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Transversals and Independence in Linear Hypergraphs with Maximum Degree Two

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Abstract

For $k \geq 2$, let H be a k -uniform hypergraph on n vertices and m edges. Let S be a set of vertices in a hypergraph H . The set S is a transversal if S intersects every edge of H , while the set S is strongly independent if no two vertices in S belong to a common edge. The transversal number, $\tau(H)$, of H is the minimum cardinality of a transversal in H , and the strong independence number of H , $\alpha(H)$, is the maximum cardinality of a strongly independent set in H . The hypergraph H is linear if every two distinct edges of H intersect in at most one vertex. Let \mathcal{H}_k be the class of all connected, linear, k -uniform hypergraphs with maximum degree 2. It is known [European J. Combin. 36 (2014), 231–236] that if $H \in \mathcal{H}_k$, then $(k+1)\tau(H) \leq n+m$, and there are only two hypergraphs that achieve equality in the bound. In this paper, we prove a much more powerful result, and establish tight upper bounds on $\tau(H)$ and tight lower bounds on $\alpha(H)$ that are achieved for infinite families of hypergraphs. More precisely, if $k \geq 3$ is odd and $H \in \mathcal{H}_k$ has n vertices and m edges, then we prove that $k(k^2-3)\tau(H) \leq (k-2)(k+1)n + (k-1)^2m + k - 1$ and $k(k^2-3)\alpha(H) \geq (k^2+k-4)n - (k-1)^2m - (k-1)$. Similar bounds are proven in the case when $k \geq 2$ is even.

Keywords: Transversal; Hypergraph; Linear hypergraph; Strong independence

AMS subject classification: 05C65;05C69

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1 Introduction

In this paper, we study transversals and independence in hypergraphs. Hypergraphs are systems of sets which are conceived as natural extensions of graphs. A *hypergraph* $H = (V, E)$ is a finite set $V = V(H)$ of elements, called *vertices*, together with a finite multiset $E = E(H)$ of subsets of V , called *hyperedges* or simply *edges*. The *order* of H is $n(H) = |V|$ and the *size* of H is $m(H) = |E|$. The hypergraph H is said to be *k-uniform* if every edge of H is of size k . Every (simple) graph is a 2-uniform hypergraph. Thus graphs are special hypergraphs. The *degree* of a vertex v in H , denoted by $d_H(v)$, is the number of edges of H which contain v . A vertex of degree r in H is called a *degree- r vertex*. The *rank* of H is the maximum size of an edge in H . The hypergraph H is *r-regular* if $d_H(v) = r$ for all $v \in V(H)$. The minimum and maximum degrees among the vertices of H is denoted by $\delta(H)$ and $\Delta(H)$, respectively. We use the standard notation $[k] = \{1, 2, \dots, k\}$.

Two vertices x and y of H are *adjacent* if there is an edge e of H such that $\{x, y\} \subseteq V(e)$. Two vertices x and y of H are *connected* if there is a sequence $x = v_0, v_1, v_2, \dots, v_k = y$ of vertices of H in which v_{i-1} is adjacent to v_i for $i \in [k]$. A *connected hypergraph* is a hypergraph in which every pair of vertices is connected. A maximal connected subhypergraph of H is a *component* of H . Thus, no edge in H contains vertices from different components.

For a subset $X \subseteq V(H)$ of vertices in H , let $H[X]$ denote the hypergraph induced by the vertices in X , in the sense that $V(H[X]) = X$ and $E(H[X]) = \{e \cap X \mid e \in E(H) \text{ and } |e \cap X| \geq 1\}$; that is, $E(H[X])$ is obtained from $E(H)$ by shrinking edges $e \in E(H)$ that intersect X to the edges $e \cap X$. For a subset $X \subset V(H)$ of vertices in H , we define $H - X$ to be the hypergraph obtained from H by deleting the vertices in X and all edges incident with X , and deleting all isolated vertices, if any, from the resulting hypergraph.

A subset T of vertices in a hypergraph H is a *transversal* (also called *vertex cover* or *hitting set* in many papers) if T intersects every edge of H . Equivalently, a set of vertices S is transversal in H if and only if $V(H) \setminus S$ is a weakly independent set in H . That is, no edge lies completely within $V(H) \setminus S$. The *transversal number* $\tau(H)$ of H is the minimum size of a transversal in H . Transversals in hypergraphs are well studied in the literature (see, for example, [5, 6, 14, 18, 26, 30]).

A set S of vertices in a hypergraph H is *strongly independent* if no two vertices in S belong to a common edge. The *strong independence number* of H , which we denote by $\alpha(H)$, is the maximum cardinality of a strongly independent set in H . The independence number is one of the most fundamental and well-studied graph and hypergraph parameters (see, for example, [1, 2, 4, 9, 11, 10, 12, 13, 15, 16, 17, 21, 22, 23, 25, 27]).

A hypergraph H is called an *intersecting hypergraph* if every two distinct edges of H have a non-empty intersection, while H is called a *linear hypergraph* if every two distinct edges of H intersect in at most one vertex. Intersecting and linear hypergraphs are well studied in the literature (see, for example, [8, 20]).

Two edges in a graph G are *independent* if they are not adjacent in G . A set of

pairwise independent edges of G is called a *matching* in G , while a matching of maximum cardinality is a *maximum matching*. The number of edges in a maximum matching of G is the *matching number* of G which we denote by $\alpha'(G)$. Matchings in graphs are extensively studied in the literature (see, for example, the classical book on matchings by Lovász and Plummer [24], and the excellent survey articles by Plummer [28] and Pulleyblank [29]).

Given a graph G , we define a hypergraph H_G as follows. Let the edges of G become vertices in H_G and the vertices of G become hyperedges in H_G , containing all edges that are incident with that vertex in the graph. Thus, $V(H_G) = E(G)$ and $E(H_G)$ contains a hyperedge for every vertex $v \in V(G)$ which consists of all elements of $V(H_G)$ that correspond with edges incident with v in G . Therefore, $n(H_G) = m(G)$ and $m(H_G) = n(G)$. We call H_G the *dual hypergraph* of G .

2 Known Matching Results

We shall need the following results by the authors [19] which establish a tight lower bound on the matching number of a graph in terms of its maximum degree, order, and size.

Theorem 1. ([19]) *If $k \geq 2$ is an even integer and G is a connected graph of order n , size m and maximum degree $\Delta(G) \leq k$, then*

$$\alpha'(G) \geq \frac{n}{k(k+1)} + \frac{m}{k+1} - \frac{1}{k(k+1)},$$

unless the following holds.

- (a) G is k -regular and $n = k + 1$, in which case $\alpha'(G) = \frac{n-1}{2} = \frac{n}{k(k+1)} + \frac{m}{k+1} - \frac{1}{k}$.
- (b) G is k -regular and $n = k + 3$, in which case $\alpha'(G) = \frac{n-1}{2} = \frac{n}{k(k+1)} + \frac{m}{k+1} - \frac{3}{k(k+1)}$.

Let $k \geq 4$ be even and let $r \geq 1$ be arbitrary and let $\ell = r(k-1)+1$. Let X_1, X_2, \dots, X_ℓ be a number of vertex disjoint graphs such that each X_i where $i \in [\ell]$ is either a single vertex or it is a K_{k+1} where an arbitrary edge has been deleted. Let $Y = \{y_1, y_2, \dots, y_r\}$ and build the graph $G_{k,r}$ as follows. Let $G_{k,r}$ be obtained from the disjoint union of the graphs X_1, X_2, \dots, X_ℓ by adding to it the vertices in Y and furthermore, for every $i \in [r]$, adding an edge from y_i to a vertex in each graph $X_{(i-1)(k-1)+1}, X_{(i-1)(k-1)+2}, X_{(i-1)(k-1)+3}, \dots, X_{(i-1)(k-1)+k}$ in such a way that no vertex degree becomes more than k . Let $\mathcal{G}_{k,r}$ be the family of all such graph $G_{k,r}$. When $k = 4$ and $r = 2$, an example of a graph G in the family $\mathcal{G}_{k,r}$ is illustrated in Figure 1, where G has order $n = 21$, size $m = 35$ and matching number $\alpha'(G) = 8$.

Proposition 2. ([19]) *For $k \geq 4$ an even integer and $r \geq 1$ arbitrary, if $G \in \mathcal{G}_{k,r}$ has order n and size m , then*

$$\alpha'(G) = \frac{n}{k(k+1)} + \frac{m}{k+1} - \frac{1}{k(k+1)}.$$

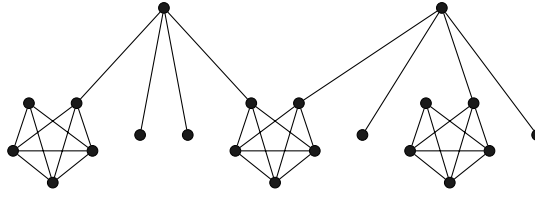


Figure 1: A graph G in the family $\mathcal{G}_{4,2}$

Theorem 3. ([19]) *If $k \geq 3$ is an odd integer and G is a connected graph of order n , size m , and with maximum degree $\Delta(G) \leq k$, then*

$$\alpha'(G) \geq \left(\frac{k-1}{k(k^2-3)} \right) n + \left(\frac{k^2-k-2}{k(k^2-3)} \right) m - \frac{k-1}{k(k^2-3)}.$$

For $k \geq 3$ odd, let H_{k+2} be the graph of (odd) order $k+2$ whose complement $\overline{H_{k+2}}$ is isomorphic to $P_3 \cup \binom{k-1}{2} P_2$. We note that every vertex in H_{k+2} has degree k , except for exactly one vertex, which has degree $k-1$. We call the vertex of degree $k-1$ in H_{k+2} the *link vertex* of H_{k+2} .

For $k \geq 3$ odd and $r \geq 1$ arbitrary, let $T_{k,r}$ be a tree with maximum degree at most k and with partite sets V_1 and V_2 , where $|V_2| = r$. Let $H_{k,r}$ be obtained from $T_{k,r}$ as follows: For every vertex x in V_2 with $d_{T_{k,r}}(x) < k$, add $k - d_{T_{k,r}}(x)$ copies of the subgraph H_{k+2} to $T_{k,r}$ and in each added copy of H_{k+2} , join the link vertex of H_{k+2} to x . We note that every vertex in the resulting graph $H_{k,r}$ has degree k , except possibly for vertices in the set V_1 whose degrees belong to the set $\{1, 2, \dots, k\}$. Let $\mathcal{F}_{k,r}$ be the family of all such graphs $H_{k,r}$.

When $k = 3$ and $r = 4$, an example of a graph G in the family $\mathcal{F}_{k,r}$ is illustrated in Figure 2, where G has order $n = 29$, size $m = 40$ and matching number $\alpha'(G) = 12$.

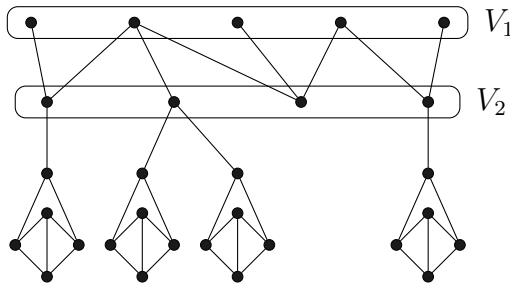


Figure 2: A graph G in the family $\mathcal{F}_{3,4}$

Proposition 4. ([19]) *For $k \geq 3$ an odd integer and $r \geq 1$ arbitrary, if $G \in \mathcal{F}_{k,r}$ has order n and size m , then*

$$\alpha'(G) = \left(\frac{k-1}{k(k^2-3)} \right) n + \left(\frac{k^2-k-2}{k(k^2-3)} \right) m - \frac{k-1}{k(k^2-3)}.$$

3 Three Families of Hypergraphs

In this section, we define three families of hypergraphs, \mathcal{H}_k , \mathcal{H}'_k and \mathcal{H}''_k . For a hypergraph H with maximum degree at most 2 we let $V_1(H)$ and $V_2(H)$ denote the set of vertices in H of degree 1 and 2, respectively. Further, we let $n_i(H) = |V_i(H)|$ for $i \in [2]$.

3.1 The Family \mathcal{H}_k

Definition 5. Let \mathcal{H}_k be the class of all connected, linear, k -uniform hypergraphs with maximum degree 2.

For a hypergraph $H \in \mathcal{H}_k$ we define a graph G_H as follows. Let the vertices of G_H be the edges of H and let the edges of G_H correspond to the $n_2(H)$ vertices of degree 2 in H : if a vertex of H is contained in the edges e and f of H , then the corresponding edge of the multigraph G_H joins vertices e and f of G_H . Thus, $V(G_H) = E(H)$ and for every $v \in V_2(H)$, contained in the two edges e and f , add an edge between e and f in G_H . By the linearity of H , the multigraph G_H is indeed a graph, called the *dual graph* of H . Since H is k -uniform and $\Delta(H) = 2$, the maximum degree, $\Delta(G_H)$, in G_H is at most k . Since H is connected, so too is G_H . By construction, $n(G_H) = m(H)$ and $m(G_H) = n_2(H)$. We note that if $H \in \mathcal{H}_k$ is 2-regular, then the dual graph, G_H , of H is k -regular.

3.2 The Family \mathcal{H}'_k

In order to define the family \mathcal{H}'_k , we first define a hypergraph, which we call L_k .

The Hypergraph L_k . For $k \geq 2$, let L_k be the 2-regular, k -uniform hypergraph of size $k+1$ and order $k(k+1)/2$ defined inductively as follows. We define $L_2 = K_3$ and we define L_3 to be the hypergraph with $V(L_3) = \{v_1, v_2, \dots, v_6\}$ and let $E(L_3) = \{e_1, e_2, e_3, e_4\}$, where $e_1 = \{v_1, v_2, v_3\}$, $e_2 = \{v_1, v_4, v_5\}$, $e_3 = \{v_2, v_4, v_6\}$ and $e_4 = \{v_3, v_5, v_6\}$. For $k \geq 2$, suppose the hypergraph L_k has been constructed and that $E(L_k) = \{e_1, e_2, \dots, e_{k+1}\}$. Let L_{k+2} be the hypergraph of order $n(L_k) + 2k + 3$ with $V(L_{k+2}) = V(L_k) \cup \{v\} \cup \{u_1, u_2, \dots, u_{k+1}\} \cup \{w_1, w_2, \dots, w_{k+1}\}$ and with edge set $E(L_{k+2}) = \{f_1, f_2, \dots, f_{k+3}\}$, where $f_i = e_i \cup \{u_i, w_i\}$ for $i \in [k+1]$ and where $f_{k+2} = \{v, u_1, \dots, u_{k+1}\}$ and $f_{k+3} = \{v, w_1, \dots, w_{k+1}\}$. The hypergraphs L_2 , L_4 and L_6 are illustrated in Figure 3(a), 3(b), and 3(c), respectively.

We shall need the following result from [7].

Theorem 6. ([7]) *For $k \geq 2$, the hypergraph L_k is the unique k -uniform, 2-regular, linear, intersecting hypergraph.*

Definition 7. Let $\mathcal{H}'_k = \mathcal{H}_k \setminus \{L_k\}$.

3.3 The Family \mathcal{H}''_k

For a hypergraph $H \in \mathcal{H}_k$, let $\alpha_2(H)$ be the maximum cardinality of a strongly independent set in H consisting only of degree-2 vertices in H . Every strongly independent set in H corresponds to a matching in the dual graph G_H of H . Conversely, every matching

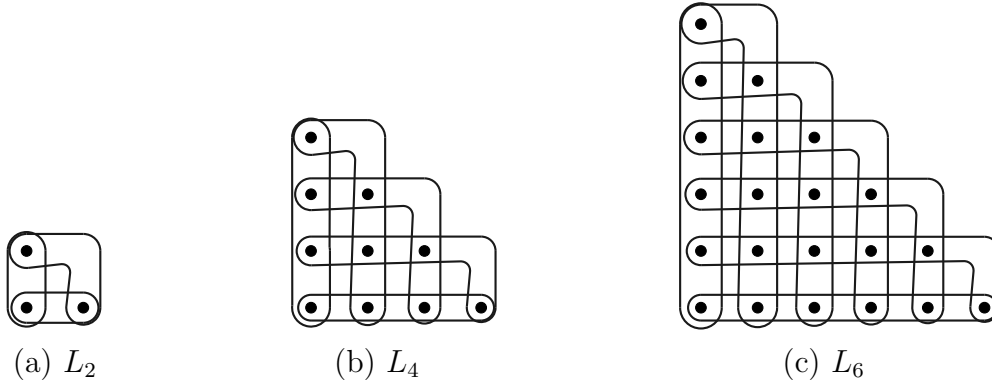


Figure 3: The hypergraphs L_2 , L_4 and L_6 .

M in the dual graph G_H of H corresponds to a strongly independent set $V_M \subseteq V_2(H)$ in H . This immediately implies the following observation.

Observation 8. *If $H \in \mathcal{H}_k$ and G_H is the dual graph of H , then $\alpha'(G_H) = \alpha_2(H)$.*

The following result is well-known (see, for example, [7]). However, since it is central to our discussions, we give a short proof for completeness.

Proposition 9. *If $H \in \mathcal{H}_k$ and G_H is the dual graph of H , then $\alpha'(G_H) = |E(H)| - \tau(H)$.*

Proof. Let $H \in \mathcal{H}_k$ and let G_H be the dual graph of H . If M is a maximum matching, then the corresponding set $V_M \subseteq V_2(H)$ is a maximum strong independent set in $V_2(H)$ by Observation 8. Therefore, V_M covers $2|V_M|$ distinct edges in H . Using an additional $|E(H)| - 2|V_M|$ vertices in H , one from each of the edges not covered by V_M , we can extend the set V_M to a transversal in H . Therefore, $\tau(H) \leq |V_M| + (|E(H)| - 2|V_M|) = |E(H)| - \alpha'(G_H)$, or, equivalently, $\alpha'(G_H) \leq |E(H)| - \tau(H)$.

Conversely, let T be a minimum transversal in H , and so, $\tau(H) = |T|$. If a vertex $x \in T$ covers only one hyperedge in H that is not covered by $T \setminus \{x\}$, then delete this vertex from T and the edge it covers from H . We continue this process removing r vertices from T , resulting in a set T' , and r associated edges from H , resulting in a hypergraph H' , until every vertex in T' covers two distinct edges in H' that are not covered by any other vertex of T' . Therefore, T' corresponds to a matching in G_H , and $|E(H)| = |E(H')| + r = 2|T'| + r = 2|T'| + (|T| - |T'|) = |T'| + |T|$. Thus, $\alpha'(G_H) \geq |T'| = |E(H)| - |T| = |E(H)| - \tau(H)$. As observed earlier, $\alpha'(G_H) \leq |E(H)| - \tau(H)$. Consequently, $\alpha'(G_H) = |E(H)| - \tau(H)$. \square

The Family \mathcal{M}_k . Let \mathcal{M}_k be the class of all connected, linear, k -uniform, 2-regular hypergraphs H with $k + 3$ edges. We note that \mathcal{M}_k is a subclass of \mathcal{H}_k . The dual graph, G_H , of a hypergraph $H \in \mathcal{M}_k$ is a k -regular graph of order $k + 3$. We note that the complement $\overline{G_H}$ of G_H is a 2-regular graph on $k + 3$ vertices. Thus, G_H can be constructed from K_{k+3} by removing the edges of a cycle factor of K_{k+3} . Using this approach, we observe that the number of non-isomorphic hypergraphs in \mathcal{M}_k is equal

to the number of non-isomorphic cycle factors in K_{k+3} . For example, $|\mathcal{M}_4| = 2$ (the cycle factors in K_7 are either a Hamilton cycle or the union of a 3-cycle and a 4-cycle) and $|\mathcal{M}_6| = 4$ (consider cycle factors with cycle lengths (9), (6, 3), (5, 4) and (3, 3, 3)). We state this formally as follows.

Observation 10. *The following holds.*

- (a) *If $H \in \mathcal{M}_k$, then the dual graph of H is a k -regular graph of order $k + 3$.*
- (b) *If G is a k -regular graph of order $k + 3$, then the dual hypergraph of G belongs to \mathcal{M}_k and has order $k(k + 3)/2$.*

Definition 11. Let $\mathcal{H}_k'' = \mathcal{H}_k' \setminus \mathcal{M}_k = \mathcal{H}_k \setminus (\mathcal{M}_k \cup \{L_k\})$.

4 Main Results

In what follows, we adopt the following notation. If $H \in \mathcal{H}_k$, we let H have order n and size m , and so $n = n(H)$ and $m = m(H)$. Further, we let $n_i = n_i(H)$ for $i \in [2]$, and so n_1 and n_2 denote the number of vertices of degree 1 and 2, respectively, in H . We note that $km = n_1 + 2n_2$. We denote the number of components of a hypergraph H by $c(H)$.

4.1 Transversal Number

Our first result establishes an upper bound on the transversal number of a connected, linear, k -uniform hypergraph with maximum degree 2 for $k \geq 2$ even.

Theorem 12. *For all even $k \geq 2$ the following holds.*

- (a) *If $H \in \mathcal{H}_k$, then $\tau(H) \leq \frac{kn+(k-1)m+k+1}{k(k+1)}$.*
- (b) *If $H \in \mathcal{H}_k'$, then $\tau(H) \leq \frac{kn+(k-1)m+3}{k(k+1)}$.*
- (c) *If $H \in \mathcal{H}_k''$, then $\tau(H) \leq \frac{kn+(k-1)m+1}{k(k+1)}$.*

Proof. Let $k \geq 2$ be even and let $H \in \mathcal{H}_k$. Let G_H be the dual graph of H . If $H = L_k$, then, by Theorem 6, we note that $m = k + 1$ and G_H is a k -regular graph of order $k + 1$. If $H \in \mathcal{M}_k$, then $m = k + 3$ and, by Observation 10, the graph G_H is a k -regular graph of order $k + 3$. If $H \in \mathcal{H}_k''$, then G_H has maximum degree $\Delta(G) \leq k$. Further, if G_H is k -regular (and still $H \in \mathcal{H}_k''$), then $n(G_H) \notin \{k + 1, k + 3\}$. In all cases, we note that G_H is a connected graph of order $n(G_H) = m$ and size $m(G_H) = n_2$. Let

$$\theta = \begin{cases} 1 & \text{if } H \in \mathcal{H}_k'' \\ 3 & \text{if } H \in \mathcal{M}_k \\ k + 1 & \text{if } H = L_k. \end{cases}$$

By Theorem 1 and our definition of θ , the following holds.

$$\alpha'(G_H) \geq \frac{m}{k(k+1)} + \frac{n_2}{k+1} - \frac{\theta}{k(k+1)}.$$

By Proposition 9, we note that the following therefore holds.

$$\begin{aligned}
\tau(H) &= m - \alpha'(G_H) \\
&\leq m - \left(\frac{m}{k(k+1)} + \frac{n_2}{k+1} - \frac{\theta}{k(k+1)} \right) \\
&= \left(1 - \frac{1}{k(k+1)} \right) \left(\frac{n_1 + 2n_2}{k} \right) - \frac{n_2}{k+1} + \frac{\theta}{k(k+1)} \\
&= \left(\frac{k(k+1) - 1}{k^2(k+1)} \right) n_1 + \left(\frac{2(k(k+1) - 1) - k^2}{k^2(k+1)} \right) n_2 + \frac{\theta}{k(k+1)}.
\end{aligned}$$

Simplifying and multiplying through with $k^2(k+1)$ we obtain the following.

$$\begin{aligned}
k^2(k+1)\tau(H) &\leq (k^2 + k - 1)n_1 + (k^2 + 2k - 2)n_2 + k\theta \\
&= k^2(n_1 + n_2) + (k - 1)(n_1 + 2n_2) + k\theta \\
&= k^2n + (k - 1)(km) + k\theta.
\end{aligned}$$

This implies the desired result. □

We discuss next the hypergraphs $H \in \mathcal{H}_k$ for $k \geq 4$ even that achieve the upper bound for the transversal number in the statement of Theorem 12. If $H = L_k$, then $m(H) = k + 1$ and $n(H) = k(k + 1)/2$, and the dual graph of H is the graph K_{k+1} . Therefore, by Proposition 9,

$$\tau(H) = m(H) - \alpha'(K_{k+1}) = (k + 1) - \frac{k}{2} = \frac{k + 2}{2} = \frac{kn + (k - 1)m + k + 1}{k(k + 1)},$$

and equality holds in the statement of Theorem 12(a). If $H \in \mathcal{M}_k$, then $m(H) = k + 3$ and $n(H) = k(k + 3)/2$. By Observation 10, the dual graph, G_H , of H is a k -regular graph of order $k + 3$. Therefore, by Proposition 9,

$$\tau(H) = m(H) - \alpha'(G_H) = (k + 3) - \frac{k + 2}{2} = \frac{k + 4}{2} = \frac{kn + (k - 1)m + 3}{k(k + 1)},$$

and equality holds in the statement of Theorem 12(b). We show next that there is an infinite family of hypergraphs $H \in \mathcal{H}_k''$ that satisfy

$$\tau(H) = \frac{kn + (k - 1)m + 1}{k(k + 1)}.$$

For $k \geq 4$ an even integer and $r \geq 1$, let G be an arbitrary graph in the family $\mathcal{G}_{k,r}$. We show that associated with the graph G , there exists a hypergraph $H \in \mathcal{H}_k''$ for which equality holds in the statement of Theorem 12(c), constructed as follows. Let H_G be the dual hypergraph of G , and so the edges of G become vertices in H_G and the vertices of G become hyperedges in H_G , containing all edges that are incident with that vertex in the graph. We note that $n(H_G) = m(G)$ and $m(H_G) = n(G)$.

Since $\Delta(G) = k$, we note that the rank of H_G is k . We note further that the edges of size 1 in H_G , if any, correspond to the pendant edges in G (that are incident with a vertex of degree 1). The edges of size 2 in H_G , if any, correspond to vertices of degree 2 in G (that have both neighbors in Y). All other edges in H_G have size $k - 1$ or k .

We now expand all edges of H_G of size less than k to edges of size k by adding new vertices of degree 1 to each such edge. For example, if e_v is an edge of size 1 in H_G containing the vertex v , then we add $k - 1$ new vertices and expand the edge e_v to an edge of size k that contains these new vertices and the vertex v . Let H_G^k denote the resulting hypergraph, and let $\mathcal{H}_{k,r}^{\text{even}}$ be the family of all such hypergraphs H_G^k . For example, given the graph $G \in \mathcal{G}_{4,2}$ shown in Figure 1 we obtain the associated hypergraph $H \in \mathcal{H}_{4,2}^{\text{even}}$ shown in Figure 4.

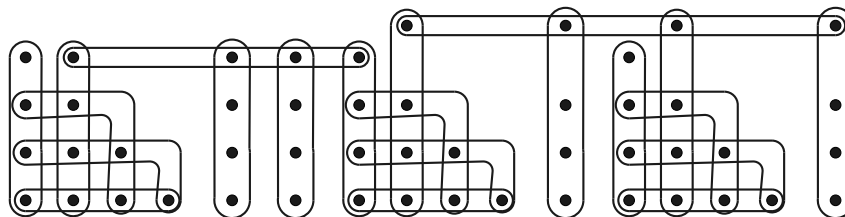


Figure 4: The hypergraph $H \in \mathcal{H}_{4,2}^{\text{even}}$ associated with the graph $G \in \mathcal{G}_{4,2}$ shown in Figure 1.

Proposition 13. For $k \geq 4$ an even integer and $r \geq 1$ arbitrary, if $H \in \mathcal{H}_{k,r}^{\text{even}}$ has order n and size m , then

$$\tau(H) = \frac{kn + (k - 1)m + 1}{k(k + 1)}.$$

Proof. We consider the graph $G \in \mathcal{G}_{k,r}$ used to construct the hypergraph $H \in \mathcal{H}_{k,r}^{\text{even}}$, and so $H = H_G^k$. Assume that when building the graph G , we have ℓ_1 single vertices and ℓ_2 copies of K_{k+1} 's minus an edge in X_1, X_2, \dots, X_ℓ . We note that $\ell_1 + \ell_2 = \ell = r(k - 1) + 1$ and $n(G) = r + \ell_1 + \ell_2(k + 1)$. Further,

$$\alpha'(G) = r + \binom{k}{2} \ell_2 = \frac{\ell_1 + \ell_2 - 1}{k - 1} + \binom{k}{2} \ell_2 = \frac{2\ell_1 + (k^2 - k + 2)\ell_2 - 2}{2(k - 1)}.$$

The order of H_G^k is

$$n(H_G^k) = k\ell_1 + \left(\frac{k^2 + k + 2}{2}\right) \ell_2.$$

Further, $m(H_G^k) = m(H_G) = n(G) = r + \ell_1 + \ell_2(k + 1)$, implying that the size of H_G^k is

$$m(H_G^k) = \left(\frac{k}{k - 1}\right) \ell_1 + \left(\frac{k^2}{k - 1}\right) \ell_2 - \frac{1}{k - 1}.$$

We remark that the graph $G \in \mathcal{G}_{k,r}$ used to construct the hypergraph $H_G^k \in \mathcal{H}_{k,r}^{\text{even}}$ is in fact the dual graph (see Section 3.1) of H_G^k . Therefore, letting $H = H_G^k$, $n = n(H_G^k)$ and $m = m(H_G^k)$, and applying Proposition 9 to H and its dual graph G , we have

$$\begin{aligned} \tau(H) &= m - \alpha'(G) \\ &= \left(\binom{k}{k-1} \ell_1 + \binom{k^2}{k-1} \ell_2 - \frac{1}{k-1} \right) \\ &\quad - \left(\binom{1}{k-1} \ell_1 + \binom{k^2 - k + 2}{2(k-1)} \ell_2 - \frac{1}{k-1} \right) \\ &= \ell_1 + \binom{k^2 + k - 2}{2(k-1)} \ell_2 \\ &= \ell_1 + \binom{k+2}{2} \ell_2 \end{aligned}$$

and

$$\begin{aligned} \frac{kn + (k-1)m + 1}{k(k+1)} &= \left(\frac{k}{k(k+1)} \right) \left(k\ell_1 + \binom{k^2 + k + 2}{2} \ell_2 \right) \\ &\quad + \left(\frac{k-1}{k(k+1)} \right) \left(\binom{k}{k-1} \ell_1 + \binom{k^2}{k-1} \ell_2 - \frac{1}{k-1} \right) \\ &\quad + \frac{1}{k(k+1)} \\ &= \ell_1 + \binom{k+2}{2} \ell_2. \end{aligned}$$

Equality therefore holds in the statement of Theorem 12(c). □

Next we consider the case when $k \geq 3$ is odd.

Theorem 14. *For $k \geq 3$ an odd integer, if $H \in \mathcal{H}_k$, then*

$$\tau(H) \leq \frac{(k-2)(k+1)n + (k-1)^2m + k-1}{k(k^2-3)}.$$

Proof. Let $k \geq 3$ be odd and let $H \in \mathcal{H}_k$. Let G_H be the dual graph of H and note that G_H has maximum degree $\Delta(G) \leq k$. Further, we note that G_H is a connected graph of order $n(G_H) = m$ and size $m(G_H) = n_2$. By Theorem 3, the following holds.

$$\alpha'(G_H) \geq \left(\frac{k-1}{k(k^2-3)} \right) m + \left(\frac{k^2 - k - 2}{k(k^2-3)} \right) n_2 - \frac{k-1}{k(k^2-3)}.$$

By Proposition 9, we note that the following therefore holds.

$$\begin{aligned}
\tau(H) &= m - \alpha'(G_H) \\
&\leq m - \left(\left(\frac{k-1}{k(k^2-3)} \right) m + \left(\frac{k^2-k-2}{k(k^2-3)} \right) n_2 - \frac{k-1}{k(k^2-3)} \right) \\
&= \left(1 - \frac{k-1}{k(k^2-3)} \right) \left(\frac{n_1+2n_2}{k} \right) - \left(\frac{k^2-k-2}{k(k^2-3)} \right) n_2 + \frac{k-1}{k(k^2-3)} \\
&= \left(\frac{k^3-4k+1}{k^2(k^2-3)} \right) n_1 + \left(\frac{2(k^3-4k+1) - k(k^2-k-2)}{k^2(k^2-3)} \right) n_2 + \frac{k-1}{k(k^2-3)}.
\end{aligned}$$

Simplifying and multiplying through with $k^2(k^2-3)$ we obtain the following.

$$\begin{aligned}
k^2(k^2-3)\tau(H) &\leq (k^3-4k+1)n_1 + (k^3+k^2-6k+2)n_2 + k(k-1) \\
&= (k^3-2k)(n_1+n_2) - (2k-1)(n_1+2n_2) + k^2 \cdot n_2 + k(k-1) \\
&= (k^3-2k)n - (2k-1)(km) + k^2 \cdot n_2 + k(k-1) \\
&= (k^3-2k)n - (2k-1)(km) + k^2(km-n) + k(k-1) \\
&= k(k-2)(k+1)n + k(k-1)^2m + k(k-1).
\end{aligned}$$

This implies the desired result. □

We discuss next the hypergraphs $H \in \mathcal{H}_k$ for $k \geq 3$ odd that achieve the upper bound for the transversal number in the statement of Theorem 14. For $k \geq 3$ an even integer and $r \geq 1$, let G be an arbitrary graph in the family $\mathcal{F}_{k,r}$. Analogously as with the case when k is even, we let H_G be the dual hypergraph of G , and we let H_G^k be the hypergraph obtained from H_G by expanding all edges of H_G of size less than k to edges of size k by adding new vertices of degree 1 to each such edge. Let H_G^k denote the resulting hypergraph, and let $\mathcal{H}_{k,r}^{\text{odd}}$ be the family of all such hypergraphs H_G^k . For example, given the graph $G \in \mathcal{F}_{3,2}$ shown in Figure 5(a) we obtain the associated hypergraph $H \in \mathcal{H}_{3,2}^{\text{odd}}$ shown in Figure 5(b).

Proposition 15. *For $k \geq 3$ an odd integer and $r \geq 1$ arbitrary, if $H \in \mathcal{H}_{k,r}^{\text{odd}}$ has order n and size m , then*

$$\tau(H) = \frac{(k-2)(k+1)n + (k-1)^2m + k-1}{k(k^2-3)}.$$

Proof. We consider the graph $G \in \mathcal{F}_{k,r}$ used to construct the hypergraph $H \in \mathcal{H}_{k,r}^{\text{odd}}$, and so $H = H_G^k$. Assume that ℓ copies of the graph H_{k+2} were added when constructing the graph G . Thus, as observed in [19],

$$\ell = (k-1)|V_2| - |V_1| + 1.$$

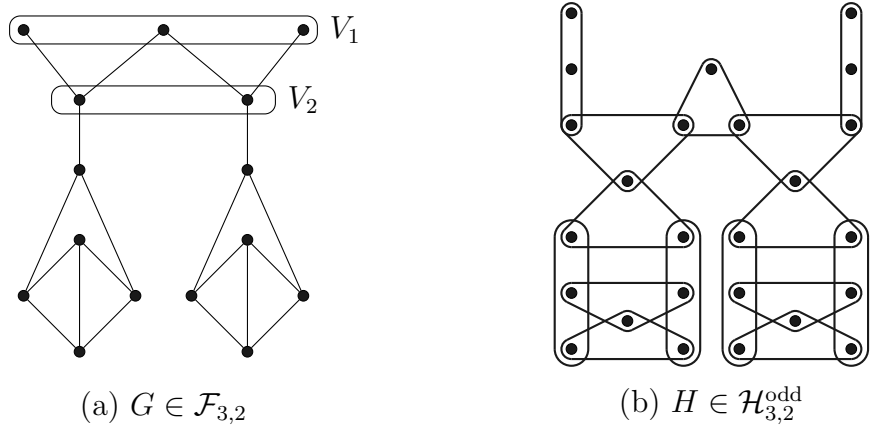


Figure 5: The hypergraph $H \in \mathcal{H}_{3,2}^{\text{odd}}$ associated with the graph $G \in \mathcal{F}_{3,2}$.

Further, the order, size and matching number of G are as follows.

$$\begin{aligned}
 n(G) &= (k^2 + k - 1)|V_2| - (k + 1)|V_1| + (k + 2) \\
 2m(G) &= (k^3 + k^2 - k + 1)|V_2| - (k^2 + 2k - 1)|V_1| + (k^2 + 2k - 1) \\
 2\alpha'(G) &= (k^2 + 1)|V_2| - (k + 1)|V_1| + (k + 1).
 \end{aligned}$$

For $i \in [k]$, let $n_{1,i}$ be the number of vertices in V_1 that have degree i in G . Thus, if $[V_1, V_2]$ denotes the set of edges between V_1 and V_2 in G , then

$$\sum_{i=1}^k n_{1,i} = |V_1| \quad \text{and} \quad \sum_{i=1}^k i \cdot n_{1,i} = |[V_1, V_2]| = k|V_2| - \ell = |V_1| + |V_2| - 1.$$

Recall that $H = H_G^k$, $n = n(H_G^k)$ and $m = m(H_G^k)$. The order of H is

$$\begin{aligned}
 n &= m(G) + \sum_{i=1}^k (k - i) \cdot n_{1,i} \\
 &= m(G) + k \left(\sum_{i=1}^k n_{1,i} \right) - \left(\sum_{i=1}^k i \cdot n_{1,i} \right) \\
 &= m(G) + (k - 1)|V_1| - |V_2| + 1 \\
 &= \left(\frac{k^3 + k^2 - k - 1}{2} \right) |V_2| - \left(\frac{k^2 + 1}{2} \right) |V_1| + \left(\frac{k^2 + 2k + 1}{2} \right).
 \end{aligned}$$

Further, H has size $m = m(H_G^k) = m(H_G) = n(G)$, and so

$$m = (k^2 + k - 1)|V_2| - (k + 1)|V_1| + (k + 2).$$

We remark that the graph $G \in \mathcal{F}_{k,r}$ used to construct the hypergraph $H_G^k \in \mathcal{H}_{k,r}^{\text{odd}}$ is in fact the dual graph (see Section 3.1) of H_G^k . Therefore, applying Proposition 9 to H and its dual graph G , we have

$$\begin{aligned} \tau(H) &= m - \alpha'(G) \\ &= ((k^2 + k - 1)|V_2| - (k + 1)|V_1| + (k + 2)) \\ &\quad - \frac{1}{2} ((k^2 + 1)|V_2| - (k + 1)|V_1| + (k + 1)) \\ &= \left(\frac{k^2 + 2k - 3}{2} \right) |V_2| - \left(\frac{k + 1}{2} \right) |V_1| + \frac{k + 3}{2} \end{aligned}$$

and

$$\begin{aligned} &\frac{(k - 2)(k + 1)n + (k - 1)^2m + k - 1}{k(k^2 - 3)} \\ &= \left(\frac{(k - 2)(k + 1)}{k(k^2 - 3)} \right) \left(\left(\frac{k^3 + k^2 - k - 1}{2} \right) |V_2| - \left(\frac{k^2 + 1}{2} \right) |V_1| + \left(\frac{k^2 + 2k + 1}{2} \right) \right) \\ &\quad + \left(\frac{(k - 1)^2}{k(k^2 - 3)} \right) ((k^2 + k - 1)|V_2| - (k + 1)|V_1| + (k + 2)) \\ &\quad + \frac{k - 1}{k(k^2 - 3)} \\ &= \left(\frac{k^2 + 2k - 3}{2} \right) |V_2| - \left(\frac{k + 1}{2} \right) |V_1| + \frac{k + 3}{2}. \end{aligned}$$

Equality therefore holds in the statement of Theorem 14. \square

4.2 Strong Independence Number

In this section we establish a lower bound on the strong independence number of a connected, linear, k -uniform hypergraph H with maximum degree 2 for $k \geq 2$. For this purpose, we first establish a lower bound on a maximum strong independent set consisting only of degree-2 vertices in H .

Theorem 16. *For all even $k \geq 2$ the following holds.*

- (a) *If $H \in \mathcal{H}_k$, then $\alpha_2(H) \geq \frac{n_1 + (k^2 + 2)n_2 - k(k + 1)}{k^2(k + 1)}$.*
- (b) *If $H \in \mathcal{H}'_k$, then $\alpha_2(H) \geq \frac{n_1 + (k^2 + 2)n_2 - 3k}{k^2(k + 1)}$.*
- (c) *If $H \in \mathcal{H}''_k$, then $\alpha_2(H) \geq \frac{n_1 + (k^2 + 2)n_2 - k}{k^2(k + 1)}$.*

Proof. Let $k \geq 2$ be even and let $H \in \mathcal{H}_k$, and let G_H be the dual graph of H . We adopt the notation in the proof of Theorem 12. Analogously as in the proof of Theorem 12,

$$\alpha'(G_H) \geq \frac{m}{k(k + 1)} + \frac{n_2}{k + 1} - \frac{\theta}{k(k + 1)}.$$

By Observation 8, we note that the following therefore holds.

$$\begin{aligned}\alpha_2(H) &= \alpha'(G_H) \\ &\geq \frac{m}{k(k+1)} + \frac{n_2}{k+1} - \frac{\theta}{k(k+1)} \\ &= \left(\frac{1}{k(k+1)}\right) \left(\frac{n_1 + 2n_2}{k}\right) + \frac{n_2}{k+1} - \frac{\theta}{k(k+1)}.\end{aligned}$$

Multiplying through with $k^2(k+1)$ we obtain the following.

$$k^2(k+1)\alpha_2(H) \geq n_1 + (k^2 + 2)n_2 - k\theta.$$

This implies the desired result. □

We proceed further with the following simple lemma.¹

Lemma 17. ([3]) *If H is a k -uniform hypergraph of order n and size m with $\delta(H) \geq 1$ and with c components, then $(k-1)m + c \geq n$.*

Proof. Replace each hyperedge $e \in E(H)$ by a star of $k-1$ edges on the vertex set of e to produce a graph G . If H has c components, then so too does G . Since G has $(k-1)m$ edges, n vertices and c components, we have that $(k-1)m + c \geq n$. □

As a special case of Lemma 17, we note that if H is a connected k -uniform hypergraph of order n and size m , then $(k-1)m + 1 \geq n$.

Theorem 18. *For all even $k \geq 2$ the following holds.*

- (a) *If $H \in \mathcal{H}_k$, then $\alpha(H) \geq \frac{(k+2)n - (k-1)m - (k+1)}{k(k+1)}$.*
- (b) *If $H \in \mathcal{H}'_k$, then $\alpha(H) \geq \frac{(k+2)n - (k-1)m - 3}{k(k+1)}$.*
- (c) *If $H \in \mathcal{H}''_k$, then $\alpha(H) \geq \frac{(k+2)n - (k-1)m - 1}{k(k+1)}$.*

Proof. Let $k \geq 2$ be even and let $H \in \mathcal{H}''_k$ be arbitrary. Let $V_1(H)$ denote the set of all vertices of degree 1 in H , and let S be the set of all edges of H that contain at least one vertex in $V_1(H)$. Let R be the vertices in H which belong to two edges of S , and let $r = |R|$. Let $X = V(H) \setminus (V_1(H) \cup R)$ and consider the hypergraph $H[X]$ induced by the vertices in X . Let S' be the set of edges in $H[X]$ of size less than k . We note that each edge in S' was obtained by shrinking an edge in S by removing from it vertices in $V_1(H) \cup R$. We note that $H[X]$ contains at most $r + 1$ components; that is, $c(H[X]) \leq r + 1$.

¹We have not been able to find the original source of this lemma, but as remarked in [3], “it definitely seems to have been known already at least in the early 1960’s.” For completeness, we provide the short proof given in [3].

Let H' be obtained from $H[X]$ by removing all edges in $H[X]$ of size less than k . Equivalently, H' is obtained from H by removing all edges in S and all resulting isolated vertices. We note that H' has order

$$n(H') = n(H) - n_1(H) - r$$

and may possibly be the empty hypergraph. For every $i = \{0\} \cup [k-1]$, let T_i denote the subset of edges of S which contain vertices from exactly i different components in H' and let $t_i = |T_i|$. We note that for $i \in [k-1] \setminus \{1\}$, the removal of all edges in T_i from $H[X]$ gives rise to at most $(i-1)t_i$ additional components. Thus,

$$c(H') \leq c(H[X]) + \sum_{i=2}^{k-1} (i-1)t_i.$$

As observed earlier, $c(H[X]) \leq r+1$, implying that

$$\sum_{i=2}^{k-1} (i-1)t_i \geq c(H') - r - 1.$$

Every edge in T_i contains at most $k-i$ vertices of degree 1 in H , and at least i vertices from different components of H' , in addition to possibly some vertices of R . Thus,

$$\begin{aligned} n_1(H) &\leq k|S| - \left(\sum_{i=1}^{k-1} i \cdot t_i \right) - 2r \\ &= k|S| - \left(\sum_{i=0}^{k-1} t_i \right) - \left(\sum_{i=2}^{k-1} (i-1)t_i \right) - 2r + t_0 \\ &\leq k|S| - |S| - (c(H') - r - 1) - 2r + t_0 \\ &= (k-1)|S| - c(H') - r + t_0 + 1. \end{aligned}$$

We now obtain a strong independent set in H by taking a maximum strong independent set of degree-2 vertices in H' and adding to this set a vertex of degree one from each edge in S . Therefore the following holds by Theorem 16, as no component belongs to $\{L_k\} \cup \mathcal{M}_k$ (recall that $H \in \mathcal{H}_k''$).

$$\alpha(H) \geq |S| + \alpha_2(H') \geq |S| + \frac{n_1(H') + (k^2 + 2)n_2(H') - k \cdot c(H')}{k^2(k+1)}.$$

As $n_1(H') + n_2(H') = n(H') = n(H) - n_1(H) - r$, we note that

$$n_2(H') = n_2(H) - n_1(H') - r.$$

Furthermore,

$$n_1(H') = k|S| - n_1(H) - 2r,$$

as the $|S|$ edges in S each have k vertices and every vertex with degree 1 in H' belongs to an edge in S and does not have degree 1 in H and does not belong to R , and every

vertex in R counts two in $k|S| - n_1(S)$ but does not belong to H' . The following now holds by the above observations.

$$\begin{aligned}
& k^2(k+1)\alpha(H) \\
& \geq k^2(k+1)|S| + n_1(H') + (k^2+2)n_2(H') - k \cdot c(H') \\
& = k^2(k+1)|S| + n_1(H') + (k^2+2)(n_2(H) - r - n_1(H')) - k \cdot c(H') \\
& = k^2(k+1)|S| + n_1(H')(1 - (k^2+2)) + (k^2+2)n_2(H) - (k^2+2)r - k \cdot c(H') \\
& = k^2(k+1)|S| + (k|S| - n_1(H) - 2r)(-k^2 - 1) + (k^2+2)n_2(H) - (k^2+2)r - k \cdot c(H') \\
& = (k^3 + k^2 - k^3 - k)|S| + n_1(H)(k^2 + 1) + (k^2+2)n_2(H) + k^2r - k \cdot c(H') \\
& = (k(k-1)|S| - k \cdot c(H') - kr + kt_0 + k) \\
& \quad - kt_0 + n_1(H)(k^2 + 1) + (k^2+2)n_2(H) + (k^2+k)r - k \\
& \geq (k \cdot n_1(H)) - kt_0 + n_1(H)(k^2 + 1) + (k^2+2)n_2(H) + (k^2+k)r - k \\
& = (k^2+k+1)n_1(H) + (k^2+2)n_2(H) + (k^2+k)r - kt_0 - k \\
& = (k^2+2k)(n_1(H) + n_2(H)) - (k-1)(n_1(H) + 2n_2(H)) + (k^2+k)r - kt_0 - k \\
& = (k^2+2k)n(H) - (k-1)(k \cdot m(H)) + (k^2+k)r - kt_0 - k.
\end{aligned}$$

Note that every edge in T_0 must contain a vertex from R . In particular, if $r = 0$, then $t_0 = 0$. In this case, dividing through by k the above simplifies to the following.

$$k(k+1)\alpha(H) \geq (k+2)n(H) - (k-1)m(H) - 1.$$

Suppose that $r \geq 1$. We note that every edge in T_0 contains at most $k-1$ vertices from R , and so $t_0 \leq (k-1)r$. Dividing through by k above we get the following.

$$\begin{aligned}
k(k+1)\alpha(H) & \geq (k+2)n(H) - (k-1)m(H) + (k+1)r - t_0 - 1 \\
& \geq (k+2)n(H) - (k-1)m(H) + (k+1)r - (k-1)r - 1 \\
& = (k+2)n(H) - (k-1)m(H) + 2r - 1 \\
& \geq (k+2)n(H) - (k-1)m(H) - 1.
\end{aligned}$$

This implies the theorem in the case when $H \in \mathcal{H}_k''$.

Suppose next that $H \in \mathcal{H}_k'$. If $H \notin \mathcal{M}_k$, then as shown above we have $k(k+1)\alpha(H) \geq (k+2)n(H) - (k-1)m(H) - 1$. Suppose, therefore, that $H \in \mathcal{M}_k$. We note that, by Theorem 16,

$$\alpha_2(H) \geq \frac{n_1(H) + (k^2+2)n_2(H) - 3k}{k^2(k+1)}.$$

As H is 2-regular, we have $\alpha(H) = \alpha_2(H)$ and $n_1(H) = 0$, and therefore $n(H) = n_2(H) = k(k+3)/2$ and $k \cdot m(H) = 2n_2(H) = k(k+3)$. Therefore,

$$\alpha(H) \geq \frac{(k^2+2)n_2(H) - 3k}{k^2(k+1)}$$

$$\begin{aligned}
&= \frac{k(k+2)n_2(H) - 2(k-1)n_2(H) - 3k}{k^2(k+1)} \\
&= \frac{k(k+2)n(H) - (k-1)(k \cdot m(H)) - 3k}{k^2(k+1)} \\
&= \frac{(k+2)n(H) - (k-1)m(H) - 3}{k(k+1)}.
\end{aligned}$$

This implies the theorem in the case when $H \in \mathcal{H}'_k$.

Suppose finally that $H \in \mathcal{H}_k$. From the above, it remains for us to consider the case when $H = L_k$. In this case Theorem 16 implies that

$$\alpha_2(H) \geq \frac{n_1(H) + (k^2 + 2)n_2(H) - k(k+1)}{k^2(k+1)}.$$

As H is 2-regular, we have $\alpha(H) = \alpha_2(H)$ and $n_1(H) = 0$, and therefore $n(H) = n_2(H) = k(k+1)/2$ and $k \cdot m(H) = 2n_2(H) = k(k+1)$. Analogous to the discussion in the previous argument,

$$\alpha(H) \geq \frac{(k+2)n(H) - (k-1)m(H) - (k+1)}{k(k+1)},$$

This implies the theorem in the case when $H \in \mathcal{H}_k$. □

We discuss next the hypergraphs $H \in \mathcal{H}_k$ for $k \geq 2$ even that achieve the lower bound for the strong independence number in the statement of Theorem 18. If $H = L_k$, then, by Observation 8 and Theorem 1(a), equality holds in the statement of Theorem 18(a). If $H \in \mathcal{M}_k$, then, by Observation 10 and Theorem 1(b), equality holds in the statement of Theorem 18(b).

We show next that there is an infinite family of hypergraphs $H \in \mathcal{H}''_k$ for which equality holds in the statement of Theorem 18(c). For $k \geq 4$ an even integer and $r \geq 1$, let G be an arbitrary graph in the family $\mathcal{G}_{k,r}$, and let H_G^k be the associated hypergraph in the family $\mathcal{H}_{k,r}^{\text{even}}$. For each vertex v of degree 1 in H_G^k , we add $k-1$ new vertices and an edge (of size k) containing v and these new vertices. Let R_G^k denote the resulting hypergraph, and let $\mathcal{R}_{k,r}$ be the family of all such hypergraphs R_G^k .

Proposition 19. *For $k \geq 4$ an even integer and $r \geq 1$ arbitrary, if $H \in \mathcal{R}_{k,r}^{\text{even}}$ has order n and size m , then*

$$\alpha(H) = \frac{(k+2)n - (k-1)m - 1}{k(k+1)}.$$

Proof. Let $G \in \mathcal{G}_{k,r}$ be the graph and $H_G^k \in \mathcal{H}_{k,r}^{\text{even}}$ the associated hypergraph used to construct the hypergraph $H \in \mathcal{R}_{k,r}^{\text{even}}$, and so $H = R_G^k$.

We show firstly that $\alpha(R_G^k) = n_1(H_G^k) + \alpha'(G)$. Let S be a maximum independent set in $H = R_G^k$ that contains the maximum number of vertices of degree 1 in H . For each vertex v of degree 1 in H_G^k , let e_v be the associated edge containing v that was

added to H_G^k when constructing H . We note that every vertex in e_v different from v has degree 1 in H . Let v' be an arbitrary vertex in e_v different from v . If $v \in S$ or if S contains no vertex from e_v , then the set $(S \setminus \{v\}) \cup \{v'\}$ is a maximum independent set containing more vertices of degree 1 than does S , a contradiction. Hence, the set S contains $n_1(H_G^k)$ vertices of degree 1, one from each edge added to H_G^k when constructing H . The remaining vertices of S belong to $V(H_G)$ and have degree 2 in H_G^k , and so $\alpha(H) \leq n_1(H_G^k) + \alpha_2(H_G^k)$. Conversely, every maximum independent set of degree-2 vertices in H_G^k can be extended to an independent set in H by adding to it $n_1(H_G^k)$ vertices of degree 1, one vertex from each edge added to H_G^k when constructing H , implying that $\alpha(H) \geq n_1(H_G^k) + \alpha_2(H_G^k)$. Consequently, $\alpha(H) = n_1(H_G^k) + \alpha_2(H_G^k)$. We note that G is the dual graph of the hypergraph $H_G^k \in \mathcal{H}_k$, and so, by Observation 8, $\alpha_2(H_G^k) = \alpha'(G)$. Therefore, $\alpha(H_G^k) = n_1(H_G^k) + \alpha'(G)$.

Let G be constructed from ℓ_1 single vertices and ℓ_2 copies of $K_{k+1} - e$. Further, let $\ell_{1,1}$ and $\ell_{1,2}$ be the number of single vertices of degree 1 and degree 2 in G , and let $\ell_{2,1}$ and $\ell_{2,2}$ be the number of copies of $K_{k+1} - e$ joined to one or two vertices in Y , respectively. We note that $(\ell_{1,1} + \ell_{1,2}) + (\ell_{2,1} + \ell_{2,2}) = \ell_1 + \ell_2 = \ell = r(k-1) + 1$ and $n(G) = r + \ell_1 + \ell_2(k+1)$. Recall that $n = n(H)$ and $m = m(H)$. We note that

$$\begin{aligned} n &= n(H_G^k) + (k-1)n_1(H_G^k) \\ m &= m(H_G^k) + n_1(H_G^k) \\ n_1(H_G^k) &= (k-1)\ell_{1,1} + (k-2)\ell_{1,2} + \ell_{2,1} \\ r &= \ell_{1,2} + \ell_{2,2} + 1 \end{aligned}$$

Recall (see the proof of Proposition 13) that

$$\begin{aligned} n(H_G^k) &= k\ell_1 + \frac{1}{2}(k^2 + k + 2)\ell_2 \\ m(H_G^k) &= \frac{1}{k-1}(k\ell_1 + k^2\ell_2 - 1) \\ \alpha'(G) &= r + \frac{1}{2}k\ell_2. \end{aligned}$$

We note that

$$\begin{aligned} &\frac{1}{k}(\ell_{1,1} + 2\ell_{1,2} + \ell_{2,1} + \ell_{2,2}) \\ &= \frac{1}{k}(\ell + \ell_{1,2} + 2\ell_{2,2}) \\ &= \frac{1}{k}(\ell + r - 1) \\ &= \frac{1}{k}((r(k-1) + 1) + r - 1) \\ &= r. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{(k+2)n - (k-1)m - 1}{k(k+1)} &= \left(\frac{k+2}{k(k+1)}\right) \left(n(H_G^k) + (k-1)n_1(H_G^k)\right) \\ &\quad - \left(\frac{k-1}{k(k+1)}\right) \left(m(H_G^k) + n_1(H_G^k)\right) \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{k(k+1)} \\
& = \left(\frac{k+2}{k(k+1)}\right) \left(k\ell_1 + \left(\frac{k^2+k+2}{2}\right)\ell_2 + (k-1)n_1(H_G^k)\right) \\
& \quad - \left(\frac{k-1}{k(k+1)}\right) \left(\left(\frac{k}{k-1}\right)\ell_1 + \left(\frac{k^2}{k-1}\right)\ell_2 - \frac{1}{k-1} + n_1(H_G^k)\right) \\
& \quad -\frac{1}{k(k+1)} \\
& = \left(\frac{k(k+2)-k}{k(k+1)}\right)\ell_1 \\
& \quad + \left(\frac{k^3+k^2+4k+4}{2k(k+1)}\right)\ell_2 \\
& \quad + \left(\frac{k^2-1}{k(k+1)}\right)n_1(H_G^k) \\
& = \ell_1 + \left(\frac{k^2+4}{2k}\right)\ell_2 + \left(\frac{k-1}{k}\right)n_1(H_G^k) \\
& = (\ell_{1,1} + \ell_{1,2}) + \left(\frac{k^2+4}{2k}\right)(\ell_{2,1} + \ell_{2,2}) \\
& \quad + \left(\frac{k-1}{k}\right)((k-1)\ell_{1,1} + (k-2)\ell_{1,2} + \ell_{2,1}) \\
& = \left(\frac{k^2-k+1}{k}\right)\ell_{1,1} + \left(\frac{k^2-2k+2}{2}\right)\ell_{1,2} \\
& \quad + \left(\frac{k^2+2k+2}{2k}\right)\ell_{2,1} + \left(\frac{k^2+4}{2k}\right)\ell_{2,1} \\
& = (k-1)\ell_{1,1} + (k-2)\ell_{1,2} + \ell_{2,1} \\
& \quad + \frac{1}{k}(\ell_{1,1} + 2\ell_{1,2} + \ell_{2,1} + 2\ell_{2,2}) + \frac{1}{2}k(\ell_{2,1} + \ell_{2,2}) \\
& = n_1(H_G^k) + r + \frac{1}{2}k\ell \\
& = n_1(H_G^k) + \alpha'(G) \\
& = \alpha(H). \quad \square
\end{aligned}$$

Next we consider the case when $k \geq 3$ is odd.

Theorem 20. For $k \geq 3$ odd, if $H \in \mathcal{H}_k$, then

$$\alpha_2(H) \geq \frac{(k-1)n_1 + (k^3 - k^2 - 2)n_2 - k(k-1)}{k^2(k^2 - 3)}.$$

Proof. Let $k \geq 3$ be odd and let $H \in \mathcal{H}_k$. Let G_H be the dual graph of H and note that G_H has maximum degree $\Delta(G) \leq k$. Further, we note that G_H is a connected graph of order $n(G_H) = m$ and size $m(G_H) = n_2$. By Theorem 3, the following holds.

$$\alpha'(G_H) \geq \left(\frac{k-1}{k(k^2-3)}\right)m + \left(\frac{k^2-k-2}{k(k^2-3)}\right)n_2 - \frac{k-1}{k(k^2-3)}.$$

By Observation 8, and noting that $km = n_1 + 2n_2$, the following therefore holds.

$$\alpha_2(H) = \alpha'(G_H) \geq \left(\frac{k-1}{k(k^2-3)} \right) \left(\frac{n_1+2n_2}{k} \right) + \left(\frac{k^2-k-2}{k(k^2-3)} \right) n_2 - \frac{k-1}{k(k^2-3)}.$$

Multiplying through with $k^2(k^2-3)$, and simplifying, we obtain the following.

$$k^2(k^2-3)\alpha_2(H) \geq (k-1)n_1 + (k^3-k^2-2)n_2 - k(k-1).$$

This implies the desired result. □

Theorem 21. For $k \geq 3$ odd, if $H \in \mathcal{H}_k$, then

$$\alpha(H) \geq \frac{(k^2+k-4)n(H) - (k-1)^2m(H) - (k-1)}{k(k^2-3)}.$$

Proof. Let $k \geq 3$ be odd and let $H \in \mathcal{H}_k$. We follow the same notation as introduced in the proof of Theorem 18. Proceeding exactly as in the proof of Theorem 18, we have

$$\begin{aligned} n_1(H) &\leq (k-1)|S| - c(H') - r + t_0 + 1 \\ n_1(H') &= k|S| - n_1(H) - 2r \\ n_2(H') &= n_2(H) - n_1(H') - r \end{aligned}$$

The following holds by Theorem 20.

$$\alpha(H) \geq |S| + \alpha_2(H') \geq |S| + \frac{(k-1)n_1(H') + (k^3-k^2-2)n_2(H') - k(k-1)c(H')}{k^2(k^2-3)}.$$

Therefore,

$$\begin{aligned} k^2(k^2-3)\alpha(H) &\geq k^2(k^2-3)|S| + (k-1)n_1(H') + (k^3-k^2-2)n_2(H') - k(k-1)c(H') \\ &= k^2(k^2-3)|S| + (k-1)n_1(H') + (k^3-k^2-2)(n_2(H) - n_1(H') - r) \\ &\quad - k(k-1)c(H') \\ &= k^2(k^2-3)|S| + (-k^3+k^2+k+1)n_1(H') \\ &\quad + (k^3-k^2-2)n_2(H) - (k^3-k^2-2)r - k(k-1)c(H') \\ &= k^2(k^2-3)|S| + (-k^3+k^2+k+1)(k|S| - n_1(H) - 2r) \\ &\quad + (k^3-k^2-2)n_2(H) - (k^3-k^2-2)r - k(k-1)c(H') \\ &= (k^4-3k^2-k^4+k^3+k^2+k)|S| + (k^3-k^2-k-1)n_1(H) \\ &\quad + (k^3-k^2-2)n_2(H) + (k^3-k^2-2k)r - k(k-1)c(H') \\ &= k(k-1)^2|S| + (k^3-k^2-k-1)n_1(H) \\ &\quad + (k^3-k^2-2)n_2(H) + (k^3-k^2-2k)r - k(k-1)c(H') \\ &= k(k-1)((k-1)|S| - c(H') - r + t_0 + 1) \\ &\quad - k(k-1)t_0 - k(k-1) + (k^3-k^2-k-1)n_1(H) \end{aligned}$$

$$\begin{aligned}
& + (k^3 - k^2 - 2)n_2(H) + (k^3 - 3k)r \\
\geq & k(k-1)n_1(H) - k(k-1)t_0 - k(k-1) + (k^3 - k^2 - k - 1)n_1(H) \\
& + (k^3 - k^2 - 2)n_2(H) + k(k^2 - 3)r \\
= & (k^3 - 2k - 1)n_1(H) + (k^3 - k^2 - 2)n_2(H) \\
& + k(k^2 - 3)r - k(k-1)t_0 - k(k-1) \\
= & k(k^2 + k - 4)(n_1(H) + n_2(H)) - (k-1)^2(n_1(H) + 2n_2(H)) \\
& + k(k^2 - 3)r - k(k-1)t_0 - k(k-1) \\
= & k(k^2 + k - 4)n(H) - k(k-1)^2m(H) + k(k^2 - 3)r - k(k-1)t_0 - k(k-1).
\end{aligned}$$

Dividing through by k , the above simplifies to

$$k(k^2 - 3)\alpha(H) \geq (k^2 + k - 4)n(H) - (k-1)^2m(H) + (k^2 - 3)r - (k-1)t_0 - (k-1).$$

As observed in the proof of Theorem 18, if $r = 0$, then $t_0 = 0$, while if $r \geq 1$, then $t_0 \leq (k-1)r$. If $r = 0$, then the above simplifies to the following.

$$k(k^2 - 3)\alpha(H) \geq (k^2 + k - 4)n(H) - (k-1)^2m(H) - (k-1).$$

If $r \geq 1$, then the above simplifies to the following.

$$\begin{aligned}
k(k^2 - 3)\alpha(H) & \geq (k^2 + k - 4)n(H) - (k-1)^2m(H) \\
& + (k^2 - 3)r - (k-1)^2r - (k-1) \\
& = (k^2 + k - 4)n(H) - (k-1)^2m(H) + 2(k-2)r - (k-1) \\
& \geq k(k^2 + k - 4)n(H) - (k-1)^2m(H) + k - 3 \\
& \geq k(k^2 + k - 4)n(H) - (k-1)^2m(H) \\
& > k(k^2 + k - 4)n(H) - (k-1)^2m(H) - (k-1).
\end{aligned}$$

This completes the proof of Theorem 21. □

We show next that there is an infinite family of hypergraphs $H \in \mathcal{H}_k$ for which equality holds in the statement of Theorem 21. For $k \geq 3$ an odd integer and $r \geq 1$, let G be an arbitrary graph in the family $\mathcal{F}_{k,r}$, and let H_G^k be the associated hypergraph in the family $\mathcal{H}_{k,r}^{\text{odd}}$. For each vertex v of degree 1 in H_G^k , we add $k-1$ new vertices and an edge (of size k) containing v and these new vertices. Let R_G^k denote the resulting hypergraph, and let $\mathcal{R}_{k,r}^{\text{odd}}$ be the family of all such hypergraphs R_G^k .

Proposition 22. *For $k \geq 3$ an odd integer and $r \geq 1$ arbitrary, if $H \in \mathcal{R}_{k,r}^{\text{odd}}$ has order n and size m , then*

$$\alpha(H) = \frac{(k^2 + k - 4)n(H) - (k-1)^2m(H) - (k-1)}{k(k^2 - 3)}.$$

Proof. Let $G \in \mathcal{F}_{k,r}$ be the graph and $H_G^k \in \mathcal{H}_{k,r}^{\text{odd}}$ the associated hypergraph used to construct the hypergraph $H \in \mathcal{R}_{k,r}^{\text{odd}}$, and so $H = R_G^k$. Analogous to the proof of Proposition 19, we have that

$$\alpha(H) = n_1(H_G^k) + \alpha'(G).$$

For $i \in [k]$, let $n_{1,i}$ be the number of vertices in V_1 that have degree i in G . As shown in the proof of Proposition 13,

$$\sum_{i=1}^k n_{1,i} = |V_1| \quad \text{and} \quad \sum_{i=1}^k i \cdot n_{1,i} = |V_1| + |V_2| - 1,$$

implying that

$$n_1(H_G^k) = \sum_{i=1}^k (k-i)n_{1,i} = k \sum_{i=1}^k n_{1,i} - \sum_{i=1}^k i \cdot n_{1,i} = (k-1)|V_1| - |V_2| + 1.$$

Recall that $n = n(H)$ and $m = m(H)$. We note that

$$\begin{aligned} n &= n(H_G^k) + (k-1)n_1(H_G^k) \\ m &= m(H_G^k) + n_1(H_G^k) \end{aligned}$$

Recall (see the proof of Proposition 13) that

$$\begin{aligned} n(H_G^k) &= \left(\frac{k^3 + k^2 - k - 1}{2} \right) |V_2| - \left(\frac{k^2 + 1}{2} \right) |V_1| + \left(\frac{k^2 + 2k + 1}{2} \right) \\ m(H_G^k) &= (k^2 + k - 1)|V_2| - (k + 1)|V_1| + (k + 2) \\ \alpha'(G) &= \frac{1}{2} ((k^2 + 1)|V_2| - (k + 1)|V_1| + (k + 1)) \end{aligned}$$

Therefore,

$$\begin{aligned} & \frac{(k^2 + k - 4)n - (k - 1)^2 m - (k - 1)}{k(k^2 - 3)} \\ &= \left(\frac{k^2 + k - 4}{k(k^2 - 3)} \right) (n(H_G^k) + (k - 1)n_1(H_G^k)) \\ & \quad - \left(\frac{(k - 1)^2}{k(k^2 - 3)} \right) (m(H_G^k) + n_1(H_G^k)) \\ & \quad - \frac{k - 1}{k(k^2 - 3)} \\ &= \left(\frac{k^2 + k - 4}{k(k^2 - 3)} \right) \left(\left(\frac{k^3 + k^2 - k - 1}{2} \right) |V_2| - \left(\frac{k^2 + 1}{2} \right) |V_1| + \left(\frac{k^2 + 2k + 1}{2} \right) \right. \\ & \quad \left. + (k - 1)n_1(H_G^k) \right) - \left(\frac{(k - 1)^2}{k(k^2 - 3)} \right) ((k^2 + k - 1)|V_2| - (k + 1)|V_1| \\ & \quad + (k + 2) + n_1(H_G^k)) - \frac{k - 1}{k(k^2 - 3)} \\ &= \left(\frac{k^5 - 2k^3 - 2k^2 - 3k + 6}{2k(k^2 - 3)} \right) |V_2| - \left(\frac{k^4 - k^3 - k^2 + 3k - 6}{2k(k^2 - 3)} \right) |V_1| \\ & \quad + \left(\frac{k^3 - k^2 - 3k + 3}{k(k^2 - 3)} \right) n_1(H_G^k) + \frac{k^4 + k^3 - k^2 - 3k - 6}{2k(k^2 - 3)} \end{aligned}$$

$$\begin{aligned}
&= \left(\frac{k^3 + k - 2}{2k}\right) |V_2| - \left(\frac{k^2 - k + 2}{2k}\right) |V_1| + \left(\frac{k - 1}{k}\right) n_1(H_G^k) + \frac{k^2 + k + 2}{2k} \\
&= \left(\frac{k^3 + k - 2}{2k}\right) |V_2| - \left(\frac{k^2 - k + 2}{2k}\right) |V_1| + n_1(H_G^k) - \frac{1}{k} ((k - 1)|V_1| - |V_2| + 1) \\
&\quad + \frac{k^2 + k + 2}{2k} \\
&= n_1(H_G^k) + \left(\frac{k^2 + 1}{2}\right) |V_2| - \left(\frac{k + 1}{2}\right) |V_1| + \frac{k + 1}{2} \\
&= n_1(H_G^k) + \alpha'(G) \\
&= \alpha(H).
\end{aligned}$$

This completes the proof of Proposition 22. □

5 Summary

For small values of $k \geq 3$, the results in this paper are summarized in Table 1 and Table 2 below.

k even	H has order n and size m .		
	$H \in \mathcal{H}_k$	$H \in \mathcal{H}'_k$	$H \in \mathcal{H}''_k$
$k = 4$	$20\tau(H) \leq 4n + 3m + 5$ $20\alpha(H) \geq 6n - 3m - 5$	$20\tau(H) \leq 4n + 3m + 3$ $20\alpha(H) \geq 6n - 3m - 3$	$20\tau(H) \leq 4n + 3m + 1$ $20\alpha(H) \geq 6n - 3m - 1$
$k = 6$	$42\tau(H) \leq 6n + 5m + 7$ $42\alpha(H) \geq 8n - 5m - 7$	$42\tau(H) \leq 6n + 5m + 3$ $42\alpha(H) \geq 8n - 5m - 3$	$42\tau(H) \leq 6n + 5m + 1$ $42\alpha(H) \geq 8n - 5m - 1$
$k = 8$	$72\tau(H) \leq 8n + 7m + 9$ $72\alpha(H) \geq 10n - 7m - 9$	$72\tau(H) \leq 8n + 7m + 3$ $72\alpha(H) \geq 10n - 7m - 3$	$72\tau(H) \leq 8n + 7m + 1$ $72\alpha(H) \geq 10n - 7m - 1$

Table 1. Results for small values of even $k \geq 4$

k odd	$H \in \mathcal{H}_k$ has order n and size m .
$k = 3$	$9\tau(H) \leq 2n + 2m + 1$ $9\alpha(H) \geq 4n - 2m - 1$
$k = 5$	$55\tau(H) \leq 9n + 8m + 2$ $55\alpha(H) \geq 13n - 8m - 2$
$k = 7$	$161\tau(H) \leq 20n + 18m + 3$ $161\alpha(H) \geq 26n - 18m - 3$

Table 2. Results for small values of odd $k \geq 3$

We have further shown that in each of the inequality statements involving the transversal number or the independence number, there is an infinite family of hypergraphs $H \in \mathcal{H}_k$ for which equality holds, implying that all the bounds are tight.

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