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Higher order scheme-independent calculations of physical quantities in the conformal phase of a gauge theory

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We consider an asymptotically free vectorial $\mathrm{SU}(N_c)$ gauge theory with N_f massless fermions in a representation R, having an infrared fixed point (IRFP) of the renormalization group at α_{IR} in the conformal non-Abelian Coulomb phase. The cases with R equal to the fundamental, adjoint, and symmetric rank-2 tensor representation are considered. We present scheme-independent calculations of the anomalous dimension $\gamma_{\bar{\psi}\psi,\mathrm{IR}}$ to $O(\Delta_f^4)$ and β'_{IR} to $O(\Delta_f^5)$ at this IRFP, where Δ_f is an N_f-dependent expansion parameter. Comparisons are made with conventional n-loop calculations and lattice measurements. As a test of the accuracy of the Δ_f expansion, we calculate $\gamma_{\bar{\psi}\psi,\mathrm{IR}}$ to $O(\Delta_f^3)$ in $\mathcal{N}=1$ $\mathrm{SU}(N_c)$ supersymmetric quantum chromodynamics and find complete agreement, to this order, with the exactly known expression. The Δ_f expansion also avoids a problem in which an IRFP may not be manifest as an IR zero of a higher n-loop beta function.

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A fundamental problem in quantum field theory concerns the properties at a conformal fixed point of the renormalization group. A specific question under intensive current investigation concerns the properties of an asymptotically free (AF) non-Abelian Yang-Mills vectorial gauge theory (in $d=4$ spacetime dimensions) with a set of massless fermions at an IRFP of the renormalization group in the Coulomb phase, where it exhibits scale and conformal invariance [1,2]. Here we consider a theory of this type, with gauge group $G=\mathrm{SU}(N_c)$ and N_f massless fermions ψ_j, $1\le j\le N_f$, in a representation R, where R is the fundamental (F), adjoint (adj), or symmetric rank-2 tensor (S). The dependence of the gauge coupling $g=g(\mu)$ on the Euclidean momentum scale μ is described by the beta function, $\beta=d\alpha/dt$, where $\alpha(\mu)=g(\mu)^2/(4\pi)$ and $dt=d\ln\mu$. The IRFP occurs at an IR zero of β at α_{IR}. At this fixed point, an operator $\mathcal{O}$ for a physical quantity exhibits scaling behavior with a dimension $D_{\mathcal{O}}=D_{\mathcal{O},\mathrm{free}}-\gamma_{\mathcal{O}}$, where $D_{\mathcal{O},\mathrm{free}}$ is the free-field dimension and $\gamma_{\mathcal{O}}$ is the anomalous dimension.

Two important quantities that characterize the properties at the IRFP α_{IR} are $\gamma_{\bar{\psi}\psi}$ [3] and $\beta'\equiv d\beta/d\alpha$, denoted $\gamma_{\bar{\psi}\psi,\mathrm{IR}}$ and β'_{IR}. Here, β'_{IR} is equivalent to the anomalous dimension of $F_{a,\mu\nu}F_a^{\mu\nu}$, where $F_a^{\mu\nu}$ is the (rescaled) field-strength tensor [4]. As physical quantities, $\gamma_{\bar{\psi}\psi,\mathrm{IR}}$ and β'_{IR} are scheme-independent (SI) [5]. However, conventional series expansions of these quantities in powers of α, calculated to a finite order, do not maintain this scheme independence beyond the lowest orders. Clearly, it is very valuable to calculate and analyze series expansions for $\gamma_{\bar{\psi}\psi,\mathrm{IR}}$ and β'_{IR} that are scheme-independent at each order. Some early work was in [6,7]. A natural expansion variable is

$$\Delta_f = N_u - N_f, \tag{1}$$

where, for a given N_c and R, N_u is the upper (u) limit to N_f allowed by asymptotic freedom. Scheme-independent series expansions of $\gamma_{\bar{\psi}\psi,\mathrm{IR}}$ and β'_{IR} are [8]

$$\gamma_{\bar{\psi}\psi,\mathrm{IR}} = \sum_{j=1}^{\infty}\kappa_j\Delta_f^j \tag{2}$$

and [9]

$$\beta'_{\mathrm{IR}} = \sum_{j=1}^{\infty} d_j\Delta_f^j, \tag{3}$$

where $d_1=0$ for all G and R. For general G and R, the κ_j were calculated to order $j=3$ in [8] and the d_j to order $j=4$ in [9], and for $G=\mathrm{SU}(3)$ and $R=F$, κ_4 was computed in [10] and d_5 in [9].

Here we report our calculations of these scheme-independent expansions of $\gamma_{\bar{\psi}\psi,\mathrm{IR}}$ and β'_{IR} to the highest orders yet achieved, presenting κ_4 and d_5 for an asymptotically free $\mathrm{SU}(N_c)$ gauge theory with a conformal IR fixed point, for $R=F,\mathrm{adj},S$. We also report our calculation of κ_3 for supersymmetric quantum chromodynamics (SQCD). We believe that our new results are a substantial advance in the knowledge of conformal field theory. Our results have the advantage of scheme independence at each order in Δ_f, in contrast to scheme-dependent (SD) series expansions of $\gamma_{\bar{\psi}\psi,\mathrm{IR}}$ and β'_{IR} in powers of α [11–16], and they complement other approaches to understanding conformal and superconformal field theory, such as the bootstrap [17] and lattice simulations [18].

The conventional power-series expansions of β and $\gamma_{\bar\psi\psi}$ are

$$\beta = -2\alpha \sum_{\ell=1}^{\infty} b_\ell \left(\frac{\alpha}{4\pi}\right)^\ell \tag{4}$$

and

$$\gamma_{\bar\psi\psi} = \sum_{\ell=1}^{\infty} c_\ell \left(\frac{\alpha}{4\pi}\right)^\ell, \tag{5}$$

where b_ℓ and c_ℓ are the ℓ-loop coefficients; b_1 [19], b_2 [20], and $c_1 = 6C_f$ are scheme-independent, while the b_ℓ with $\ell \geq 3$ and the c_ℓ with $\ell \geq 2$ are scheme-dependent, i.e. they depend on the scheme used for regularization and renormalization [5]. We denote the n-loop ($n\ell$) β and $\gamma_{\bar\psi\psi}$ as $\beta_{n\ell}$ and $\gamma_{\bar\psi\psi,n\ell}$ and the IR zero of $\beta_{n\ell}$ as $\alpha_{\mathrm{IR},n\ell}$.

The calculation of κ_j requires, as inputs, the values of the b_ℓ for $1 \leq \ell \leq j+1$ and the c_ℓ for $1 \leq \ell \leq j$. The calculation of d_j requires, as inputs, the values of the b_ℓ for $1 \leq \ell \leq j$. Thus, importantly, κ_j does not receive any corrections from b_ℓ with $\ell > j+1$ or c_ℓ with $\ell > j$, and similarly, d_j does not receive any corrections from any b_ℓ with $\ell > j$.

The coefficients κ_j were calculated in [8] for an (AF vectorial) supersymmetric gauge theory (SGT) with gauge group G and N_f pairs of chiral superfields in the R and $\bar R$ representation, for $j = 1, 2$. Complete agreement was found, to the order calculated, with the exactly known result in the conformal non-Abelian Coulomb phase (NACP) [21–23]

$$\gamma_{\mathrm{IR,SGT}} = \frac{\frac{2T_f}{3C_A}\Delta_f}{1 - \frac{2T_f}{3C_A}\Delta_f}. \tag{6}$$

In this theory, $N_u = 3C_A/(2T_f)$, and the conformal NACP is the interval $N_\ell < N_f < N_u$, where $N_\ell = N_u/2$, so that Δ_f varies from 0 to a maximum of $(\Delta_f)_{\max} = 3C_A/(4T_f)$ in the NACP [24]. Hence, $\gamma_{\mathrm{IR,SGT}}$ increases monotonically from 0 to 1 as N_f decreases from N_u to N_ℓ, saturating the upper bound $\gamma_{\bar\psi\psi,\mathrm{IR,SGT}} < 1$ from conformal invariance in this SGT [25].

As a test of the accuracy of the Δ_f expansion, we have now calculated κ_3 for SQCD with $R = F$, using inputs from [26]. We find $\kappa_3 = 1/(3N_c)^3$, in perfect agreement, to this order, with the exact result, Eq. (6). This agreement explicitly illustrates the scheme independence of the κ_j, since our calculations in [8] and here used inputs computed in the $\overline{\mathrm{DR}}$ scheme, while (6) was derived in the NSVZ scheme [21]. Our new result has a far-reaching implication: it strongly suggests that $\kappa_j = [2T_f/(3C_A)]^j$ for all j, so that the expansion (2) for this supersymmetric gauge theory, calculated to order $O(\Delta_f^p)$, agrees with the exact result to the given order for all p.

Because of electric-magnetic duality [22], as $N_f \to N_\ell$ in the NACP, the physics is described by a magnetic theory with coupling strength going to zero, or equivalently, by an electric theory with divergent α_{IR}. Hence, another important finding here is that the complete agreement that we obtain in SQCD to $O(\Delta_f^3)$ between Eq. (2) and the exact Eq. (6) holds for arbitrarily strong α_{IR}. Even apart from the issue of scheme dependence in Eq. (5), this agreement could not be achieved with the conventional expansion (5) of $\gamma_{\bar\psi\psi,\mathrm{IR}}$ in powers of α.

The Δ_f expansion also avoids a problem in which an IRFP may not be manifest as a physical IR zero of the n-loop beta function for some n. Indeed, although $\beta_{n\ell}$ has a physical $\alpha_{\mathrm{IR},n\ell}$ in SQCD for $n = 2, 3$ loops [27], we have analyzed $\beta_{4\ell}$ (in the $\overline{\mathrm{DR}}$ scheme), and we find that for a range of N_f in the NACP, it does not exhibit a physical $\alpha_{\mathrm{IR},4\ell}$. This is analogous to the situation that we found for $\alpha_{\mathrm{IR},5\ell}$ in the nonsupersymmetric gauge theory [16]. In both cases, the Δ_f expansions (2) and (3) circumvent this problem of a possible unphysical $\alpha_{\mathrm{IR},n\ell}$ that one may encounter in using the convention expansions (4) and (5).

We next present our results for κ_4 and d_5 for a (nonsupersymmetric) $\mathrm{SU}(N_c)$ gauge theory, making use of the impressive recent computation of b_5 in [28]. (We have actually calculated κ_4 and d_5 for general G and R [29], but only present results here for $R = F, \mathrm{adj}, S$.) The two-loop beta function has an IR zero (IRZ) in the interval $I_{\mathrm{IRZ}}: N_\ell < N_f < N_u$, with upper and lower ($\ell$) ends at $N_u = 11N_c/(4T_f)$ and $N_\ell = 17C_A^2/[2T_f(5C_A + 3C_f)]$ [24]. The non-Abelian Coulomb phase extends downward in I_{IRZ} from N_u to a lower value denoted $N_{f,cr}$ [30]. Since chiral symmetry is exact in the NACP, one can classify the bilinear fermion operators according to their flavor transformation properties. These operators include the flavor-singlet $\bar\psi\psi$ and the flavor-adjoint $\bar\psi T_a \psi$, where T_a is a generator of $\mathrm{SU}(N_f)$. These have the same anomalous dimension [31], which we write simply as $\gamma_{\bar\psi\psi}$. For general G and R, the coefficients b_ℓ were computed up to loop order $\ell = 4$ [32] (checked in [33]) and the c_ℓ also up to loop order $\ell = 4$ [34], in the widely used $\overline{\mathrm{MS}}$ scheme [35]. These results were used in [8] to calculate the κ_j to order $j = 3$ and in [9] to calculate d_j to order $j = 4$. For $N_c = 3$ and $R = F$, b_5 was computed in [36], and this was used to calculate κ_4 in [10] and d_5 in [9] for this case (see also [16]).

We first report our results for κ_4 and d_5 for $R = F$, using b_5 from [28]. We denote the Riemann zeta function as $\zeta_s = \sum_{n=1}^{\infty} n^{-s}$. We obtain

$$\kappa_{4,F} = \frac{4(N_c^2-1)}{3^4 N_c^4 (25N_c^2-11)^7}[(263345440N_c^{12} - 673169750N_c^{10} + 256923326N_c^8 - 290027700N_c^6 + 557945201N_c^4 - 208345544N_c^2 + 6644352) + 384(25N_c^2-11)(4400N_c^{10} - 123201N_c^8 + 480349N_c^6 - 486126N_c^4 + 84051N_c^2 + 1089)\zeta_3 + 211200N_c^2(25N_c^2-11)^2(N_c^6 + 3N_c^4 - 16N_c^2 + 22)\zeta_5] \quad (7)$$

and

$$d_{5,F} = \frac{2^5}{3^6 N_c^3 (25N_c^2-11)^7}[N_c^{12}(-298194551 - 423300000\zeta_3 + 528000000\zeta_5) + N_c^{10}(414681770 + 1541114400\zeta_3 - 821040000\zeta_5) + N_c^8(80227411 - 4170620256\zeta_3 + 2052652800\zeta_5) + N_c^6(210598856 + 5101712352\zeta_3 - 4268183040\zeta_5) + N_c^4(-442678324 - 2250221952\zeta_3 + 2744628480\zeta_5) + N_c^2(129261880 + 304571520\zeta_3 - 534103680\zeta_5) + 3716152 + 1022208\zeta_3], \quad (8)$$

where the simple factorizations of the denominators have been indicated. For this $R = F$ case, we find that $\kappa_4 > 0$, as was also true of κ_j with $1 \le j \le 3$ (indeed, κ_1 and κ_2 are manifestly positive for any G and R). We also find the same positivity results for $R = \text{adj}$ and $R = S$. The property that for all of these representations R, $\kappa_j > 0$ for $1 \le j \le 4$ and for all N_c implies two important monotonicity results. First, for these R, and with a fixed p in the interval $1 \le p \le 4$, $\gamma_{\bar\psi\psi,IR,\Delta_f^p}$ is a monotonically increasing function of Δ_f for $N_f \in I_{IRZ}$. Second, for these R, and with a fixed $N_f \in I_{IRZ}$, $\gamma_{\bar\psi\psi,IR,\Delta_f^p}$ is a monotonically increasing function of p in the range $1 \le p \le 4$. In addition to the manifestly positive κ_1 and κ_2, a plausible conjecture is that, for these R, $\kappa_j > 0$ for all $j \ge 3$. Note that the exact result (6) for the supersymmetric gauge theory shows that in that theory, $\kappa_j > 0$ for all j and for any G and R.

In Figs. 1 and 2 we plot γ_{IR,Δ_f^p} for $R = F$, $N_c = 2$, 3 and $1 \le p \le 4$. In Table I we list values of these γ_{IR,Δ_f^p} [37]. These all satisfy the upper bound $\gamma_{IR} < 2$ from conformal invariance [25]. Below, we will often omit the $\bar\psi\psi$ subscript, writing $\gamma_{\bar\psi\psi,IR} \equiv \gamma_{IR}$ and $\gamma_{\bar\psi\psi,IR,\Delta_f^p} \equiv \gamma_{IR,\Delta_f^p}$.

For this $R = F$ case we first remark on the comparison of γ_{IR,Δ_f^4} with calculations of $\gamma_{IR,n\ell}$ from analyses of power series in α, which were performed to $n = 4$ loop level in [11–14] using b_ℓ and c_ℓ in the $\overline{\text{MS}}$ scheme (with studies of scheme dependence in [15]) and extended to $n = 5$ loop level for $N_c = 3$ in [16]. We have noted that $\beta_{5\ell}$ does not have a physical $\alpha_{IR,5\ell}$ for N_f in the lower part of the interval I_{IRZ} [16]. Although we were able to surmount this problem via Padé approximants in [16], these are still scheme-dependent, while the Δ_f expansion has the advantage of being scheme-independent. In general, we find that for a given N_c and N_f, the value of γ_{IR,Δ_f^p} that we calculate to highest order, namely $p = 4$, is somewhat larger than $\gamma_{IR,n\ell}$ calculated to its highest order [10,13]. For example, for $N_c = 3$, $N_f = 12$, $\gamma_{IR,4\ell} = 0.253$, $\gamma_{IR,5\ell} \simeq 0.255$ (using a value of $\alpha_{IR,5\ell}$ from a Padé approximant [10,16]), while $\gamma_{IR,\Delta_f^4} = 0.338$ and an extrapolation yields the estimate 0.400(5) for $\gamma_{IR} = \lim_{p\to\infty}\gamma_{IR,\Delta_f^p}$ [10]. Similarly, for $N_c = 2$ and $N_f = 8$, $\gamma_{IR,4\ell} = 0.204$, while $\gamma_{IR,\Delta_f^4} = 0.298$; and for $N_c = 4$, $N_f = 16$, $\gamma_{IR,4\ell} = 0.269$, while $\gamma_{IR,\Delta_f^4} = 0.352$.

We next compare our new results with lattice measurements, restricting to cases where the lattice studies are consistent with the theories being IR-conformal [18,30]. For $N_c = 3$, we compared our calculations of γ_{IR,Δ_f^4} with lattice measurements for $N_f = 12$ in [10], finding general consistency with the range of lattice results, although our γ_{IR,Δ_f^4} and extrapolation to the exact γ_{IR} were higher than

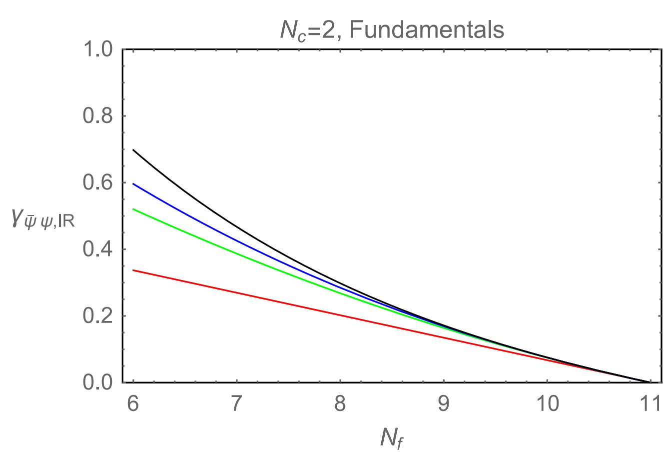

FIG. 1. Plot of $\gamma_{\bar\psi\psi,IR,\Delta_f^p}$ for $R = F$, $N_c = 2$, and $1 \le p \le 4$ as a function of $N_f \in I_{IRZ}$. From bottom to top, the curves (with colors online) refer to $\gamma_{\bar\psi\psi,IR,\Delta_f}$ (red), $\gamma_{\bar\psi\psi,IR,\Delta_f^2}$ (green), $\gamma_{\bar\psi\psi,IR,\Delta_f^3}$ (blue), and $\gamma_{\bar\psi\psi,IR,\Delta_f^4}$ (black).

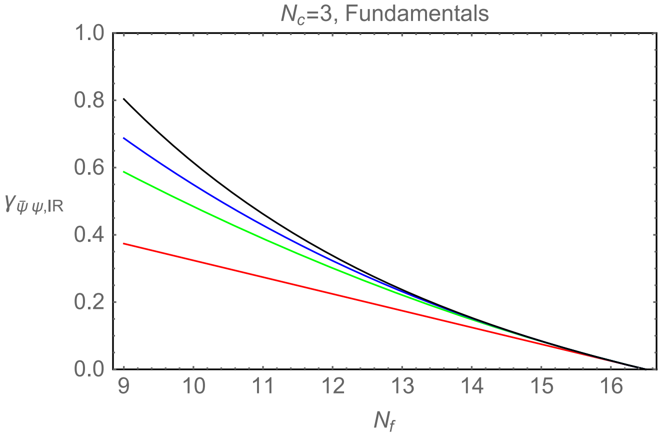

FIG. 2. Plot of $\gamma_{\bar\psi\psi,\mathrm{IR},\Delta_f^p}$ for $R=F$, $N_c=3$, and $1\le p\le 4$ as a function of $N_f\in I_{\mathrm{IRZ}}$. From bottom to top, the curves (with colors online) refer to $\gamma_{\bar\psi\psi,\mathrm{IR},\Delta_f}$ (red), $\gamma_{\bar\psi\psi,\mathrm{IR},\Delta_f^2}$ (green), $\gamma_{\bar\psi\psi,\mathrm{IR},\Delta_f^3}$ (blue), and $\gamma_{\bar\psi\psi,\mathrm{IR},\Delta_f^4}$ (black).

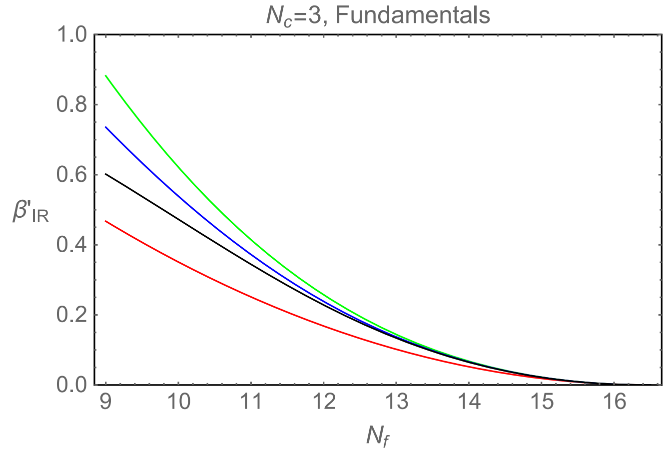

FIG. 3. Plot of $\beta'_{\mathrm{IR},\Delta_f^p}$ for $R=F$, $N_c=3$, and $2\le p\le 4$ as a function of $N_f\in I_{\mathrm{IRZ}}$. From bottom to top, the curves (with colors online) refer to $\beta'_{\mathrm{IR},\Delta_f^2}$ (red), $\beta'_{\mathrm{IR},\Delta_f^3}$ (green), $\beta'_{\mathrm{IR},\Delta_f^4}$ (blue), and $\beta'_{\bar\psi\psi,\mathrm{IR},\Delta_f^5}$ (black).

some of the lattice values. We also found consistency for the cases $N_f=10$ and $N_f=8$ [10]. Here, we compare with lattice results for γ_{IR} in the case $N_c=2$, $N_f=8$. [It is not clear from lattice studies if the SU(2), $R=F$, $N_f=6$ theory has a conformal IRFP or not [18,30,38].] Following lattice studies of the SU(2), $R=F$, $N_f=8$ theory by several groups [18,39], a recent measurement is $\gamma_{\mathrm{IR}}=0.15\pm 0.02\equiv 0.15(2)$ [40]. Our value $\gamma_{\mathrm{IR},\Delta_f^4}=0.298$ is somewhat higher than this lattice result.

We proceed to discuss d_5 for $R=F$. In Fig. 3 we plot $\beta'_{\mathrm{IR},\Delta_f^p}$ for $R=F$, $N_c=3$, and $2\le p\le 5$. In Table II we list values of $\beta'_{\mathrm{IR},\Delta_f^p}$ for $R=F$, $N_c=2$, 3 and $2\le p\le 5$. For $R=F$ and general N_c, d_2 and d_3 are positive, while d_4 and d_5 are negative. For the case SU(3), $N_f=12$, we get $\beta'_{\mathrm{IR},\Delta_f^5}=0.228$. The conventional n-loop calculation yielded $\beta'_{\mathrm{IR},3\ell}=0.2955$ and $\beta'_{\mathrm{IR},4\ell}=0.282$ [41], so $\beta'_{\mathrm{IR},\Delta_f^5}$ is slightly smaller than $\beta'_{\mathrm{IR},4\ell}$. A recent lattice measurement yields $\beta'_{\mathrm{IR}}=0.26(2)$ [42], consistent with both our $\beta'_{\mathrm{IR},\Delta_f^5}$ and $\beta'_{\mathrm{IR},4\ell}$.

We next discuss the case $R=\mathrm{adj}$, for which $N_u=11/4$ and $N_\ell=17/16$, so I_{IRZ} includes the single integer value $N_f=2$ (whence $\Delta_f=N_u-2=3/4$). Results for this case were given for κ_p with $1\le p\le 3$ in [8] and for d_p with $1\le p\le 4$ in [9]. Here we find

$$\kappa_{4,\mathrm{adj}}=\frac{53389393}{2^7\cdot 3^{14}}+\frac{368}{3^{10}}\zeta_3+\left(-\frac{2170}{3^{10}}+\frac{33952}{3^{11}}\zeta_3\right)N_c^{-2}$$
$$=0.0946976+0.193637N_c^{-2} \tag{9}$$

TABLE I. Values of the scheme-independent anomalous dimension $\gamma_{\mathrm{IR},\Delta_f^p}$ with $1\le p\le 4$ for $R=F$ and $N_c=2$, 3.

N_c	N_f	$\gamma_{\mathrm{IR},\Delta_f}$	$\gamma_{\mathrm{IR},\Delta_f^2}$	$\gamma_{\mathrm{IR},\Delta_f^3}$	$\gamma_{\mathrm{IR},\Delta_f^4}$
2	6	0.337	0.520	0.596	0.698
2	7	0.270	0.387	0.426	0.467
2	8	0.202	0.268	0.285	0.298
2	9	0.135	0.164	0.169	0.172
2	10	0.0674	0.07475	0.07535	0.0755
3	9	0.374	0.587	0.687	0.804
3	10	0.324	0.484	0.549	0.615
3	11	0.274	0.389	0.428	0.462
3	12	0.224	0.301	0.323	0.338
3	13	0.174	0.221	0.231	0.237
3	14	0.125	0.148	0.152	0.153
3	15	0.0748	0.0833	0.0841	0.0843
3	16	0.0249	0.0259	0.0259	0.0259

TABLE II. Scheme-independent values of $\beta'_{\mathrm{IR},\Delta_f^p}$ with $2\le p\le 4$ for $R=F$, $N_c=2$, 3 as functions of N_f in the respective intervals I_{IRZ}. The notation ae-n means $a\times 10^{-n}$.

N_c	N_f	$\beta'_{\mathrm{IR},\Delta_f^2}$	$\beta'_{\mathrm{IR},\Delta_f^3}$	$\beta'_{\mathrm{IR},\Delta_f^4}$	$\beta'_{\mathrm{IR},\Delta_f^5}$
2	6	0.499	0.957	0.734	0.6515
2	7	0.320	0.554	0.463	0.436
2	8	0.180	0.279	0.250	0.243
2	9	0.0799	0.109	0.1035	0.103
2	10	0.0200	0.0236	0.0233	0.0233
3	9	0.467	0.882	0.7355	0.602
3	10	0.351	0.621	0.538	0.473
3	11	0.251	0.415	0.3725	0.344
3	12	0.168	0.258	0.239	0.228
3	13	0.102	0.144	0.137	0.134
3	14	0.0519	0.0673	0.0655	0.0649
3	15	0.0187	0.0220	0.0218	0.0217
3	16	2.08e-3	2.20e-3	2.20e-3	2.20e-3

and

$$d_{5,\mathrm{adj}} = -\frac{7141205}{2^3 \cdot 3^{16}} + \frac{5504}{3^{12}}\zeta_3 - \left(\frac{30928}{3^{14}} + \frac{465152}{3^{13}}\zeta_3\right)N_c^{-2} = -(0.828739 \times 10^{-2}) - 0.357173 N_c^{-2}. \quad (10)$$

We remark on the SU(2), $N_f = 2$, $R = \mathrm{adj}$ theory, which has been of interest [43]. Extensive lattice studies of this theory have been performed and are consistent with IR conformality [18]. We get $\beta'_{\mathrm{IR},\Delta_f^5} = 0.147$; and $\gamma_{\mathrm{IR},\Delta_f^2} = 0.465$, $\gamma_{\mathrm{IR},\Delta_f^3} = 0.511$, and $\gamma_{\mathrm{IR},\Delta_f^4} = 0.556$. These $\gamma_{\mathrm{IR},\Delta_f^p}$ values are close to our n-loop calculations in [13] for this theory, namely $\gamma_{\mathrm{IR},3\ell} = 0.543$, $\gamma_{\mathrm{IR},4\ell} = 0.500$. Lattice measurements of this theory have yielded a wide range of values of γ_{IR} including 0.49(13) [44], 0.22(6) [45], 0.31(6) [46], 0.17(5) [47], 0.20(3) [48], 0.50(26) [49], and 0.15(2) [40] (see references for details of uncertainty estimates).

Finally, we discuss the case $R = S$. For SU(2), $S = \mathrm{adj}$, already discussed above. For SU(3), we focus on the $N_f = 2$ theory, for which we find $\beta'_{\mathrm{IR},\Delta_f^5} = 0.333$; and $\gamma_{\mathrm{IR},\Delta_f^2} = 0.789$, $\gamma_{\mathrm{IR},\Delta_f^3} = 0.960$, and $\gamma_{\mathrm{IR},\Delta_f^4} = 1.132$ [37]. For comparison, our n-loop results from [13] for this case are $\gamma_{\mathrm{IR},3\ell} = 0.500$ and $\gamma_{\mathrm{IR},4\ell} = 0.470$. Lattice studies of this theory include one that concludes that it is IR-conformal and gets $\gamma_{\mathrm{IR}} < 0.45$ [50] and another that concludes that it is not IR-conformal and gets an effective $\gamma_{\mathrm{IR}} \simeq 1$ [51].

In summary, we have presented calculations of $\gamma_{\bar\psi\psi,\mathrm{IR}}$ and β'_{IR} at a conformal IR fixed point of an asymptotically free gauge theory with fermions, to the highest orders yet achieved. We believe that these results are of fundamental value for the understanding of conformal field theory, especially because they are scheme-independent.

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[1] The assumption of massless fermions incurs no loss of generality, since if a fermion had a nonzero mass m_0, it would be integrated out of the effective field theory at scales $\mu < m_0$, and hence would not affect the IR limit $\mu \to 0$.

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