j)} := 0, \quad j = 1, \ldots, N,
\]
\[r_k^{(j)} := \sum_{r=2}^{k} m_r^{(j)}, \quad j = 1, \ldots, N, \quad k = 2, \ldots, n_j.
\]
\[\square\]

**Theorem 3.2.** Let \(u \in U(O_n)\) be such that \(\lambda_u(F_n) \subseteq F_n\) and \(\dim \lambda_u(F_n)' \cap F_n < \infty\). Put \(\alpha := \text{Ad} u \circ \varphi\). If \(\alpha\) satisfies the following two conditions:

(i) \(\alpha(\lambda_u(F_n)' \cap F_n) = \lambda_u(F_n)' \cap F_n\), and

(ii) \(\alpha|_{\lambda_u(F_n) \cap F_n}\) preserves the \(\tau\)-trace,

then \(u \in F_n\).

**Proof.** At first, we observe that there exists a unitary group \(\{u_z^*\}_{z \in \mathbb{T}}\) in \(Z(\lambda_u(F_n)' \cap F_n)\) such that \(\text{Ad} u_z^*(x) = \gamma_z(x)\) for all \(x \in \lambda_u(F_n)' \cap O_n\) and \(\gamma_z(u) = u_z^* \alpha(u_z^{*\star}) u\). Indeed, it suffices to put \(u_z^* := q_z u_z\), with \(q_z\) as in Lemma 3.1 above. Then \(\alpha(u_z^*) \in Z(\lambda_u(F_n)' \cap F_n)\) by condition (i) of the theorem, and hence \(\{u_z^* \alpha(u_z^{*\star})\}_{z \in \mathbb{T}}\) is a unitary group. Thus, \(u_z^* \alpha(u_z^{*\star}) = \sum z^{k_j} p_j\) for some integers \(k_j\) and a partition of unity by projections \(p_j\) from \(Z(\lambda_u(F_n)' \cap F_n)\).

Now, we claim that \(p_j = 0\) whenever \(k_j \neq 0\). To this end, suppose first that \(k_j > 0\) for some index \(j\), and put \(R := p_{k_j} u(S_{1})^{k_j}\). We have \(\gamma_z(R) = R\) for all
\[ z \in \mathbb{T}, \text{ and thus } R \in \mathcal{F}_n. \text{ However, an easy calculation shows that } RR^* = pk_j \text{ and } R^*R = \varphi^{k+1}(\alpha^1(p_k))S_1^kS_1^{k+1}. \text{ In view of condition (ii) of the theorem, this would imply } \tau(RR^*) \neq \tau(R^*R) \text{ if } p_j \neq 0, \text{ a contradiction. Therefore } p_j = 0 \text{ for all } k_j > 0. \text{ A similar argument shows that } p_j = 0 \text{ if } k_j < 0.

Consequently, \( z \in \mathbb{T} \), and the theorem is proved. \( \square \)

We note that Theorem 3.2 gives a necessary and sufficient condition for \( u \in \mathcal{F}_n \), since the reverse implication is trivial. Likewise, Corollary 3.3 below, gives a necessary and sufficient condition for \( u_k \in \mathcal{F}_n \).

**Corollary 3.3.** Let \( u \in \mathcal{U}(\mathcal{O}_n) \) be such that \( \lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n \) and \( \dim \lambda_u(\mathcal{F}_n)^\prime \cap \mathcal{F}_n < \infty \). Put \( \alpha := (\operatorname{Ad} u \circ \varphi)^k \), for some positive integer \( k \). If \( \alpha \) satisfies the following two conditions:

(i) \( \alpha(\lambda_u(\mathcal{F}_n)^\prime \cap \mathcal{F}_n) = \lambda_u(\mathcal{F}_n)^\prime \cap \mathcal{F}_n \), and

(ii) \( \alpha|_{\lambda_u(\mathcal{F}_n) \cap \mathcal{F}_n} \) preserves the \( \tau \)-trace,

then \( u_k \in \mathcal{F}_n \).

Now, we are ready to give the following decomposability criterion.

**Corollary 3.4.** Let \( u \in \mathcal{U}(\mathcal{O}_n) \) be such that \( \lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n \) and \( \dim \lambda_u(\mathcal{F}_n)^\prime \cap \mathcal{F}_n < \infty \). Put \( \alpha := \operatorname{Ad} u \circ \varphi \). Then the following two conditions are equivalent:

1. There exist unitaries \( w \in \lambda_u(\mathcal{F}_n)^\prime \cap \mathcal{O}_n \) and \( v \in \mathcal{F}_n \) such that \( u = wv \).
2. For each minimal projection \( p \in \mathcal{Z}(\lambda_u(\mathcal{F}_n)^\prime \cap \mathcal{O}_n) \) there exists a \( \tau \)-preserving isomorphism

\[ p(\lambda_u(\mathcal{F}_n)^\prime \cap \mathcal{F}_n) \cong \alpha(p)(\lambda_u(\mathcal{F}_n)^\prime \cap \mathcal{F}_n). \]

Now, we show how to construct examples of endomorphisms \( \lambda_u \) of \( \mathcal{O}_n \) globally preserving the core UHF-subalgebra \( \mathcal{F}_n \) but such that no unitary \( v \in \mathcal{F}_n \) exists for which \( \lambda_u|_{\mathcal{F}_n} = \lambda_v|_{\mathcal{F}_n} \).

To begin with, take two non-zero, orthogonal projections \( q_1, q_2 \) in \( \mathcal{F}_n \) such that \( \tau(q_2)/\tau(q_1) = n^r \) for some non-zero integer \( r \). Let \( A_1 \) be a partial isometry in \( \mathcal{O}_n^{(r)} \) with domain projection \( \varphi(q_1) \) and range projection \( q_2 \). Likewise, let \( A_2 \) be a partial isometry in \( \mathcal{O}_n^{(r)} \) with domain projection \( \varphi(q_2) \) and range projection \( q_1 \). Finally, let \( A_3 \) be a partial isometry in \( \mathcal{F}_n \) with domain projection \( 1 - \varphi(q_1) - \varphi(q_2) \) and range projection \( 1 - q_1 - q_2 \). Put \( u := A_1 + A_2 + A_3 \). Then \( u \) is a unitary in \( \mathcal{O}_n \) such that

\[ \operatorname{Ad} u \circ \varphi(q_1) = q_2 \quad \text{and} \quad \operatorname{Ad} u \circ \varphi(q_2) = q_1. \]

Now, \( u^{\gamma_z}(u^*) = z^r q_1 + z^{-r} q_2 + 1 - q_1 - q_2 \) belongs to \( \text{span}\{1, q_1, q_2\} \), and \( \text{span}\{1, q_1, q_2\} \subseteq \lambda_u(\mathcal{F}_n)^\prime \cap \mathcal{O}_n \) by [4, Proposition 2.3] and [7] above. Thus \( \lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n \) by Proposition 2.1 above.
More generally, Let \( 1 = \sum q_j \) be a partition of unity by projections in \( \mathcal{O}_n \). Let \( u \) be any unitary in \( \mathcal{O}_n \) such that \( \text{Ad} \, u \circ \varphi \) permutes projections \( \{q_j\} \) and for each \( j \) there is a \( k_j \in \mathbb{Z} \) such that \( q_j u \in \mathcal{O}_n^{(k_j)} \). Then \( u \gamma_z(u^*) \in \text{span}\{q_j\} \subseteq \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n \) for all \( z \in \mathbb{T} \). This simple construction gives a large class of examples of unitaries \( u \in \mathcal{O}_n \setminus \mathcal{F}_n \) such that \( \lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n \). However, to verify the conditions of Corollary 3.4 one needs more detailed information on the relative commutants \( \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n \subseteq \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n \). Exact determination of these relative commutants is rather difficult and does not seem possible in general, despite the identity from [4, Proposition 2.3]. However, it is quite doable in concrete cases.

Now, we illustrate the above discussion with two concrete examples in \( \mathcal{O}_2 \). In these examples, along with the main algebra \( C^*(S_1, S_2) \cong \mathcal{O}_2 \), we consider its other subalgebras, also isomorphic to \( \mathcal{O}_2 \). For example, if \( T_1, T_2 \) are isometries in \( C^*(S_1, S_2) \) generating a copy of \( \mathcal{O}_2 \), then we use subscript \( T \) along with the standard notation to indicate that the object comes from \( C^*(T_1, T_2) \) and its generators. Thus \( \varphi_T \) denotes the usual shift on \( C^*(T_1, T_2) \), that is a map \( \varphi : C^*(T_1, T_2) \to C^*(T_1, T_2) \) such that \( \varphi(x) = T_1 x T_1^* + T_2 x T_2^* \). Similarly, \( \mathcal{D}_T \) denotes the diagonal MASA of \( C^*(T_1, T_2) \), and so on. The proof of one technical lemma needed in Example 3.6 is given afterwards.

**Example 3.5.** Take \( q_1 = P_{11}, q_2 = P_{222} \), and set
\[
A_1 = S_{2221}S_{111}^* + S_{2222}S_{211}^*
\]
\[
A_2 = S_{11}S_{122}^* + S_{112}S_{222}^*
\]
\[
A_3 = S_{1222}S_{2221}^* + S_{211}S_{112}^* + P_{121} + P_{1221} + P_{212} + P_{221}
\]

We note that unitary \( u := A_1 + A_2 + A_3 \) falls within the class of polynomial unitaries considered in [4, Section 5]. In particular, its graph \( E_u \), as defined therein, admits the \( \{-1, 0, +1\} \) labelling:

![Graph](image)

This labelled graph satisfies the path condition defined in [4, p. 616], and this is an alternative way of showing that \( \lambda_u(\mathcal{F}_2) \subseteq \mathcal{F}_2 \).
Now, we have $P_{11}O_2P_{11} \cong O_2 = C^*(T_1, T_2)$, under the isomorphism sending $T_1$ to $S_{111}S_{11}$ and $T_2$ to $S_{112}S_{11}$. Similarly, $P_{222}O_2P_{222} \cong O_2 = C^*(R_1, R_2)$, under the isomorphism sending $R_1$ to $S_{2221}S_{222}$ and $R_2$ to $S_{2222}S_{222}^*$. Then an easy calculation shows that

$$\text{Ad } u \circ \varphi(T_j) = \varphi_R(R_j),$$

$$\text{Ad } u \circ \varphi(R_j) = \varphi_T(T_j),$$

for $j = 1, 2$. Consequently, the restriction of $(\text{Ad } u \circ \varphi)^2$ to $P_{11}O_2P_{11}$ is conjugate to $\varphi_R^2$. Likewise, the restriction of $(\text{Ad } u \circ \varphi)^2$ to $P_{222}O_2P_{222}$ is conjugate to $\varphi_T^2$. This immediately implies

$$\lambda_u(F_2)' \cap P_{11}O_2P_{11} \subseteq \bigcap_{k=1}^{\infty} (\text{Ad } u \circ \varphi)^{2k}(P_{11}O_2P_{11}) = CP_{11},$$

$$\lambda_u(F_2)' \cap P_{222}O_2P_{222} \subseteq \bigcap_{k=1}^{\infty} (\text{Ad } u \circ \varphi)^{2k}(P_{222}O_2P_{222}) = CP_{222}.$$ 

That is, both $P_{11}$ and $P_{222}$ are minimal projections in $\lambda_u(F_2)' \cap O_2$. One easily checks that $\text{Ad } u \circ \varphi(S_{11}S_{222}^*) = S_{222}S_{11}^*$. Thus $S_{11}S_{222}^*$ is in $\lambda_u(F_2)' \cap O_2$, and we see that $(P_{11} + P_{222})\lambda_u(F_2)' \cap O_2(P_{11} + P_{222}) \cong \mathbb{M}_2(\mathbb{C})$. We remark that the restriction of $\text{Ad } u \circ \varphi$ to $(P_{11} + P_{222})O_2(P_{11} + P_{222})$ is conjugate to endomorphism $\rho_{1342}$ from $[5]$. Let

$$w := S_{11}S_{222}^* + S_{222}S_{11}^* + 1 - P_{11} - P_{222}.$$ 

Then $w$ is a unitary in $\lambda_u(F_2)' \cap O_2$ such that $w^*u \in F_2$. \hfill \Box

Example 3.6. Take $q_1 = P_1$, $q_2 = P_{21}$, and set

$$A_1 = S_{211}S_{21}^* + S_{2121}S_{112}^* + S_{2122}S_{111}^*,$$

$$A_2 = S_{11}S_{221}^* + S_{11}S_{222}^*,$$

$$A_3 = S_{221}S_{222}^*.$$ 

We put $u := A_1 + A_2 + A_3$. By construction, $\text{Ad } u \circ \varphi(P_1) = P_{21}$ and also $\text{Ad } u \circ \varphi(P_{21}) = P_1$. Hence $\text{Ad } u \circ \varphi(P_{22}) = P_{22}$ as well.

We have $P_{22}C^*(S_1, S_2)P_{22} \cong O_2 = C^*(R_1, R_2)$, under the identification of $S_{221}S_{222}^*$ with $R_1$ and $S_{222}S_{221}^*$ with $R_2$. This isomorphism yields a conjugation between the restriction of $\text{Ad } u \circ \varphi$ to $P_{22}C^*(S_1, S_2)P_{22}$ and the shift $\varphi_R$. Consequently,

$$\lambda_u(F_2)' \cap P_{22}C^*(S_1, S_2)P_{22} = \bigcap_{k=1}^{\infty} (\text{Ad } u \circ \varphi)^k(P_{22}C^*(S_1, S_2)P_{22}) = CP_{22}.$$ 

We have $P_1C^*(S_1, S_2)P_1 \cong O_2 = C^*(T_1, T_2)$, under the identification of $S_{11}S_{11}^*$ with $T_1$ and $S_{12}S_{12}^*$ with $T_2$. This isomorphism carries the restriction of $(\text{Ad } u \circ \varphi)^2$ to $P_1C^*(S_1, S_2)P_1$ to the endomorphism of $C^*(T_1, T_2)$ given as composition $\varphi_T \circ \psi_T$, where $\psi_T$ is an endomorphism of $C^*(T_1, T_2)$ such that

$$\psi_T(x) = T_1xT_1^* + T_2(\text{Ad } F_T(x))T_2^*,$$
where \( F_T := T_2T_1^* + T_1T_2^* \). By Lemma \ref{lem:iso}, we have

\[
\lambda_u(\mathcal{F}_2)' \cap P_1 C^*(S_1, S_2) P_1 \subseteq \bigcap_{k=1}^{\infty} (\Ad u \circ \varphi)^{2k}(P_1 C^*(S_1, S_2) P_1) = \mathbb{C} P_1.
\]

We have \( P_1 C^*(S_1, S_2) P_2 \cong \mathcal{O}_2 = C^*(V_1, V_2) \), under the identification of \( S_{211}S_{21} \) with \( V_1 \) and \( S_{212}S_{21} \) with \( V_2 \). This isomorphism carries the restriction of \( (\Ad u \circ \varphi)^2 \) to \( P_1 C^*(S_1, S_2) P_2 \) to \( \psi_V \circ \varphi_V \). An argument similar to that from Lemma \ref{lem:iso} shows that \( \lambda_u(\mathcal{F}_2)' \cap P_2 C^*(S_1, S_2) P_2 = \mathbb{C} P_2 \). Alternatively, this also easily follows from the preceding argument and the fact that \( \Ad u \circ \varphi(P_21) = P_1 \).

In view of the above, either \( \lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2 = \text{span}\{P_1, P_21, P_22\} \cong \mathbb{C}^3 \), or \( P_1 \) and \( P_21 \) are equivalent in \( \lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2 \). In the latter case, \( \lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2 \) contains a subalgebra isomorphic to \( M_2(\mathbb{C}) \) which is invariant under \( \Ad u \circ \varphi \) and has \( P_1 \) and \( P_21 \) as its minimal projections. Suppose for a moment that this is the case. Then \( \Ad u \circ \varphi \) restricts to a non-trivial automorphism of \( M_2(\mathbb{C}) \), by necessity inner. The implementing unitary matrix \( g \) is fixed by \( \Ad u \circ \varphi \) and thus belongs to \( \lambda_u(\mathcal{O}_2)' \cap \mathcal{O}_2 \). Matrix \( g \) has both diagonal entries equal to 0. Multiplying \( g \) by a suitable scalar of modulus 1, we can find such \( g \) that is self-adjoint. Now we see that there is a unitary element \( x \) of \( \mathcal{O}_2 \) such that

\[
g = S_{21} x^* S_1^* + S_1 x S_{21}^* \in \lambda_u(\mathcal{O}_2)' \cap \mathcal{O}_2.
\]

Now, writing \( F := S_1 S_2^* + S_2 S_1^* \), we compute

\[
\Ad u \circ \varphi(g) = u(S_{11}x S_{12}^* + S_{12}x^* S_{11}^* + S_{21}x S_{21}^* + S_{21}x^* S_{21}^*) u^*
\]

\[
= S_{212} F x S_{12}^* + S_{12} x^* F S_{212}^* + S_{211} x S_{11}^* + S_{11} x^* S_{211}^*.
\]

and hence we get

\[
S_1 x S_{21}^* + S_{21} x^* S_1^* = S_{212} F x S_{12}^* + S_{12} x^* F S_{212}^* + S_{211} x S_{11}^* + S_{11} x^* S_{211}^*.
\]

Multiplying by \( S_1^* \) from the left-side and by \( S_{21} \) from the right-side, we obtain

\[
x = S_2 x^* F S_{21}^* + S_1 x^* S_1^*.
\]

Equation (8) implies \( x S_1 = S_1 x^* \) and \( S_1^* x = x^* S_1^* \). These two combined then yield \( (x + x^*) S_1 = S_1 (x + x^*) \) and \( (x - x^*) S_1 = -S_1 (x - x^*) \). By \cite[Proposition 4]{14}, both \( x + x^* \) and \( x - x^* \) are scalars, and thus so is \( x \). This however contradicts (8).

Thus \( \lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2 = \text{span}\{P_1, P_21, P_22\} \) and since \( \tau(P_1) \neq \tau(P_21) \), we conclude from Corollary \ref{cor:no-unitaries} that there are no unitaries \( w \in \lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2 \) and \( v \in \mathcal{F}_2 \) such that \( u = wv \).

\end{proof}

\begin{lemma}
Let \( \psi_T \) be an endomorphism of \( C^*(T_1, T_2) \cong \mathcal{O}_2 \) such that

\[
\psi_T(x) = T_1 x T_1^* + T_2 (\Ad F_T(x)) T_2^*.
\]

\end{lemma}
where \( F_T := T_2 T_1^* + T_1 T_2^* \). Then we have

\[
\bigcap_{k=1}^{\infty} (\varphi_T \psi_T)^k(C^*(T_1, T_2)) = \mathbb{C}1.
\]

**Proof.** We note that

\[
\varphi_T \psi_T(x) = T_{11} x T_{11}^* + T_{21} x T_{21}^* + T_{12} (\text{Ad } F_T(x)) T_{12}^* + T_{22} (\text{Ad } F_T(x)) T_{22}^*.
\]

Also, we clearly have \( F_T T_1 = T_2 \) and \( F_T T_2 = T_1 \). Thus \( (\varphi_T \psi_T)^k(x) \) may be written as a finite sum of elements of the form \( T_\mu X T_\mu^* \) with \( |\mu| = 2k \). This gives

\[
\bigcap_{k=1}^{\infty} (\varphi_T \psi_T)^k(C^*(T_1, T_2)) \subseteq D_T' \cap C^*(T_1, T_2) = D_T.
\]

For a positive integer \( k \), let

\[
Q_k := \sum_{|\mu|=k-1} T_\mu T_\mu^*.
\]

Then a straightforward induction on \( k \) shows that

\[
Q_{2k} (\varphi_T \psi_T)^k(x) = Q_{2k} \varphi_T^{2k}(x)
\]

for all \( x \in C^*(T_1, T_2) \). Take a \( d = d^* \in D_T \) that belongs to \( \bigcap_{k=1}^{\infty} (\varphi_T \psi_T)^k(C^*(T_1, T_2)) \).

Suppose \( d \) is not a constant multiple of 1. Then there exist \( k \in \mathbb{N}, t \in \mathbb{R}, \epsilon > 0 \) and \( \mu, \nu \in W_2^{2k-1} \) such that

\[
T_\mu T_\mu^* d \geq (t + \epsilon) T_\mu T_\mu^* \quad \text{and} \quad T_\nu T_\nu^* d \leq (t - \epsilon) T_\nu T_\nu^*.
\]

Let \( x = x^* \in D_2 \) be such that \( d = (\varphi_T \psi_T)^k(x) \). Then \( Q_{2k} d = Q_{2k} \varphi_T^{2k}(x) \). Since \( T_\mu T_\mu^* \leq Q_{2k} \) and \( T_\nu T_\nu^* \leq Q_{2k} \), we get

\[
T_\mu T_\mu^* = T_\mu T_\mu^* Q_{2k} \varphi_T^{2k}(x) \geq (t + \epsilon) T_\mu T_\mu^*, \quad \text{and}
\]

\[
T_\nu T_\nu^* = T_\nu T_\nu^* Q_{2k} \varphi_T^{2k}(x) \leq (t - \epsilon) T_\nu T_\nu^*.
\]

This, however, is a contradiction. Indeed, since \( T_\mu \) and \( T_\nu \) are isometries, the above two inequalities would imply that both \( x \geq (t + \epsilon) \) and \( x \leq (t - \epsilon) \). Consequently,

\[
\bigcap_{k=1}^{\infty} (\varphi_T \psi_T)^k(C^*(T_1, T_2)) = \mathbb{C}1,
\]

as required. \( \square \)

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