

## Crossed products for interactions and graph algebras

Kwasniewski, Bartosz

*Published in:*  
Integral Equations and Operator Theory

*DOI:*  
[10.1007/s00020-014-2166-5](https://doi.org/10.1007/s00020-014-2166-5)

*Publication date:*  
2014

*Document version*  
Final published version

*Citation for published version (APA):*  
Kwasniewski, B. (2014). Crossed products for interactions and graph algebras. *Integral Equations and Operator Theory*, 80(3), 415-451. <https://doi.org/10.1007/s00020-014-2166-5>

### Terms of use

This work is brought to you by the University of Southern Denmark through the SDU Research Portal. Unless otherwise specified it has been shared according to the terms for self-archiving. If no other license is stated, these terms apply:

- You may download this work for personal use only.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying this open access version

If you believe that this document breaches copyright please contact us providing details and we will investigate your claim. Please direct all enquiries to [puresupport@bib.sdu.dk](mailto:puresupport@bib.sdu.dk)

# Crossed Products for Interactions and Graph Algebras

B. K. Kwaśniewski

**Abstract.** We consider Exel's interaction  $(\mathcal{V}, \mathcal{H})$  over a unital  $C^*$ -algebra  $A$ , such that  $\mathcal{V}(A)$  and  $\mathcal{H}(A)$  are hereditary subalgebras of  $A$ . For the associated crossed product, we obtain a uniqueness theorem, ideal lattice description, simplicity criterion and a version of Pimsner–Voiculescu exact sequence. These results cover the case of crossed products by endomorphisms with hereditary ranges and complemented kernels. As model examples of interactions not coming from endomorphisms we introduce and study in detail interactions arising from finite graphs.

The interaction  $(\mathcal{V}, \mathcal{H})$  associated to a graph  $E$  acts on the core  $\mathcal{F}_E$  of the graph algebra  $C^*(E)$ . By describing a partial homeomorphism of  $\widehat{\mathcal{F}_E}$  dual to  $(\mathcal{V}, \mathcal{H})$  we find the fundamental structure theorems for  $C^*(E)$ , such as Cuntz–Krieger uniqueness theorem, as results concerning reversible noncommutative dynamics on  $\mathcal{F}_E$ . We also provide a new approach to calculation of  $K$ -theory of  $C^*(E)$  using only an induced partial automorphism of  $K_0(\mathcal{F}_E)$  and the six-term exact sequence.

**Mathematics Subject Classification (2010).** Primary 46L55; Secondary 46L80.

**Keywords.** Interaction, graph algebra, endomorphism, topological freeness, crossed product, Hilbert bimodule,  $K$ -theory, Pimsner–Voiculescu exact sequence.

## Contents

1. Introduction	416
1.1. Preliminaries on Hilbert Bimodules	420
2. Corner Interactions and Their Crossed Products	423
2.1. Interactions and $C^*$ -Dynamical Systems	423
2.2. Crossed Product for Corner Interactions	426
2.3. Topological Freeness, Ideal Structure and Simplicity Criteria	428
2.4. $K$ -Theory	430

3. Graph $C^*$ -Algebras via Interactions	431
3.1. Graph $C^*$ -Algebra $C^*(E)$ and its AF-Core	432
3.2. Interactions Arising from Graphs	433
3.3. Dynamical Systems Dual to Graph Interactions	437
3.4. Topological Freeness of Graph Interactions	442
3.5. $K$ -Theory	445
Open Access	448
References	449

## 1. Introduction

In [12] Exel extended celebrated Pimsner’s construction [38] of the (nowadays called) Cuntz–Pimsner algebras by introducing an intriguing new concept of a generalized  $C^*$ -correspondence. The leading example in [12] arises from an *interaction*—a pair  $(\mathcal{V}, \mathcal{H})$  of positive linear maps on a  $C^*$ -algebra  $A$  that are mutual generalized inverses and such that the image of one map is in the multiplicative domain of the other. An interaction can be considered a ‘symmetrized’ generalization of a  $C^*$ -dynamical system, i.e. a pair  $(\alpha, \mathcal{L})$  consisting of an endomorphism  $\alpha : A \rightarrow A$  and its transfer operator  $\mathcal{L} : A \rightarrow A$  [11]. One can think of many examples of interactions naturally appearing in various problems, cf. [13, 14, 17]. However, at present there is only one significant application of an interaction  $(\mathcal{V}, \mathcal{H})$  which is not a  $C^*$ -dynamical system. Namely, in the recent paper [14] Exel showed that the  $C^*$ -algebra  $\mathcal{O}_{n,m}$  introduced in [3], is Morita equivalent to the crossed product  $C^*(A, \mathcal{V}, \mathcal{H})$  for an interaction  $(\mathcal{V}, \mathcal{H})$ , over a commutative  $C^*$ -algebra  $A$ , where neither  $\mathcal{V}$  nor  $\mathcal{H}$  is multiplicative. Moreover, for crossed products under consideration general structure theorems known so far concern only the case when the initial object is an injective endomorphism, cf. [7, 18, 35, 36, 43]. In particular, there are no such theorems for genuine interactions, i.e. when both  $\mathcal{V}$  and  $\mathcal{H}$  are not multiplicative.

The purpose of the present article is twofold.

Firstly, we establish general tools to study the structure of  $C^*(A, \mathcal{V}, \mathcal{H})$  for an accessible and, as the  $C^*$ -dynamical system case indicates, important class of interactions  $(\mathcal{V}, \mathcal{H})$ . Thus this might be a considerable step in understanding these new objects. More precisely, crossed products associated with  $C^*$ -dynamical systems  $(\alpha, \mathcal{L})$  on a unital  $C^*$ -algebra  $A$  boast their greatest successes in the case  $\alpha(A)$  is a hereditary subalgebra of  $A$ , cf. [2, 11, 35, 36, 42]. Then  $\mathcal{L}$  is a *corner retraction*, see [43, page 424], [26]. It is uniquely determined by  $\alpha$  and it is called a *complete transfer operator* in [2], see [23, 26]. In the present paper we focus on interactions  $(\mathcal{V}, \mathcal{H})$  for which both  $\mathcal{V}(A)$  and  $\mathcal{H}(A)$  are hereditary subalgebras of  $A$ . Then  $\mathcal{V}(A)$  and  $\mathcal{H}(A)$  are automatically corners in  $A$ . We call such interactions *corner interactions*. It turns out that each mapping in such an interaction  $(\mathcal{V}, \mathcal{H})$  is completely determined by the other. This plus the obvious connotation to complete transfer operators

make it tempting to call  $(\mathcal{V}, \mathcal{H})$  a complete interaction [28], but we resist this temptation here.

We show that for a corner interaction  $(\mathcal{V}, \mathcal{H})$  the crossed product  $C^*(A, \mathcal{V}, \mathcal{H})$  defined in [12] is the universal  $C^*$ -algebra generated by a copy of  $A$  and a partial isometry  $s$  subject to relations

$$\mathcal{V}(a) = s(a)s^*, \quad \mathcal{H}(a) = s^*(a)s, \quad a \in A.$$

As a consequence  $C^*(A, \mathcal{V}, \mathcal{H})$  can be modeled as the crossed product  $A \rtimes_X \mathbb{Z}$ , [1], of  $A$  by a Hilbert bimodule  $X = AsA$ . It also follows that  $C^*(A, \mathcal{V}, \mathcal{H}) \cong C^*(A, \mathcal{V}) \cong C^*(A, \mathcal{H})$  where  $C^*(A, \mathcal{V})$  (resp.  $C^*(A, \mathcal{H})$ ) is the crossed product of  $A$  by the completely positive mapping  $\mathcal{V}$  (resp.  $\mathcal{H}$ ), as introduced in [26]. We study  $C^*(A, \mathcal{V}, \mathcal{H})$  by applying general methods developed for Hilbert bimodules [25] and  $C^*$ -correspondences [22]. For instance, we have a naturally defined partial homeomorphism  $\widehat{\mathcal{V}}$  of  $\widehat{A}$  dual to  $(\mathcal{V}, \mathcal{H})$ . Identifying it with the inverse to the induced partial homeomorphism  $X$ -Ind studied in [25] we obtain: *uniqueness theorem*—topological freeness of  $\widehat{\mathcal{V}}$  implies faithfulness of every representation of  $C^*(A, \mathcal{V}, \mathcal{H})$  which is faithful on  $A$ ; ideal lattice description via  $\widehat{\mathcal{V}}$ -invariant open sets, when  $\widehat{\mathcal{V}}$  is free; and the simplicity of  $C^*(A, \mathcal{V}, \mathcal{H})$  when  $\widehat{\mathcal{V}}$  is minimal and topologically free (see Theorem 2.20 below). Similarly, identifying the abstract morphisms in Katsura’s version of Pimsner–Voiculescu exact sequence [22] we get a natural cyclic exact sequence for  $K$ -groups of  $C^*(A, \mathcal{V}, \mathcal{H})$  (Theorem 2.25). It generalizes the corresponding exact sequence obtained by Paschke for injective endomorphisms [36], which plays a crucial role, for instance, in [42].

Secondly, we provide a detailed analysis of nontrivial corner interactions with an interesting *noncommutative dynamics* related to Markov shifts, and graph  $C^*$ -algebras as crossed products. More specifically, already in [9] Cuntz considered his  $C^*$ -algebras  $\mathcal{O}_n$  as crossed products of the core UHF-algebras by injective endomorphisms implemented by one of the generating isometries. As noticed by Rørdam [42, Example 2.5], a similar reasoning can be performed for Cuntz–Krieger algebras  $\mathcal{O}_A$  by considering an isometry given by the sum of all generating partial isometries with properly restricted initial spaces. An analogous isometry in  $\mathcal{O}_A$ , but in a sense canonically associated with the underlying dynamics of Markov shifts, was found in [11, proof of Theorem 4.3], cf. [2, formula (4.18)]. For the graph  $C^*$ -algebra  $C^*(E)$  associated with a row-finite graph  $E$  with no sources<sup>1</sup> the corresponding isometry appears implicitly in [7, Theorem 5.1] and explicitly in [19, Theorem 5.2], see formula (3.2) below. In particular, if we assume  $E$  is finite, i.e. the sets of vertices and edges are finite, and  $E$  has no sources, we know from [19, Theorem 5.2] that  $C^*(E)$  is naturally isomorphic to the crossed product of the AF-core  $C^*$ -algebra  $\mathcal{F}_E$  by an injective endomorphism with hereditary range implemented by the aforementioned isometry  $s$ . Thus we have

$$C^*(E) = C^*(\mathcal{F}_E \cup \{s\}), \quad s\mathcal{F}_E s^* \subset \mathcal{F}_E, \quad s^*\mathcal{F}_E s \subset \mathcal{F}_E.$$

<sup>1</sup> We follow here the original conventions of [4, 29] and hence in the context of representations of graphs we consider different orientation of edges than in [7, 19, 39].

Moreover, one can notice that the above picture remains valid for arbitrary finite graphs, possibly with sources. The only difference is that  $s$  may be no longer an isometry but a partial isometry. Hence the mapping  $\mathcal{F}_E \ni a \rightarrow sas^* \in \mathcal{F}_E$  may be no longer multiplicative (at least not on its whole domain) and then a natural framework for  $C^*(E)$  is the crossed product for an interaction  $(\mathcal{V}, \mathcal{H})$  over  $\mathcal{F}_E$  where  $\mathcal{V}(\cdot) := s(\cdot)s^*$ ,  $\mathcal{H}(\cdot) := s^*(\cdot)s$ . We call the pair  $(\mathcal{V}, \mathcal{H})$  arising in this way a *graph interaction*. It can be viewed from many different perspectives as a model example illustrating and giving new insight, for instance, to the following objects and issues that we hope to be pursued in the future.

- *Interactions with nontrivial algebras and not multiplicative dynamics.* The crossed product  $C^*(\mathcal{F}_E, \mathcal{V}, \mathcal{H})$  is naturally isomorphic to the graph  $C^*$ -algebra  $C^*(E)$  (Proposition 3.2). In general,  $(\mathcal{V}, \mathcal{H})$  is not a  $C^*$ -dynamical system and is not a part of a group interaction [13]. We precisely identify the values of  $n \in \mathbb{N}$  for which  $(\mathcal{V}^n, \mathcal{H}^n)$  is an interaction (see Proposition 3.5), and it turns out that such  $n$ 's might have almost arbitrary distribution. Moreover,  $(\mathcal{V}^n, \mathcal{H}^n)$  is an interaction for all  $n \in \mathbb{N}$  if and only if  $(\mathcal{V}, \mathcal{H})$  is a  $C^*$ -dynamical system (which may happen even if  $E$  has sources).
- *Noncommutative Markov shifts.* The main motivation in [11] for introducing  $C^*$ -dynamical systems  $(\alpha, \mathcal{L})$  was to realize Cuntz–Krieger algebras  $\mathcal{O}_A$  as crossed products of the underlying Markov shifts, which was in turn suggested by [10, Proposition 2.17]. In terms of graph  $C^*$ -algebras the relevant statement, see [7, Theorem 5.1], says that when  $E$  is finite and has no sinks, then  $C^*(E)$  is isomorphic to Exel crossed product  $\mathcal{D}_E \rtimes_{\phi_E, \mathcal{L}} \mathbb{N}$  where  $\mathcal{D}_E \cong C(E^\infty)$  is a canonical masa in  $\mathcal{F}_E$ . The spectrum of  $\mathcal{D}_E$  is identified with the space of infinite paths  $E^\infty$ ,  $\phi_E$  is a transpose to the Markov shift on  $E^\infty$  and  $\mathcal{L}$  is its classical Ruelle–Perron–Frobenius operator. Both  $\phi_E$  and  $\mathcal{L}$  extend naturally to completely positive maps on  $C^*(E)$  and the extension of  $\phi_E$  is called the noncommutative Markov shift, cf. e.g. [20]. However, from the point of view of the crossed product construction the predominant role is played by  $\mathcal{L}$ , see [26]. In particular,  $\mathcal{L} = \mathcal{H}$  where  $(\mathcal{V}, \mathcal{H})$  is the graph interaction, and  $\mathcal{F}_E$  is a minimal  $C^*$ -algebra invariant under  $\mathcal{V}$  and containing  $\mathcal{D}_E$ . Thus there are good reasons to regard the graph interaction  $(\mathcal{V}, \mathcal{H})$  as an alternative candidate for the noncommutative counterpart of the Markov shift. Our dual and  $K$ -theoretic pictures of  $(\mathcal{V}, \mathcal{H})$  (see Theorem 3.9 and Proposition 3.22, respectively) support this point of view.
- *Graph  $C^*$ -algebras.* The structure of graph algebras was originally studied via grupoids [29, 30], and  $K$ -theory was calculated using a dual Pimsner–Voiculescu exact sequence and skew products of initial graphs [39, 40]. The corresponding results can also be achieved in the realm of partial actions of free groups on certain commutative  $C^*$ -algebras, see [15, 16]. We present here another approach, based on interactions. We show that the partial homeomorphism  $\hat{\mathcal{V}}$  dual to  $\mathcal{V}$  is topologically free if and only if  $E$  satisfies the so-called condition (L) [4]. Hence we derive the Cuntz–Krieger

uniqueness theorem [4, 30, 39] from our general uniqueness theorem for interactions. Similarly, we see that freeness of  $\widehat{\mathcal{V}}$  is equivalent to condition (K) for  $E$  [4, 30]. Thus minimality and freeness of  $\widehat{\mathcal{V}}$  is equivalent to the known simplicity criteria for  $C^*(E)$ . Moreover, it turns out that pure infiniteness of  $C^*(E)$ , as defined in [29, 31], is equivalent to a very strong version of topological freeness of  $\widehat{\mathcal{V}}$  (see Remark 3.20), which therefore might be considered an instance of a noncommutative version of local boundary action, see [31]. Finally, our approach to calculation of  $K$ -groups for  $C^*(E)$  seems to be the most direct upon the existing ones; it uses only direct limit description of the AF-core  $\mathcal{F}_E$  and the cyclic six-term exact sequence.

- *Topological freeness.* The condition known as topological freeness was for the first time explicitly stated in [32] where the author use it to show, what we call here, uniqueness theorem. Namely, he proved that topological freeness of a homeomorphism dual to an automorphisms  $\alpha$  of a  $C^*$ -algebra  $A$  implies that any representation of  $A \rtimes_{\alpha} \mathbb{Z}$  whose restriction to  $A$  is injective, is automatically faithful. The converse implication (equivalence between topological freeness and the aforementioned uniqueness property) in the case  $A$  is noncommutative turned out to be a difficult problem. It was proved in [34, Theorem 10.4] combined with [33, Theorem 2.5], see also [33, Remark 4.8], under the assumption that  $A$  is separable. The proof is nontrivial and passes through conditions involving such notions as Connes spectrum, inner derivations, or proper outerness. Since it is known that condition (L) is necessary for Cuntz–Krieger uniqueness theorem to hold, our explicit characterization of topological freeness for graph interactions (see Theorem 3.19) serves as a good illustration and a starting point for further generalizations of the aforementioned notions and facts.
- *Dilations of completely positive maps.* Let us consider a  $C^*$ -algebra  $C^*(A \cup \{s\})$  generated by a  $C^*$ -algebra  $A$  and a partial isometry  $s$  such that  $sAs^* \subset A$ . Also assume that  $A$  and  $C^*(A \cup \{s\})$  have a common unit. Then  $\mathcal{V}(\cdot) = s(\cdot)s^*$  is a completely positive map on  $A$  sending the unit to an idempotent (this is a general form of such mappings, cf. [26]). We may put  $\mathcal{H}(\cdot) := s^*(\cdot)s$  and then one can see that

$$B := \overline{\text{span}} \{a_0 s^* a_1 s^* a_2 \dots s^* a_n s b_1 s b_2 \dots s b_n : a_i, b_i \in A, \quad n \in \mathbb{N}\}$$

is the smallest  $C^*$ -algebra preserved by  $\mathcal{H}$  and containing  $A$ . Plainly, the pair  $(\mathcal{V}, \mathcal{H})$  is a corner interaction on  $B$ . Hence, potentially, our results could be applied to study the structure of  $C^*(A \cup \{s\}) = C^*(B \cup \{s\})$ . Nevertheless, the dilation of  $\mathcal{V}$  from  $A$  to  $B$  is a nontrivial procedure and in general depends on the initial representation of  $\mathcal{V}$  via  $s$ . The core algebras  $B$  arising in this way are studied in detail for instance in [21, 24, 27]. Our analysis of the graph interaction  $(\mathcal{V}, \mathcal{H})$  can be viewed as a case study of the above situation when  $A \cong \mathbb{C}^N$  is a finite dimensional commutative  $C^*$ -algebra, see Remark 3.10. In particular, Theorem 3.9 can be interpreted as that the partial homeomorphism dual to a dilation of the Ruelle–Perron–Frobenius operator  $\mathcal{H} = \mathcal{L}$  (from  $A$  to  $B = \mathcal{F}_E$ ) is a quotient of the Markov shift.

We begin by presenting relevant notions and statements concerning Hilbert bimodules and briefly clarifying their relationship with generalized  $C^*$ -correspondences. General corner interactions are studied in Sect. 2. Section 3 is devoted to analysis of graph interactions.

**1.1. Preliminaries on Hilbert Bimodules**

Throughout  $A$  is a  $C^*$ -algebra which (starting from Sect. 2) will always be unital. By homomorphisms, epimorphisms, etc. between  $C^*$ -algebras we always mean  $*$ -preserving maps. All ideals in  $C^*$ -algebras are assumed to be closed and two sided. We adhere to the convention that

$$\beta(A, B) = \overline{\text{span}}\{\beta(a, b) \in C : a \in A, b \in B\}$$

for maps  $\beta: A \times B \rightarrow C$  such as inner products, multiplications or representations.

As in [25] we say that a partial homeomorphism  $\varphi$  of a topological space  $M$ , i.e. a homeomorphism whose domain  $\Delta$  and range  $\varphi(\Delta)$  are open subsets of  $M$ , is *topologically free* if for any  $n > 0$  the set of fixed points for  $\varphi^n$  (on its natural domain) has empty interior. A set  $V$  is  $\varphi$ -invariant if  $\varphi(V \cap \Delta) = V \cap \varphi(\Delta)$ . If there are no nontrivial closed invariant sets, then  $\varphi$  is called *minimal*, and  $\varphi$  is said to be (residually) *free*, if it is topologically free on every closed invariant set (in the Hausdorff space case this amounts to requiring that  $\varphi$  has no periodic points).

Following [6, 1.8] and [1] by a *Hilbert bimodule over  $A$*  we mean  $X$  which is both a left Hilbert  $A$ -module and a right Hilbert  $A$ -module with respective inner products  $\langle \cdot, \cdot \rangle_A$  and  ${}_A \langle \cdot, \cdot \rangle$  satisfying the so-called *imprimitivity condition*:  $x \cdot \langle y, z \rangle_A = {}_A \langle x, y \rangle \cdot z$ , for all  $x, y, z \in X$ . A *covariant representation* of  $X$  is a pair  $(\pi_A, \pi_X)$  consisting of a homomorphism  $\pi_A : A \rightarrow \mathcal{B}(H)$  and a linear map  $\pi_X : X \rightarrow \mathcal{B}(H)$  such that

$$\pi_X(ax) = \pi_A(a)\pi_X(x), \quad \pi_X(xa) = \pi_X(x)\pi_A(a), \tag{1.1}$$

$$\pi_A(\langle x, y \rangle_A) = \pi_X(x)^* \pi_X(y), \quad \pi_A({}_A \langle x, y \rangle) = \pi_X(x)\pi_X(y)^*, \tag{1.2}$$

for all  $a \in A, x, y \in X$ . The *crossed product*  $A \rtimes_X \mathbb{Z}$  is a  $C^*$ -algebra generated by a copy of  $A$  and  $X$  universal with respect to covariant representations of  $X$ , see [1]. It is equipped with the *circle gauge action*  $\gamma = \{\gamma_z\}_{z \in \mathbb{T}}$  given on generators by  $\gamma_z(a) = a$  and  $\gamma_z(x) = zx$ , for  $a \in A, x \in X, z \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ .

As it is standard, we abuse the language and denote by  $\pi$  both an irreducible representation of  $A$  and its equivalence class in the spectrum  $\widehat{A}$  of  $A$ . It should not cause confusion when we consider induced representations, as for a Hilbert bimodule  $X$  over  $A$  the induced representation functor  $X$ -Ind preserves such classes. We briefly recall, and refer to [41] for all necessary details, that  $X$ -Ind maps a representation  $\pi : A \rightarrow \mathcal{B}(H)$  to a representation  $X$ -Ind( $\pi$ ) :  $A \rightarrow \mathcal{B}(X \otimes_\pi H)$  where the Hilbert space  $X \otimes_\pi H$  is generated by simple tensors  $x \otimes_\pi h, x \in X, h \in H$ , satisfying  $\langle x_1 \otimes_\pi h_1, x_2 \otimes_\pi h_2 \rangle = \langle h_1, \pi(\langle x_1, x_2 \rangle_A) h_2 \rangle$ , and

$$X\text{-Ind}(\pi)(a)(x \otimes_\pi h) = (ax) \otimes_\pi h, \quad a \in A.$$

The spaces  $\langle X, X \rangle_A$  and  ${}_A\langle X, X \rangle$  are ideals in  $A$  and the bimodule  $X$  implements a Morita equivalence between them. Hence  $X\text{-Ind} : \widehat{\langle X, X \rangle}_A \rightarrow {}_A\widehat{\langle X, X \rangle}$  is a homeomorphism which we may naturally treat as a partial homeomorphism of  $\widehat{A}$ , see [25].

The results of [25] can be summarized as follows.

**Theorem 1.1.** *Let  $X\text{-Ind}$  be a partial homeomorphism of  $\widehat{A}$ , as described above.*

- (i) *If  $X\text{-Ind}$  is topologically free, then every faithful covariant representation  $(\pi_A, \pi_X)$  of  $X$  ‘integrates’ to the faithful representation of  $A \rtimes_X \mathbb{Z}$ .*
- (ii) *If  $X\text{-Ind}$  is free, then  $J \mapsto \widehat{J \cap A}$  is a lattice isomorphism between ideals in  $A \rtimes_X \mathbb{Z}$  and open invariant sets in  $\widehat{A}$ .*
- (iii) *If  $X\text{-Ind}$  is topologically free and minimal, then  $A \rtimes_X \mathbb{Z}$  is simple.*

*Remark 1.2.* The map  $X\text{-Ind}$  is a lift of the so-called Rieffel homeomorphism  $h_X : \text{Prim} \langle X, X \rangle_A \rightarrow \text{Prim} {}_A\langle X, X \rangle$ , cf. [41, Corollary 3.33], [25, Remark 2.3]. Plainly, topological freeness of  $(\text{Prim}(A), h_X)$  implies topological freeness of  $(\widehat{A}, X\text{-Ind})$ , but the converse is not true and as we will see, cf. Example 3.4 below, Cuntz algebras  $\mathcal{O}_n$  provide an excellent example of this phenomenon.

*Remark 1.3.* Schweizer [43] showed that if  $X$  is a full nondegenerate  $C^*$ -correspondence over a unital  $C^*$ -algebra  $A$ , then the Cuntz–Pimsner algebra  $\mathcal{O}_X$ , defined as in [38], is simple if and only if  $X$  is minimal and aperiodic [43, Definition 3.7]. Clearly, if  $X$  is a Hilbert bimodule, minimality of  $X\text{-Ind}$  is equivalent to the minimality of  $X$  and topological freeness of  $X\text{-Ind}$  implies the aperiodicity of  $X$ . Moreover, the algebras  $\mathcal{O}_X$  and  $A \rtimes_X \mathbb{Z}$  coincide if and only if  ${}_A\langle X, X \rangle$  is an essential ideal in  $A$  (which in turn is equivalent to injectivity of the left action of  $A$  on  $X$ ). In particular, if the ideal  ${}_A\langle X, X \rangle$  is essential in  $A$  and  $\langle X, X \rangle_A = A$  is unital, then [43, Theorem 3.9] implies that  $A \rtimes_X \mathbb{Z}$  is simple iff  $X$  is minimal and aperiodic.

Let us fix a Hilbert bimodule  $X$  over  $A$ . We notice that it is naturally equipped with the ternary ring operation

$$[x, y, z] := x\langle y, z \rangle_A = {}_A\langle x, y \rangle z, \quad x, y, z \in X,$$

making it into a generalized correspondence over  $A$ , as defined in [12, Definition 7.1]. Alternatively, this generalized correspondence could be described in terms of [12, Proposition 7.6] as the triple  $(X, \lambda, \rho)$  where we consider  $X$  as a  ${}_A\langle X, X \rangle$ - $\langle X, X \rangle_A$ -Hilbert bimodule and define homomorphisms  $\lambda : A \rightarrow {}_A\langle X, X \rangle$  and  $\rho : A \rightarrow \langle X, X \rangle_A$  to be (necessarily unique) extensions of the identity maps.

The following fact should be compared with [12, Proposition 7.13].

**Proposition 1.4.** *The crossed product  $A \rtimes_X \mathbb{Z}$  of the Hilbert bimodule  $X$  is naturally isomorphic to the covariance algebra  $C^*(A, X)$ , as defined in [12, 7.12], for  $X$  treated as a generalized correspondence.*



*Proof.* The Toeplitz algebra  $\mathcal{T}(A, X)$  for the generalized correspondence  $X$ , see [12, page 57], is a universal  $C^*$ -algebra generated by a copy of  $A$  and  $X$  subject to all  $A$ - $A$ -bimodule relations plus the ternary ring relations:

$$xy^*z = x\langle y, z \rangle_A = A\langle x, y \rangle z, \quad x, y, z \in X. \tag{1.3}$$

The  $C^*$ -algebra  $C^*(A, X)$  is the quotient  $\mathcal{T}(A, X)/(J_\ell + J_r)$  where  $J_\ell$  (respectively  $J_r$ ) is an ideal in  $\mathcal{T}(A, X)$  generated by the elements  $a - k$  such that  $a \in (\ker \lambda)^\perp$ ,  $k \in XX^*$  (resp.  $a \in (\ker \rho)^\perp$ ,  $k \in X^*X$ ) and

$$ax = kx \quad (\text{or resp. } xa = xk) \quad \text{for all } x \in X. \tag{1.4}$$

Note that  $(\ker \lambda)^\perp = {}_A\langle X, X \rangle$  and  $(\ker \rho)^\perp = \langle X, X \rangle_A$ . By (1.3),  $XX^*$  and  $X^*X$  are  $C^*$ -subalgebras of  $\mathcal{T}(A, X)$ . Hence using approximate units argument we see that when  $a$  is fixed relations (1.4) determine  $k$  uniquely. It follows that

$$J_\ell = \overline{\text{span}} \{ {}_A\langle x, y \rangle - xy^* : x, y \in X \}, \quad J_r = \overline{\text{span}} \{ \langle x, y \rangle_A - x^*y : x, y \in X \},$$

because if (for instance)  $a - k \in J_\ell$  where  $a = \sum_{i=1}^n {}_A\langle x_i, y_i \rangle \in (\ker \lambda)^\perp$  and  $k \in X^*X$ , then by (1.3),  $ax = \sum_{i=1}^n x_i y_i^* x$  for all  $x \in X$  and thus  $k = \sum_{i=1}^n x_i y_i^*$ .

Accordingly, both  $C^*(A, X)$  and  $A \rtimes_X \mathbb{Z}$  are universal  $C^*$ -algebras generated by copies of  $A$  and  $X$  subject to the same relations.  $\square$

Katsura [22] obtained a version of the Pimsner–Voiculescu exact sequence for general  $C^*$ -correspondences and their  $C^*$ -algebras. We recall it in the case  $X$  is a Hilbert bimodule and in a form suitable for our purposes. We consider the linking algebra  $D_X = \mathcal{K}(X \oplus A)$  in the following matrix representation

$$D_X = \begin{pmatrix} \mathcal{K}(X) & X \\ \tilde{X} & A \end{pmatrix},$$

where  $\tilde{X}$  is the dual Hilbert bimodule of  $X$ , cf. e.g. [41, pages 49, 50]. Let  $\iota : {}_A\langle X, X \rangle \rightarrow A$ ,  $\iota_{11} : \mathcal{K}(X) \rightarrow D_X$  and  $\iota_{22} : A \rightarrow D_X$  be inclusion maps;  $\iota_{11}(a) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ ,  $\iota_{22}(a) = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$ . By [22, Proposition B.3],  $(\iota_{22})_* : K_*(A) \rightarrow K_*(D_X)$  is an isomorphism and by [22, Theorem 8.6] the following sequence is exact:

$$\begin{CD} K_0({}_A\langle X, X \rangle) @>>> K_0(A) @>>> K_0(A \rtimes_X \mathbb{Z}) \\ @. @VVV @VVV \\ @. K_1(A \rtimes_X \mathbb{Z}) @<<< K_1(A) @<<< K_1({}_A\langle X, X \rangle) \end{CD} \tag{1.5}$$

$\uparrow$   $\xrightarrow{\iota_* - (X_* \circ \phi_*)}$   $\xrightarrow{(i_A)_*}$   $\xrightarrow{\iota_* - (X_* \circ \phi_*)}$   $\xleftarrow{(i_A)_*}$   $\downarrow$

where  $\phi : A \rightarrow \mathcal{L}(X)$  is the homomorphism implementing the left action of  $A$  on  $X$ , and  $X_* : K_*({}_A\langle X, X \rangle) \rightarrow K_*(A)$  is the composition of  $(\iota_{11})_* : K_*({}_A\langle X, X \rangle) \rightarrow K_*(D_X)$  and the inverse to the isomorphism  $(\iota_{22})_* : K_*(A) \rightarrow K_*(D_X)$ .

## 2. Corner Interactions and Their Crossed Products

In this section, following closely the relationship between  $C^*$ -dynamical systems and interactions, we introduce corner interactions, describe the structure of the associated crossed product and establish fundamental tools for its analysis (Theorems 2.20, 2.25).

### 2.1. Interactions and $C^*$ -Dynamical Systems

It is instructive to consider interactions as generalization of pairs  $(\alpha, \mathcal{L})$ , sometimes called Exel systems [19], consisting of an endomorphism  $\alpha : A \rightarrow A$  and its *transfer operator*, i.e. a positive linear map  $\mathcal{L} : A \rightarrow A$  such that  $\mathcal{L}(\alpha(a)b) = a\mathcal{L}(b)$ ,  $a, b \in A$ , see [11]. Then  $\mathcal{L}$  is automatically continuous,  $*$ -preserving, and we also have:  $\mathcal{L}(b\alpha(a)) = \mathcal{L}(b)a$ ,  $a, b \in A$ . We say that a transfer operator  $\mathcal{L}$  is *regular* if  $\alpha(\mathcal{L}(1)) = \alpha(1)$ , or equivalently [11, Proposition 2.3], if  $\mathcal{E}(a) := \alpha(\mathcal{L}(a))$  is a conditional expectation from  $A$  onto  $\alpha(A)$ . We note that originally [11] Exel called such transfer operators non-degenerate. However, the use of the latter term is a bit unfortunate. For instance, it is used in the related context to mean a different property in [12, page 60], and also there are historical reasons to change this name, see [26].

It is important, see [23], that the range of a regular transfer operator  $\mathcal{L}$  coincides with the annihilator  $(\ker \alpha)^\perp$  of the kernel of  $\alpha$  and  $\mathcal{L}(1)$  is the unit in  $\mathcal{L}(A) = (\ker \alpha)^\perp$ , so in particular the latter is a complemented ideal.

**Definition 2.1.** A pair  $(\alpha, \mathcal{L})$  where  $\mathcal{L} : A \rightarrow A$  is a regular transfer operator for an endomorphism  $\alpha : A \rightarrow A$  will be called a  *$C^*$ -dynamical system*.

A dissatisfaction concerning asymmetry in the  $C^*$ -dynamical system  $(\alpha, \mathcal{L})$ ;  $\alpha$  is multiplicative while  $\mathcal{L}$  is ‘merely’ positive linear, lead the author of [12] to the following more general notion.

**Definition 2.2.** ([12], Definition 3.1) The pair  $(\mathcal{V}, \mathcal{H})$  of positive linear maps  $\mathcal{V}, \mathcal{H} : A \rightarrow A$  is called an *interaction* over  $A$  if

- (i)  $\mathcal{V} \circ \mathcal{H} \circ \mathcal{V} = \mathcal{V}$ ,
- (ii)  $\mathcal{H} \circ \mathcal{V} \circ \mathcal{H} = \mathcal{H}$ ,
- (iii)  $\mathcal{V}(ab) = \mathcal{V}(a)\mathcal{V}(b)$ , if either  $a$  or  $b$  belong to  $\mathcal{H}(A)$ ,
- (iv)  $\mathcal{H}(ab) = \mathcal{H}(a)\mathcal{H}(b)$ , if either  $a$  or  $b$  belong to  $\mathcal{V}(A)$ .

*Remark 2.3.* An interaction  $(\mathcal{V}, \mathcal{H})$ , or even a  $C^*$ -dynamical system  $(\alpha, \mathcal{L})$ , in general does not generate a semigroup of interactions [28] and all the more is not an element of a group interaction in the sense of [13]. This will be a generic case in our example arising from graphs, cf. Proposition 3.5 below. Accordingly, in general the facts proved in [13, 28], can not be applied in our present context.

Let  $(\mathcal{V}, \mathcal{H})$  be an interaction. By [12, Propositions 2.6, 2.7],  $\mathcal{V}(A)$  and  $\mathcal{H}(A)$  are  $C^*$ -subalgebras of  $A$ ,  $\mathcal{E}_{\mathcal{V}} := \mathcal{V} \circ \mathcal{H}$  is a conditional expectation onto  $\mathcal{V}(A)$ ,  $\mathcal{E}_{\mathcal{H}} := \mathcal{H} \circ \mathcal{V}$  is a conditional expectation onto  $\mathcal{H}(A)$ , and the mappings

$$\mathcal{V} : \mathcal{H}(A) \rightarrow \mathcal{V}(A), \quad \mathcal{H} : \mathcal{V}(A) \rightarrow \mathcal{H}(A)$$

are isomorphisms, each being the inverse of the other. Actually we have

**Proposition 2.4.** *The relations  $\mathcal{E}_{\mathcal{V}} = \mathcal{V} \circ \mathcal{H}$ ,  $\mathcal{E}_{\mathcal{H}} = \mathcal{H} \circ \mathcal{V}$ ,  $\theta = \mathcal{V}|_{\mathcal{E}_{\mathcal{H}}(A)}$  yield a one-to-one correspondence between interactions  $(\mathcal{V}, \mathcal{H})$  and triples  $(\theta, \mathcal{E}_{\mathcal{V}}, \mathcal{E}_{\mathcal{H}})$  consisting of two conditional expectations  $\mathcal{E}_{\mathcal{V}}, \mathcal{E}_{\mathcal{H}}$  and an isomorphism  $\theta : \mathcal{E}_{\mathcal{H}}(A) \rightarrow \mathcal{E}_{\mathcal{V}}(A)$ .*

*Proof.* It suffices to verify that if  $(\theta, \mathcal{E}_{\mathcal{V}}, \mathcal{E}_{\mathcal{H}})$  is as in the assertion, then  $\mathcal{V}(a) := \theta(\mathcal{E}_{\mathcal{H}}(a))$  and  $\mathcal{H}(a) := \theta^{-1}(\mathcal{E}_{\mathcal{V}}(a))$  form an interaction. This is straightforward.  $\square$

Recall that the  $C^*$ -algebra  $A$  has the unit 1. It follows that the algebras involved in an interaction are automatically also unital.

**Lemma 2.5.** *If  $(\mathcal{V}, \mathcal{H})$  is an interaction, then  $\mathcal{V}(1) = \mathcal{E}_{\mathcal{V}}(1)$  and  $\mathcal{H}(1) = \mathcal{E}_{\mathcal{H}}(1)$  are units in  $\mathcal{V}(A)$  and  $\mathcal{H}(A)$ , respectively (in particular, they are projections).*

*Proof.* Let us observe that

$$\begin{aligned} \mathcal{E}_{\mathcal{V}}(1) &= \mathcal{V}(\mathcal{H}(1)) = \mathcal{V}(\mathcal{H}(1)1) = \mathcal{V}(\mathcal{H}(1))\mathcal{V}(1) = \mathcal{V}(\mathcal{H}(1))\mathcal{V}(\mathcal{H}(\mathcal{V}(1))) \\ &= \mathcal{V}(\mathcal{H}(1)\mathcal{H}(\mathcal{V}(1))) = \mathcal{V}(\mathcal{H}(1\mathcal{V}(1))) = \mathcal{V}(\mathcal{H}(\mathcal{V}(1))) = \mathcal{V}(1). \end{aligned}$$

Therefore we have  $\mathcal{V}(a) = \mathcal{E}_{\mathcal{V}}(\mathcal{V}(a)) = \mathcal{E}_{\mathcal{V}}(1\mathcal{V}(a)) = \mathcal{E}_{\mathcal{V}}(1)\mathcal{V}(a) = \mathcal{V}(1)\mathcal{V}(a)$  for arbitrary  $a \in A$ . It follows that  $\mathcal{V}(1)$  is the unit in  $\mathcal{V}(A)$  and a similar argument works for  $\mathcal{H}$ .  $\square$

The following statement generalizes [12, Proposition 3.4].

**Proposition 2.6.** *Any  $C^*$ -dynamical system  $(\alpha, \mathcal{L})$  is an interaction.*

*Proof.* Consider the conditions (i)-(iv) in Definition 2.2. Since  $\alpha \circ \mathcal{L} \circ \alpha = \mathcal{E} \circ \alpha = \alpha$ , (i) is satisfied. To see (ii) recall that  $\mathcal{L}(1)$  is the unit in  $\mathcal{L}(A)$ , cf. [23, Proposition 1.5], and therefore

$$\mathcal{L}(\alpha(\mathcal{L}(a))) = \mathcal{L}(1\alpha(\mathcal{L}(a))) = \mathcal{L}(1)\mathcal{L}(a) = \mathcal{L}(a).$$

Condition (iii) is trivial for  $(\alpha, \mathcal{L})$ , and (iv) holds because

$$\mathcal{L}(a\alpha(b)) = \mathcal{L}(a)b = \mathcal{L}(a)\mathcal{L}(1)b = \mathcal{L}(a)\mathcal{L}(1\alpha(b)) = \mathcal{L}(a)\mathcal{L}(\alpha(b)),$$

and by passing to adjoints we also get  $\mathcal{L}(\alpha(b)a) = \mathcal{L}(\alpha(b))\mathcal{L}(a)$ .  $\square$

As shown in [2], in the case the conditional expectation  $\mathcal{E} = \alpha \circ \mathcal{L}$  is given by

$$\mathcal{E}(a) = \alpha(1)a\alpha(1), \quad a \in A, \tag{2.1}$$

there is a very natural crossed product associated to the  $C^*$ -dynamical system  $(\alpha, \mathcal{L})$ . This crossed product coincides with the one introduced in [11] and is sufficient to cover many classic constructions, see [2].

A transfer operator for which (2.1) holds is called *complete* [2, 23]. It is a *corner retraction* [26, 43]. By [23] a given endomorphism  $\alpha$  admits a complete transfer operator  $\mathcal{L}$  if and only if  $\ker \alpha$  is a complemented ideal and  $\alpha(A)$  is a hereditary subalgebra of  $A$ . In this case  $\mathcal{L}$  is a unique regular transfer operator for  $\alpha$ , see [2, 23, 26, 43]. We naturally generalize the aforementioned concepts to interactions, cf. also [28].

**Definition 2.7.** An interaction  $(\mathcal{V}, \mathcal{H})$  will be called a *corner interaction* if  $\mathcal{V}(A)$  and  $\mathcal{H}(A)$  are hereditary subalgebras of  $A$ .

**Proposition 2.8.** *An interaction  $(\mathcal{V}, \mathcal{H})$  is corner if and only if  $\mathcal{V}(A) = \mathcal{V}(1)A\mathcal{V}(1)$  and  $\mathcal{H}(A) = \mathcal{H}(1)A\mathcal{H}(1)$  are corners in  $A$ . Moreover, for a corner interaction  $(\mathcal{V}, \mathcal{H})$  the following conditions are equivalent*

- (i)  $(\mathcal{V}, \mathcal{H})$  is a (corner)  $C^*$ -dynamical system,
- (ii)  $\mathcal{V}$  is multiplicative,
- (iii)  $\ker \mathcal{V}$  is an ideal in  $A$ ,
- (iv)  $\mathcal{H}(A)$  is an ideal in  $A$ ,
- (v)  $\mathcal{H}(1)$  lies in the center of  $A$ .

*Proof.* For the first part of the assertion apply Lemma 2.5 and notice that if  $B$  is a hereditary subalgebra of  $A$  and  $B$  has a unit  $P$ , then  $B = PAP$ . To show the second part of assertion let us suppose that  $(\mathcal{V}, \mathcal{H})$  is a corner interaction.

The implications (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii) and the equivalence (iv)  $\Leftrightarrow$  (v) are clear.

(iii)  $\Rightarrow$  (v). By the first part of the assertion  $\mathcal{V}$  is isometric on  $\mathcal{H}(1)A\mathcal{H}(1)$  and thus  $\ker \mathcal{V} \cap \mathcal{H}(1)A\mathcal{H}(1) = \{0\}$ . In view of Lemma 2.5, for any  $a \in A$  we have  $a(1 - \mathcal{H}(1)) \in \ker \mathcal{V}$ . Hence if  $\ker \mathcal{V}$  is an ideal, then  $\mathcal{H}(1)a(1 - \mathcal{H}(1))a^*\mathcal{H}(1) \in (\ker \mathcal{V}) \cap \mathcal{H}(1)A\mathcal{H}(1) = \{0\}$ , that is  $\mathcal{H}(1)a(1 - \mathcal{H}(1)) = 0$  which means that  $\mathcal{H}(1)a = a\mathcal{H}(1)$ .

(v)  $\Rightarrow$  (i). By the first part of the assertion  $\mathcal{E}_{\mathcal{H}}(a) = \mathcal{H}(1)a\mathcal{H}(1)$ . Thus, for any  $a, b \in A$ , we have

$$\begin{aligned} \mathcal{V}(ab) &= \mathcal{V}(\mathcal{E}_{\mathcal{H}}(ab)) = \mathcal{V}(\mathcal{H}(1)ab\mathcal{H}(1)) = \mathcal{V}(a\mathcal{H}(1)b\mathcal{H}(1)) \\ &= \mathcal{V}(a\mathcal{E}_{\mathcal{H}}(b)) = \mathcal{V}(a)\mathcal{V}(\mathcal{E}_{\mathcal{H}}(b)) = \mathcal{V}(a)\mathcal{V}(b). \end{aligned}$$

Hence  $\mathcal{V}$  is an endomorphism of  $A$ . The map  $\mathcal{H}$  is a transfer operator for  $\mathcal{V}$  because

$$\mathcal{H}(a\mathcal{V}(b)) = \mathcal{H}(a)\mathcal{H}(\mathcal{V}(b)) = \mathcal{H}(a)\mathcal{H}(1)b\mathcal{H}(1) = \mathcal{H}(a)b. \quad \square$$

As it is indicated by the uniqueness of the complete transfer operator, it turns out that each mapping in a corner interaction determines the other.

**Proposition 2.9.** *A positive linear map  $\mathcal{V} : A \rightarrow A$  is a part of a non-zero corner interaction  $(\mathcal{V}, \mathcal{H})$  if and only if  $\|\mathcal{V}(1)\| = 1$ ,  $\mathcal{V}(A)$  is a hereditary subalgebra of  $A$  and there is a projection  $P \in A$  such that  $\mathcal{V} : PAP \rightarrow \mathcal{V}(A)$  is an isomorphism.*

*Moreover, in the above equivalence  $P$  and  $\mathcal{H}$  are uniquely determined by  $\mathcal{V}$ , and we have*

$$\mathcal{H}(a) := \mathcal{V}^{-1}(\mathcal{V}(1)a\mathcal{V}(1)), \quad a \in A, \tag{2.2}$$

where  $\mathcal{V}^{-1}$  is the inverse to  $\mathcal{V} : PAP \rightarrow \mathcal{V}(A)$ .

*Proof.* The necessity of the stated conditions follows from Proposition 2.8 and Lemma 2.5. For the sufficiency note that  $\mathcal{V}(P)$  is a unit in  $\mathcal{V}(A)$  and therefore  $\mathcal{V}(A) = \mathcal{V}(P)A\mathcal{V}(P)$ , as  $\mathcal{V}(A)$  is hereditary in  $A$ . In particular,  $\mathcal{E}_{\mathcal{V}}(a) := \mathcal{V}(P)a\mathcal{V}(P)$  is a conditional expectation onto  $\mathcal{V}(A)$ . We define  $\mathcal{E}_{\mathcal{H}}(a) := \mathcal{V}^{-1}(\mathcal{V}(a))$  where  $\mathcal{V}^{-1}$  is the inverse to  $\mathcal{V} : PAP \rightarrow \mathcal{V}(A)$ . Then  $\mathcal{E}_{\mathcal{H}}$  is an idempotent map of norm one because  $\|\mathcal{E}_{\mathcal{H}}\| = \|\mathcal{V}\| = \|\mathcal{V}(1)\| = 1$ . Hence  $\mathcal{E}_{\mathcal{H}}$  is a conditional expectation onto  $PAP$ . By Proposition 2.4,

the triple  $(\mathcal{V}, \mathcal{E}_{\mathcal{V}}, \mathcal{E}_{\mathcal{H}})$  yields a (necessarily corner) interaction  $(\mathcal{V}, \mathcal{H})$  where  $\mathcal{H}(a) = \mathcal{V}^{-1}(\mathcal{V}(P)a\mathcal{V}(P))$ . In particular, it follows from Lemma 2.5 that  $\mathcal{V}(P) = \mathcal{V}(1)$ , that is  $\mathcal{H}$  is given by (2.2).

What remains to be shown is the uniqueness of  $P$ . Suppose then that  $(\mathcal{V}, \mathcal{H}_i), i = 1, 2$ , are two corner interactions and consider projections  $P_1 := \mathcal{H}_1(1)$  and  $P_2 := \mathcal{H}_2(1)$ . We have

$$\mathcal{V}(P_1P_2P_1) = \mathcal{V}(P_2) = \mathcal{V}(1) = \mathcal{V}(P_1) = \mathcal{V}(P_2P_1P_2),$$

and as  $\mathcal{V}$  is injective on  $\mathcal{H}_i(A) = P_iAP_i, i = 1, 2$ , it follows that  $P_1P_2P_1 = P_1$  and  $P_2 = P_2P_1P_2$ . This implies  $P_1 = P_2$ . □

**2.2. Crossed Product for Corner Interactions**

From now on  $(\mathcal{V}, \mathcal{H})$  will always stand for a corner interaction. We define the corresponding crossed product in universal terms.

**Definition 2.10.** A *covariant representation* of  $(\mathcal{V}, \mathcal{H})$  is a pair  $(\pi, S)$  consisting of a non-degenerate representation  $\pi : A \rightarrow \mathcal{B}(H)$  and an operator  $S \in \mathcal{B}(H)$  (which is necessarily a partial isometry) such that

$$S\pi(a)S^* = \pi(\mathcal{V}(a)) \text{ and } S^*\pi(a)S = \pi(\mathcal{H}(a)) \text{ for all } a \in A.$$

The *crossed product for the interaction*  $(\mathcal{V}, \mathcal{H})$  is the  $C^*$ -algebra  $C^*(A, \mathcal{V}, \mathcal{H})$  generated by  $i_A(A)$  and  $s$  where  $(i_A, s)$  is a universal covariant representation of  $(\mathcal{V}, \mathcal{H})$ . It is equipped with the *circle gauge action* determined by  $\gamma_z(i_A(a)) = i_A(a), a \in A$ , and  $\gamma_z(s) = zs$ .

Obviously, the above definition generalizes the crossed product studied in [2]. In other words  $C^*(A, \mathcal{V}, \mathcal{H})$  coincides with Exel’s crossed product [11] when  $(\mathcal{V}, \mathcal{H})$  is a  $C^*$ -dynamical system. To show it is essentially the same algebra as the one associated to (general) interactions in [12], we realize  $C^*(A, \mathcal{V}, \mathcal{H})$  as the crossed product for a Hilbert bimodule. To this end, we conveniently adopt Exel’s construction of his generalized correspondence associated to  $(\mathcal{V}, \mathcal{H})$ , [12, Section 5].

Let  $X_0 = A \odot A$  be the algebraic tensor product over the complexes, and let  $\langle \cdot, \cdot \rangle_A$  and  ${}_A\langle \cdot, \cdot \rangle$  be the  $A$ -valued sesqui-linear functions defined on  $X_0 \times X_0$  by

$$\langle a \odot b, c \odot d \rangle_A = b^*\mathcal{H}(a^*c)d, \quad {}_A\langle a \odot b, c \odot d \rangle = a\mathcal{V}(bd^*)c^*.$$

We consider the linear space  $X_0$  as an  $A$ - $A$ -bimodule with the natural module operations:  $a \cdot (b \odot c) = ab \odot c, (a \odot b) \cdot c = a \odot bc$ .

**Lemma 2.11.** *A quotient of  $X_0$  becomes naturally a pre-Hilbert  $A$ - $A$ -bimodule. More precisely,*

- (i) *the space  $X_0$  with a function  $\langle \cdot, \cdot \rangle_A$  (respectively  ${}_A\langle \cdot, \cdot \rangle$ ) becomes a right (respectively left) semi-inner product  $A$ -module;*
- (ii) *the corresponding semi-norms*

$$\|x\|_A := \|\langle x, x \rangle_A\|^{\frac{1}{2}} \quad \text{and} \quad {}_A\|x\| := \|{}_A\langle x, x \rangle\|^{\frac{1}{2}}$$

*coincide on  $X_0$  and thus the quotient space  $X_0/\|\cdot\|$  obtained by modding out the vectors of length zero with respect to the seminorm  $\|x\| := \|x\|_A = {}_A\|x\|$  is both a left and a right pre-Hilbert module over  $A$ ;*

(iii) denoting by  $a \otimes b$  the canonical image of  $a \odot b$  in the quotient space  $X_0/\|\cdot\|$  we have

$$ac \otimes b = a \otimes \mathcal{H}(c)b, \quad \text{if } c \in \mathcal{V}(A), \quad a \otimes cb = a\mathcal{V}(c) \otimes b, \quad \text{if } c \in \mathcal{H}(A),$$

and  $a \otimes b = a\mathcal{V}(1) \otimes \mathcal{H}(1)b$  for all  $a, b \in A$ ;

(iv) the inner-products in  $X_0/\|\cdot\|$  satisfy the imprimitivity condition.

*Proof.* (i) All axioms of  $A$ -valued semi-inner products for  $\langle \cdot, \cdot \rangle_A$  and  ${}_A \langle \cdot, \cdot \rangle$  except the non-negativity are straightforward, and to show the latter one may rewrite the proof of [12, Proposition 5.2] [just erase the symbol  $e_{\mathcal{H}}$  or put  $e_{\mathcal{H}} = \mathcal{H}(1)$ ].

(ii) Similarly, the proof of [12, Proposition 5.4] implies that for  $x = \sum_{i=1}^n a_i^* \odot b_i$ ,  $a_i, b_i \in A$ , we have

$$\|x\|_A = \|\mathcal{H}(aa^*)^{\frac{1}{2}}\mathcal{H}(\mathcal{V}(bb^*))^{\frac{1}{2}}\| = \|\mathcal{V}(\mathcal{H}(aa^*))^{\frac{1}{2}}\mathcal{V}(bb^*)^{\frac{1}{2}}\| = {}_A\|x\| \quad (2.3)$$

where  $a = (a_1, \dots, a_n)^T$  and  $b = (b_1, \dots, b_n)^T$  are viewed as column matrices.

(iii) For the first part consult the proof of [12, Proposition 5.6]. The second part can be proved analogously. Namely, for every  $x, y, a, b \in A$  we have

$$\langle x \otimes y, a \otimes b \rangle_A = y^* \mathcal{H}(x^* a) b = y^* \mathcal{H}(x^* a \mathcal{V}(1)) \mathcal{H}(1) b = \langle x \otimes y, a \mathcal{V}(1) \otimes \mathcal{H}(1) b \rangle_A,$$

which implies that  $\|a \otimes b - a\mathcal{V}(1) \otimes \mathcal{H}(1)b\| = 0$ .

(iv) The form of imprimitivity condition allows us to check it only on simple tensors. Using (iii), for  $a, b, c, d, e, f \in A$ , we have

$$\begin{aligned} a \otimes b \langle c \otimes d, e \otimes f \rangle_A &= a \otimes bd^* \mathcal{H}(c^* e) f = a \otimes \mathcal{H}(1) bd^* \mathcal{H}(c^* e) f \\ &= a \mathcal{V} \left( \mathcal{H}(1) bd^* \mathcal{H}(c^* e) \right) \otimes f = a \mathcal{V}(\mathcal{H}(1) bd^*) \mathcal{V}(\mathcal{H}(c^* e)) \otimes f \\ &= a \mathcal{V}(bd^*) \mathcal{V}(1) c^* e \mathcal{V}(1) \otimes f = a \mathcal{V}(bd^*) c^* e \otimes f \\ &= {}_A \langle a \otimes b, c \otimes d \rangle e \otimes f. \end{aligned} \quad \square$$

**Definition 2.12.** We call the completion  $X$  of the pre-Hilbert bimodule  $X_0$  described in Lemma 2.11 a *Hilbert bimodule associated to*  $(\mathcal{V}, \mathcal{H})$ .

*Remark 2.13.* The Hilbert bimodule  $X$  could be obtained directly from the imprimitivity  $\mathcal{K}_{\mathcal{V}}\text{-}\mathcal{K}_{\mathcal{H}}$ -bimodule  $\mathcal{X}$  constructed in [12, Section 5]. Indeed, by (2.3),  $X$  and  $\mathcal{X}$  coincide as Banach spaces, and since

$$\langle X, X \rangle_A = A\mathcal{H}(1)A, \quad {}_A \langle X, X \rangle = A\mathcal{V}(1)A,$$

$X$  can be considered an imprimitivity  $A\mathcal{V}(1)A\text{-}A\mathcal{H}(1)A$ -bimodule. Furthermore, the mappings  $\lambda_{\mathcal{V}} : A \rightarrow \mathcal{K}_{\mathcal{V}}$ ,  $\lambda_{\mathcal{H}} : A \rightarrow \mathcal{K}_{\mathcal{H}}$ , the author of [12] uses to define an  $A\text{-}A$ -bimodule structure on  $\mathcal{X}$ , when restricted respectively to  $A\mathcal{V}(1)A$  and  $A\mathcal{H}(1)A$  are isomorphisms. Hence we may use them to assume the identifications  $\mathcal{K}_{\mathcal{V}} = A\mathcal{V}(1)A$  and  $\mathcal{K}_{\mathcal{H}} = A\mathcal{H}(1)A$ , and then Exel generalized correspondence and the Hilbert bimodule  $X$  coincide.

Now we are ready to identify the structure of  $C^*(A, \mathcal{V}, \mathcal{H})$  as the Hilbert bimodule crossed product.

**Proposition 2.14.** *We have a one-to-one correspondence between covariant representations  $(\pi, S)$  of the interaction  $(\mathcal{V}, \mathcal{H})$  and covariant representations  $(\pi, \pi_X)$  of the Hilbert bimodule  $X$  associated to  $(\mathcal{V}, \mathcal{H})$ . It is given by relations*

$$\pi_X(a \otimes b) = \pi(a)S\pi(b), \quad x \in X, \quad S = \pi_X(1 \otimes 1).$$

*In particular,  $C^*(A, \mathcal{V}, \mathcal{H}) \cong A \rtimes_X \mathbb{Z}$  and the isomorphism is gauge-invariant.*

*Proof.* Let  $(\pi, S)$  be a covariant representation of  $(\mathcal{V}, \mathcal{H})$ . Since

$$\begin{aligned} \left\| \sum_i \pi(a_i)S\pi(b_i) \right\|^2 &= \left\| \sum_{i,j} \pi(a_i)S\pi(b_i b_j^*)S^* \pi(a_j^*) \right\|^2 = \left\| \pi \left( \sum_{i,j} a_i \mathcal{V}(b_i b_j^*) a_j^* \right) \right\|^2 \\ &\leq \left\| \sum_i a_i \otimes b_i \right\|^2, \end{aligned}$$

we see that  $\pi_X(\sum_i a_i \otimes b_i) := \sum_i \pi(a_i)S\pi(b_i)$  defines a contractive linear mapping on  $X_0/\|\cdot\|$ . Clearly, it satisfies (1.1) and (1.2). Hence by continuity it extends uniquely to  $X$  in a way that  $(\pi, \pi_X)$  is a covariant representation of  $X$ . Conversely suppose that  $(\pi, \pi_X)$  is a covariant representation of the Hilbert bimodule  $X$  and put  $S := \pi_X(1 \otimes 1)$ . Then for  $a \in A$  we have

$$S\pi(a)S^* = \pi_X((1 \otimes 1)a)\pi_X(1 \otimes 1)^* = \pi(A\langle 1 \otimes a, 1 \otimes 1 \rangle) = \pi(\mathcal{V}(a)).$$

Similarly,  $S^* \pi(a)S = \pi_X(1 \otimes 1)^* \pi_X(a(1 \otimes 1)) = \pi(\langle 1 \otimes 1, a \otimes 1 \rangle_A) = \pi(\mathcal{H}(a))$ . □

*Remark 2.15.* The Hilbert bimodule  $X$  is nothing but the GNS  $C^*$ -correspondence determined by the completely positive map  $\mathcal{H}$ , cf. [26]. In particular, the above proposition shows that  $C^*(A, \mathcal{V}, \mathcal{H})$  is isomorphic to the crossed product of  $A$  by the completely positive map  $\mathcal{H}$  (or  $\mathcal{V}$ , depending on preferences), see [26].

Finally, by Remark 2.13 and Propositions 1.4, 2.14 we get

**Corollary 2.16.** *Let  $\mathcal{X}$  be the generalized correspondence constructed out of  $(\mathcal{V}, \mathcal{H})$  as in [12, Section 5]. The crossed product  $C^*(A, \mathcal{V}, \mathcal{H})$  for the interaction  $(\mathcal{V}, \mathcal{H})$  and the covariance algebra  $C^*(A, \mathcal{X})$  for  $\mathcal{X}$  are naturally isomorphic.*

**2.3. Topological Freeness, Ideal Structure and Simplicity Criteria**

Let  $(\mathcal{V}, \mathcal{H})$  be a corner interaction. Since  $\mathcal{V}(A)$  and  $\mathcal{H}(A)$  are hereditary subalgebras of  $A$  we have a standard way, cf. e.g. [37, Proposition 4.1.9], of identifying their spectra with open subsets of  $\widehat{A}$ . Namely, we assume that

$$\widehat{\mathcal{V}(A)} = \{\pi \in \widehat{A} : \pi(\mathcal{V}(1)) \neq 0\}, \quad \widehat{\mathcal{H}(A)} = \{\pi \in \widehat{A} : \pi(\mathcal{H}(1)) \neq 0\}. \quad (2.4)$$

The isomorphisms  $\mathcal{V} : \mathcal{H}(A) \rightarrow \mathcal{V}(A)$  and  $\mathcal{H} : \mathcal{V}(A) \rightarrow \mathcal{H}(A)$  induce mutually inverse homeomorphisms  $\widehat{\mathcal{V}} : \widehat{\mathcal{V}(A)} \rightarrow \widehat{\mathcal{H}(A)}$  and  $\widehat{\mathcal{H}} : \widehat{\mathcal{H}(A)} \rightarrow \widehat{\mathcal{V}(A)}$ , which under identifications (2.4) become partial homeomorphisms of  $\widehat{A}$ .

**Definition 2.17.** We refer to  $\widehat{\mathcal{V}}$  and  $\widehat{\mathcal{H}}$  as *partial homeomorphisms dual to  $(\mathcal{V}, \mathcal{H})$* .

*Remark 2.18.* For an irreducible representation  $\pi : A \rightarrow \mathcal{B}(H)$  with  $\pi(\mathcal{H}(1)) \neq 0$  the element  $\widehat{\mathcal{H}}(\pi) \in \widehat{A}$  is given by the (unique up to unitary equivalence) extension of the representation

$$\widehat{\mathcal{H}}(\pi)|_{\mathcal{V}(A)} = \pi \circ \mathcal{H} : \mathcal{V}(A) \rightarrow \mathcal{B}(\pi(\mathcal{H}(1))H),$$

Moreover, in the case  $(\mathcal{V}, \mathcal{H})$  is a  $C^*$ -dynamical system  $\mathcal{H}(A)$  is an ideal and then  $\pi(\mathcal{H}(1))H = H$ .

**Proposition 2.19.** *If  $X$  is the Hilbert bimodule associated to  $(\mathcal{V}, \mathcal{H})$  and  $X$ -Ind is the partial homeomorphism of  $\widehat{A}$  associated to  $X$ , then  $X$ -Ind =  $\widehat{\mathcal{H}}$ .*

*Proof.* Let  $\pi : A \rightarrow \mathcal{B}(H)$  be an irreducible representation with  $\pi(\mathcal{H}(1)) \neq 0$ . For  $(a \otimes b) \otimes_{\pi} h \in X \otimes_{\pi} H$ ,  $a, b \in A$ ,  $h \in H$ , using Lemma 2.11 (iii) we have

$$\begin{aligned} X\text{-Ind}(\pi)(\mathcal{V}(1))(a \otimes b) \otimes_{\pi} h &= (\mathcal{V}(1)a \otimes b) \otimes_{\pi} h = (\mathcal{V}(1)a\mathcal{V}(1) \otimes b) \otimes_{\pi} h \\ &= (1 \otimes \mathcal{H}(a)b) \otimes_{\pi} h = (1 \otimes 1) \otimes_{\pi} \pi(\mathcal{H}(a)b)h. \end{aligned}$$

Hence we see that the space  $H_0 := X\text{-Ind}(\pi)(\mathcal{V}(1))(X \otimes_{\pi} H)$  consists of the vectors of the form  $(1 \otimes 1) \otimes_{\pi} h$ ,  $h \in \pi(\mathcal{H}(1))H$ . Moreover, for  $h \in \pi(\mathcal{H}(1))H$  we have

$$\begin{aligned} \|(1 \otimes 1) \otimes_{\pi} h\|^2 &= \langle (1 \otimes 1) \otimes_{\pi} h, (1 \otimes 1) \otimes_{\pi} h \rangle = \langle h, \pi(\langle 1 \otimes 1, 1 \otimes 1 \rangle_A)h \rangle \\ &= \langle h, \pi(\mathcal{H}(1))h \rangle = \|h\|^2, \end{aligned}$$

and thus the mapping  $(1 \otimes 1) \otimes_{\pi} h \mapsto h$  is a unitary  $U$  from  $H_0$  onto  $\pi(\mathcal{H}(1))H$ . For  $a \in \mathcal{V}(A)$  we have

$$\begin{aligned} X\text{-Ind}(\pi)(a)(1 \otimes 1) \otimes_{\pi} h &= (a \otimes 1) \otimes_{\pi} h = (1 \otimes \mathcal{H}(a)) \otimes_{\pi} h \\ &= (1 \otimes 1) \otimes_{\pi} \pi(\mathcal{H}(a))h, \end{aligned}$$

that is  $X\text{-Ind}(\pi)(a)U^*h = U^*\pi(\mathcal{H}(a))h$ . It follows that  $U$  establishes unitary equivalence between  $X\text{-Ind}(\pi) : \mathcal{V}(A) \rightarrow \mathcal{B}(H_0)$  and  $\pi \circ \mathcal{H} : \mathcal{V}(A) \rightarrow \mathcal{B}(\pi(\mathcal{H}(1))H)$ . Hence  $X\text{-Ind} = \widehat{\mathcal{H}}$ , cf. Remark 2.18. □

As  $\widehat{\mathcal{H}} = \widehat{\mathcal{V}}^{-1}$ , our preference for  $\widehat{\mathcal{V}}$  in the sequel is totally a subjective choice.

**Theorem 2.20.** *Let  $(\mathcal{V}, \mathcal{H})$  a corner interaction and  $\widehat{\mathcal{V}}$  the partial homeomorphism dual to  $\mathcal{V}$ .*

- (i) *If  $\widehat{\mathcal{V}}$  is topologically free, then every representation of  $C^*(A, \mathcal{V}, \mathcal{H})$  which is faithful on  $A$  is automatically faithful on  $C^*(A, \mathcal{V}, \mathcal{H})$ .*
- (ii) *If  $\widehat{\mathcal{V}}$  is free, then  $J \mapsto \widehat{J \cap A}$  is a lattice isomorphism between ideals in  $C^*(A, \mathcal{V}, \mathcal{H})$  and open  $\widehat{\mathcal{V}}$ -invariant sets in  $\widehat{A}$ .*
- (iii) *If  $\widehat{\mathcal{V}}$  is topologically free and minimal, then  $C^*(A, \mathcal{V}, \mathcal{H})$  is simple.*

*Proof.* Combine Propositions 2.14, 2.19 and Theorem 1.1. □

*Remark 2.21.* Our simplicity criterion [Theorem 2.20 (iii)] have an intersection with the criteria in [43, Theorems 4.1, 4.6] only in the case of a  $C^*$ -dynamical system  $(\alpha, \mathcal{L})$  where  $\alpha$  is an isomorphism from  $A$  onto a full corner  $\alpha(1)A\alpha(1)$  in  $A$ , cf. Remark 1.3 and Corollary 2.23 below. In this case



topological freeness implies that no power of  $\alpha$  or  $\mathcal{L}$  is inner (i.e. implemented by an isometry in  $A$ ).

In general, one can deduce from Propositions 2.14, 2.19, see [25, discussion before Theorem 2.5], that open  $\widehat{\mathcal{V}}$ -invariant sets in  $\widehat{A}$  are in a one-to-one correspondence with gauge invariant ideals in  $C^*(A, \mathcal{V}, \mathcal{H})$ . Therefore it is useful to have a convenient description of the former.

**Lemma 2.22.** *Let  $I$  be an ideal in  $A$ . The following conditions are equivalent:*

- (i) *The set  $\widehat{I} \subset \widehat{A}$  is  $\widehat{\mathcal{V}}$ -invariant,*
- (ii)  *$\mathcal{V}(I) = \mathcal{V}(1)I\mathcal{V}(1)$ ,*
- (iii)  *$\mathcal{V}(I) \subset I$  and  $\mathcal{H}(I) \subset I$ .*

*Proof.* Notice that  $\mathcal{V}(1)I\mathcal{V}(1) = I \cap \mathcal{V}(A)$  and  $\mathcal{H}(1)I\mathcal{H}(1) = I \cap \mathcal{H}(A)$ . Hence  $\mathcal{V}(I) = \mathcal{V}(\mathcal{H}(1)I\mathcal{H}(1)) = \mathcal{V}(I \cap \mathcal{H}(A))$  and (ii) reads as  $\mathcal{V}(I \cap \mathcal{H}(A)) = I \cap \mathcal{V}(A)$ . Now equivalence (i)  $\Leftrightarrow$  (ii) is clear.

(ii)  $\Rightarrow$  (iii). We have  $\mathcal{V}(I) = \mathcal{V}(1)I\mathcal{V}(1) \subset I$  and  $\mathcal{H}(I) = \mathcal{H}(\mathcal{V}(1)I\mathcal{V}(1)) = \mathcal{H}(\mathcal{V}(I)) = \mathcal{H}(1)I\mathcal{H}(1) \subset I$ .

(iii)  $\Rightarrow$  (ii). The inclusion  $\mathcal{V}(I) \subset I$  implies  $\mathcal{V}(I) \subset \mathcal{V}(1)I\mathcal{V}(1)$  and  $\mathcal{H}(I) \subset I$  implies that  $\mathcal{V}(1)I\mathcal{V}(1) = \mathcal{V}(\mathcal{H}(I)) \subset \mathcal{V}(I)$ . □

**Corollary 2.23.** *Suppose  $(\alpha, \mathcal{L})$  is a corner  $C^*$ -dynamical system. The partial homeomorphism  $\widehat{\alpha}$  is minimal if and only if there is no nontrivial ideal  $I$  in  $A$  such that  $\alpha(I) \subset I$ .*

*Proof.* The if part follows immediately from Lemma 2.22. If we suppose that  $\alpha(I) \subset I$  and  $\alpha(I) \neq \alpha(1)I\alpha(1)$  for a certain nontrivial ideal  $I$  in  $A$ , then one sees (by induction on  $n$ ) that the closure  $J$  of elements of the form  $\sum_{k=0}^n \mathcal{L}^k(a_k)$ ,  $a_k \in I$ ,  $k = 0, \dots, n$ ,  $n \in \mathbb{N}$ , is a nontrivial ideal in  $A$  (it does not contain the unit) such that  $\alpha(J) \subset J$  and  $\mathcal{L}(J) \subset J$ . Hence by Lemma 2.22,  $\widehat{\alpha}$  is not minimal. □

**2.4.  $K$ -Theory**

We retain the notation from page 8 with the additional assumption that  $X$  is the Hilbert bimodule associated to a corner interaction  $(\mathcal{V}, \mathcal{H})$ . In particular,  ${}_A\langle X, X \rangle = A\mathcal{V}(1)A$ .

**Lemma 2.24.** *The following diagram commutes and the horizontal map is an isomorphism*

$$\begin{array}{ccc}
 K_*(\mathcal{V}(A)) & \xrightarrow{\iota_*} & K_*(A\mathcal{V}(1)A) \\
 & \searrow^{(\iota_{22} \circ \mathcal{H})_*} & \swarrow_{(\iota_{11} \circ \phi)_*} \\
 & & K_*(D_X)
 \end{array}$$

*Proof.* Since  $\mathcal{V}(A)$  is a full corner in  $A\mathcal{V}(1)A$  it is known that the inclusion  $\iota : \mathcal{V}(A) \rightarrow A\mathcal{V}(1)A$  yields isomorphisms of  $K$ -groups, cf. e.g. [22, Proposition B.5]. We claim that the map

$$M_2(\mathcal{H}(A)) \ni \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \xrightarrow{\Phi} \begin{pmatrix} \phi(\mathcal{V}(a_{11})) & 1 \otimes a_{12} \\ b(1 \otimes a_{21}^*) & a_{22} \end{pmatrix} \in D_X,$$

where  $\flat : X \rightarrow \tilde{X}$  is the canonical antilinear isomorphism, is a homomorphism of  $C^*$ -algebras. Plainly, it is linear,  $*$ -preserving, and the reader easily checks that  $\Phi(ab) = \Phi(a)\Phi(b)$ , for  $a = [a_{ij}]$ ,  $b = [b_{ij}] \in M_2(\mathcal{H}(A))$ , using the following calculations

$$\begin{aligned} (1 \otimes a_{12}) \cdot \flat(1 \otimes b_{21}^*)x \otimes y &= \Theta_{(1 \otimes a_{12}), (1 \otimes b_{21}^*)}x \otimes y = 1 \otimes a_{12}b_{21}\mathcal{H}(x)y \\ &= \mathcal{V}(a_{12}b_{21})\mathcal{V}(\mathcal{H}(x)) \otimes y = \mathcal{V}(a_{12})\mathcal{V}(b_{21})x \otimes y \\ &= \phi(\mathcal{V}(a_{12}b_{21}))(x \otimes y), \end{aligned}$$

$$\phi(\mathcal{V}(a_{11}))(1 \otimes b_{12}) = \mathcal{V}(a_{11}) \otimes b_{12} = 1 \otimes a_{11}b_{12},$$

$$\flat(1 \otimes a_{21}^*) \cdot (1 \otimes b_{12}) = \langle 1 \otimes a_{21}^*, 1 \otimes b_{12} \rangle_A = a_{21}\mathcal{H}(1)b_{12} = a_{21}b_{12}.$$

This shows our claim. The following diagram commutes (it commutes on the level of  $C^*$ -algebras)

$$\begin{array}{ccc} K_*(\mathcal{V}(A)) & \xrightarrow{\iota_*} & K_*(A\mathcal{V}(1)A) \ . \\ (\iota_{11} \circ \mathcal{H})_* \downarrow & & \downarrow (\iota_{11} \circ \phi)_* \\ K_*(M_2(\mathcal{H}(A))) & \xrightarrow{\Phi_*} & K_*(D_X) \end{array}$$

However, since for any  $C^*$ -algebra  $B$  the homomorphisms  $\iota_{ii} : B \rightarrow M_2(B)$ ,  $i = 1, 2$ , induce the same mappings on the level of  $K$ -theory, the mappings  $(\iota_{11} \circ \mathcal{H})_*, (\iota_{22} \circ \mathcal{H})_* : K_*(\mathcal{V}(A)) \rightarrow K_*(M_2(\mathcal{H}(A)))$  coincide. Moreover, by the form of  $\Phi$  we see that  $\Phi \circ \iota_{22} \circ \mathcal{H} = \iota_{22} \circ \mathcal{H}$  on  $\mathcal{V}(A)$ . Hence

$$(\iota_{11} \circ \phi)_* \circ \iota_* = \Phi_* \circ (\iota_{11} \circ \mathcal{H})_* = \Phi_* \circ (\iota_{22} \circ \mathcal{H})_* = (\iota_{22} \circ \mathcal{H})_* \ . \quad \square$$

Using the above lemma we see that in sequence (1.5) we may replace  $K_*(A\langle X, X \rangle) = K_*(A\mathcal{V}(1)A)$  with  $K_*(\mathcal{V}(A))$  and then  $X_*$  turns into  $(\iota_{22})_*^{-1} \circ (\iota_{22} \circ \mathcal{H})_* = \mathcal{H}_*$ . Hence we get the following version of Pimsner–Voiculescu exact sequence, cf. [36, 42].

**Theorem 2.25.** *For any corner interaction  $(\mathcal{V}, \mathcal{H})$  we have the following exact sequence*

$$\begin{array}{ccccc} K_0(\mathcal{V}(A)) & \xrightarrow{\iota_* - \mathcal{H}_*} & K_0(A) & \xrightarrow{(i_A)_*} & K_0(C^*(A, \mathcal{V}, \mathcal{H})) \ . \\ \uparrow & & & & \downarrow \\ K_1(C^*(A, \mathcal{V}, \mathcal{H})) & \xleftarrow{(i_A)_*} & K_1(A) & \xleftarrow{\iota_* - \mathcal{H}_*} & K_1(\mathcal{V}(A)) \end{array}$$

### 3. Graph $C^*$ -Algebras via Interactions

In this section we introduce and study properties of graph interactions. We show that Theorem 2.20 applied to graph interactions is equivalent to the Cuntz–Krieger uniqueness theorem and its consequences. We use Theorem 2.25 to calculate  $K$ -theory for graph algebras straight from the dynamics on their AF-cores.

**3.1. Graph  $C^*$ -Algebra  $C^*(E)$  and its AF-Core**

Throughout we let  $E = (E^0, E^1, r, s)$  to be a fixed finite directed graph. Thus  $E^0$  is a set of vertices,  $E^1$  is a set of edges,  $r, s : E^1 \rightarrow E^0$  are range, source maps, and we assume that both sets  $E_0, E^1$  are finite. We write  $E^n, n > 0$ , for the set of paths  $\mu = \mu_1 \dots \mu_n, \mu_i \in E^1, r(\mu_i) = s(\mu_{i+1}), i = 1, \dots, n - 1$ , of length  $n$ . The maps  $r, s$  naturally extend to  $E^n$ , so that  $(E^0, E^n, s, r)$  is the graph, and  $s$  extends to the set  $E^\infty$  of infinite paths  $\mu = \mu_1 \mu_2 \mu_3 \dots$ . We also put  $s(v) = r(v) = v$  for  $v \in E^0$ . The elements of  $E^0_{sinks} := E^0 \setminus s(E^1)$  and respectively  $E^0_{sources} := E^0 \setminus r(E^1)$  are called sinks and sources. We also consider sets  $E^n_{sinks} = \{\mu \in E^n : r(\mu) \in E^0_{sinks}\}, n \in \mathbb{N}$ .

We adhere to conventions of [4, 29]. In our setting a Cuntz–Krieger  $E$ -family compose of non-zero pair-wise orthogonal projections  $\{P_v : v \in E^0\}$  and partial isometries  $\{S_e : e \in E^1\}$  satisfying

$$S_e^* S_e = P_{r(e)} \quad \text{and} \quad P_v = \sum_{e \in s^{-1}(v)} S_e S_e^* \quad \text{for all } v \in s(E^1), e \in E^1.$$

Having such a family we put  $S_\mu := S_{\mu_1} S_{\mu_2} \dots S_{\mu_n}$  for  $\mu = \mu_1 \dots \mu_n$  ( $S_\mu \neq 0 \Rightarrow \mu \in E^n$ ) and  $S_v := P_v$  for  $v \in E^0$ . The above Cuntz–Krieger relations extend to operators  $S_\mu$ , see [29, Lemma 1.1], as follows

$$S_\nu^* S_\mu = \begin{cases} S_{\mu'}, & \text{if } \mu = \nu \mu', \mu' \notin E^0, \\ S_{\nu'}, & \text{if } \nu = \mu \nu', \nu' \notin E^0, \\ 0 & \text{otherwise.} \end{cases}$$

In particular,  $C^*(\{P_v : v \in E^0\} \cup \{S_e : e \in E^1\})$  is the closure of the linear span of elements  $S_\mu S_\nu^*, \mu \in E^n, \nu \in E^m, n, m \in \mathbb{N}$ .

The graph  $C^*$ -algebra  $C^*(E)$  of  $E$  is a universal  $C^*$ -algebra generated by a universal Cuntz–Krieger  $E$ -family  $\{s_e : e \in E^1\}, \{p_v : v \in E^0\}$ . It is equipped with the natural circle gauge action  $\gamma : \mathbb{T} \rightarrow \text{Aut } C^*(E)$  established by relations  $\gamma_\lambda(p_v) = p_v, \gamma_\lambda(s_e) = \lambda s_e$ , for  $v \in E^0, e \in E^1, \lambda \in \mathbb{T}$ . The fixed point  $C^*$ -algebra for  $\gamma$  is called the core. It is an AF-algebra of the form

$$\mathcal{F}_E := \overline{\text{span}} \{s_\mu s_\nu^* : \mu, \nu \in E^n, n = 0, 1, \dots\}.$$

We recall the standard Bratteli diagram for  $\mathcal{F}_E$ . For each vertex  $v$  and  $N \in \mathbb{N}$  we set

$$\mathcal{F}_N(v) := \text{span} \{s_\mu s_\nu^* : \mu, \nu \in E^N, r(\mu) = r(\nu) = v\},$$

which is a simple  $I_n$  factor with  $n = |\{\mu \in E^N : r(\mu) = v\}|$  (if  $n = 0$  we put  $\mathcal{F}_N(v) := \{0\}$ ). The spaces

$$\mathcal{F}_N := \left( \bigoplus_{v \notin E^0_{sinks}} \mathcal{F}_N(v) \right) \oplus \left( \bigoplus_{w \in E^0_{sinks}} \bigoplus_{i=0}^N \mathcal{F}_i(w) \right), \quad N \in \mathbb{N},$$

form an increasing family of finite-dimensional algebras, cf. e.g. [4], and

$$\mathcal{F}_E = \overline{\bigcup_{N \in \mathbb{N}} \mathcal{F}_N}. \tag{3.1}$$

We denote by  $\Lambda(E)$  the corresponding Bratteli diagram for  $\mathcal{F}_E$ . If  $E$  has no sinks we can view  $\Lambda(E)$  as an infinite vertical concatenation of  $E$ : on the  $n$ -th level we have the vertices  $r(E^n), n \in \mathbb{N}$ , and multiplicities are given by

the number of edges with corresponding endings and sources. If  $E$  has sinks, one has to attach to every sink on each level an infinite tail, so on the  $n$ -th level of  $\Lambda(E)$  we have  $r(E^n) \cup \bigcup_{k=0}^{N-1} \{v^{(k)} : v \in r(E_{sinks}^k)\}$  and each  $v^{(k)}$  descends to  $v^{(k)}$  with multiplicity one. We adopt the convention that if  $V$  is a subset of  $E^0$  we treat it as a full subgraph of  $E$  and  $\Lambda(V)$  stands for the corresponding Bratteli diagram for  $\mathcal{F}_V$ . In particular, if  $V$  is *hereditary*, i.e.  $s(e) \in V \implies r(e) \in V$  for all  $e \in E^1$ , and *saturated*, i.e. every vertex which feeds into  $V$  and only  $V$  is in  $V$ , then the subdiagram  $\Lambda(V)$  of  $\Lambda(E)$  yields an ideal in  $\mathcal{F}_E$  which is naturally identified with  $\mathcal{F}_V$ . In general, viewing  $\Lambda(E)$  as an infinite directed graph the hereditary and saturated subgraphs (subdiagrams) of  $\Lambda(E)$  correspond to ideals in  $\mathcal{F}_E$ , see [5, 3.3].

### 3.2. Interactions Arising from Graphs

For each vertex  $v \in E^0$  we let  $n_v := |r^{-1}(v)|$  be the number of the edges received by  $v$ . We define an operator  $s$  in  $C^*(E)$  as the sum of the partial isometries  $\{s_e : e \in E^1\}$  "averaged" on the spaces corresponding to projections  $\{p_v : v \in r(E^0)\}$  that are not sources:

$$s := \sum_{e \in E^1} \frac{1}{\sqrt{n_{r(e)}}} s_e = \sum_{v \in r(E^1)} \frac{1}{\sqrt{n_v}} \sum_{e \in r^{-1}(v)} s_e. \tag{3.2}$$

Since  $s^*s = \sum_{v \in r(E^1)} p_v$  is a projection the operator  $s$  is a partial isometry. It is an isometry iff  $E$  has no sources. We use  $s$  to define

$$\mathcal{V}(a) := sas^*, \quad \mathcal{H}(a) := s^*as, \quad a \in C^*(E). \tag{3.3}$$

Plainly,  $(\mathcal{V}, \mathcal{H})$  is a corner interaction over  $C^*(E)$ . Moreover, one sees that  $\mathcal{V}$  and  $\mathcal{H}$  are unique bounded linear maps on  $C^*(E)$  satisfying the following formulas

$$\begin{aligned} \mathcal{V}(s_\mu s_\nu^*) &= \begin{cases} \frac{1}{\sqrt{n_{s(\mu)}n_{s(\nu)}}} \sum_{e,f \in E^1} s_{e\mu} s_{f\nu}^*, & n_{s(\mu)}n_{s(\nu)} \neq 0, \\ 0, & n_{s(\mu)}n_{s(\nu)} = 0, \end{cases} \tag{3.4} \\ \mathcal{H}(s_{e\mu} s_{f\nu}^*) &= \frac{1}{\sqrt{n_{s(\mu)}n_{s(\nu)}}} s_\mu s_\nu^*, \quad \mathcal{H}(p_v) = \begin{cases} \sum_{e \in s^{-1}(v)} \frac{p_{r(e)}}{n_{r(e)}}, & v \notin E_{sinks}^0, \\ 0, & v \in E_{sinks}^0, \end{cases} \end{aligned} \tag{3.5}$$

where  $\mu \in E^n, \nu \in E^m, n, m \in \mathbb{N}, e, f \in E^1, v \in E^0$ . It follows that  $\mathcal{V}$  and  $\mathcal{H}$  preserve the core algebra  $\mathcal{F}_E$ . Hence  $(\mathcal{V}, \mathcal{H})$  defines a corner interaction over  $\mathcal{F}_E$ . We note, however, that  $\mathcal{V}$  hardly ever preserves the canonical diagonal algebra  $\mathcal{D}_E := \overline{\text{span}} \{s_\mu s_\mu^* : \mu \in E^n, n \in \mathbb{N}\} \subset \mathcal{F}_E$ .

**Definition 3.1.** We call the pair  $(\mathcal{V}, \mathcal{H})$  of continuous linear maps on  $\mathcal{F}_E$  satisfying (3.4), (3.5) a (*corner*) *interaction of the graph  $E$*  or simply a *graph interaction*.

The following statement is one of the facts justifying the above definition.

**Proposition 3.2.** *We have a one-to-one correspondence between Cuntz–Krieger  $E$ -families  $\{P_v : v \in E^0\}$ ,  $\{S_e : e \in E^1\}$  for  $E$  and faithful covariant representations  $(\pi, S)$  of the graph interaction  $(\mathcal{V}, \mathcal{H})$ . It is given by the relations*

$$S = \sum_{e \in E^1} \frac{1}{\sqrt{n_{r(e)}}} S_e, \quad P_v = \pi(p_v), \quad S_e = \sqrt{n_{r(e)}} \pi(s_e s_e^*) S.$$

*In particular, we have a gauge-invariant isomorphism  $C^*(E) \cong C^*(\mathcal{F}_E, \mathcal{V}, \mathcal{H})$ .*

*Proof.* A Cuntz–Krieger  $E$ -family  $\{P_v : v \in E^0\}$ ,  $\{S_e : e \in E^1\}$  yields a representation  $\pi$  of  $C^*(E)$  which is well known to be faithful on  $\mathcal{F}_E$ . By the definition of  $(\mathcal{V}, \mathcal{H})$  the pair  $(\pi|_{\mathcal{F}_E}, S)$  where  $S := \pi(s) = \sum_{e \in E^1} \frac{1}{\sqrt{n_{r(e)}}} S_e$  is a covariant representation of  $(\mathcal{V}, \mathcal{H})$ . Conversely, let  $(\pi, S)$  be a faithful representation of  $(\mathcal{V}, \mathcal{H})$  and put  $P_v := \pi(p_v)$  and  $S_e := \sqrt{n_{r(e)}} \pi(s_e s_e^*) S$ . We claim that  $\{P_v : v \in E^0\}$ ,  $\{S_e : e \in E^1\}$  is a Cuntz–Krieger  $E$ -family such that  $S = \sum_{e \in E^1} \frac{S_e}{\sqrt{n_{r(e)}}}$ . Indeed, for  $e \in E^1$  we have

$$S_e^* S_e = n_{r(e)} \pi(p_{r(e)}) \pi(\mathcal{H}(s_e s_e^*)) \pi(p_{r(e)}) = \pi(p_{r(e)}) = P_{r(v)},$$

and for  $v \in s(E^1)$  we have

$$\begin{aligned} \sum_{e \in s^{-1}(v)} S_e S_e^* &= \sum_{e \in s^{-1}(v)} n_{r(e)} \pi(s_e s_e^*) \pi(\mathcal{V}(1)) \pi(s_e s_e^*) \\ &= \sum_{e \in s^{-1}(v), e_1, e_2 \in E^1} \frac{n_{r(e)}}{\sqrt{n_{r(e_1)} n_{r(e_2)}}} \pi(s_e s_e^* (s_{e_1} s_{e_2}^*) s_e s_e^*) \\ &= \sum_{e \in s^{-1}(v)} \pi(s_e s_e^*) = \pi(p_v) = P_v. \end{aligned}$$

Now note that  $S^* S = \pi(\mathcal{H}(1)) = \sum_{v \in r(E^1)} \pi(p_v)$  and thus  $S = \sum_{e \in E^1} S \pi(p_v)$ . Moreover, for each  $v \in r(E^1)$  we have

$$\begin{aligned} \left( \sum_{e \in r^{-1}(v)} \pi(s_e s_e^*) \right) S \pi(p_v) S^* &= \sum_{e \in r^{-1}(v)} \pi(s_e s_e^* \mathcal{V}(p_v)) \\ &= \sum_{e, e_1, e_2 \in r^{-1}(v)} \frac{\pi(s_e s_e^* s_{e_1} s_{e_2}^*)}{n_v} \\ &= \sum_{e_1, e_2 \in r^{-1}(v)} \frac{\pi(s_{e_1} s_{e_2}^*)}{n_v} = \pi(\mathcal{V}(p_v)) \\ &= S \pi(p_v) S^*. \end{aligned}$$

Hence the final space of the partial isometry  $S \pi(p_v)$  decomposes into the orthogonal sum of ranges of the projections  $\pi(s_e s_e^*)$ ,  $e \in r^{-1}(v)$ , and consequently

$$\sum_{e \in E^1} \frac{S_e}{\sqrt{n_{r(e)}}} = \sum_{e \in E^1} \pi(s_e s_e^*) S \pi(p_{r(e)}) = \sum_{v \in E^0} \sum_{e \in r^{-1}(v)} \pi(s_e s_e^*) S \pi(p_v) = S. \quad \square$$

*Remark 3.3.* If  $E$  has no sources then  $s$  is an isometry and  $\mathcal{V}$  is an injective endomorphism with hereditary range. In this case  $C^*(E)$  coincides with various crossed products by endomorphisms that involve isometries, cf. [2, 11, 35]. In particular, Proposition 3.2 has a nontrivial intersection with [19, Theorem 5.2] proved for locally finite graphs without sources.

*Remark 3.4.* The canonical completely positive map  $\phi_E : C^*(E) \rightarrow C^*(E)$  is given by the formula

$$\phi_E(x) = \sum_{e \in E^1} s_e x s_e^*.$$

This map (unlike  $\mathcal{V}$  but like  $\mathcal{H}$ ) always preserves both  $\mathcal{F}_E$  and  $\mathcal{D}_E$  and the pair  $(\phi_E, \mathcal{H})$  is a  $C^*$ -dynamical on  $\mathcal{D}_E$ , cf. Proposition 3.5 below. Moreover, if  $E$  has no sinks the same relations as in Proposition 3.2 yield an isomorphism between  $C^*(E)$  and the Exel’s crossed product  $\mathcal{D}_E \rtimes_{(\phi_E, \mathcal{H})} \mathbb{N}$ , see [7, Theorem 5.1]. The advantage of  $\mathcal{D}_E \rtimes_{(\phi_E, \mathcal{H})} \mathbb{N}$  over  $C^*(\mathcal{F}_E, \mathcal{V}, \mathcal{H})$  is that it starts from a commutative  $C^*$ -algebra  $\mathcal{D}_E$ . The disadvantages are that the dynamics in  $(\phi_E, \mathcal{H})$  is irreversible and involves two mappings (at least implicitly, see [26]), while in essence  $(\mathcal{V}, \mathcal{H})$  is a single map (recall Proposition 2.9) possessing a natural generalized inverse.

A natural question to ask is when the graph interaction  $(\mathcal{V}, \mathcal{H})$  is a  $C^*$ -dynamical system. It is somewhat surprising that this holds only if  $(\mathcal{V}, \mathcal{H})$  is a part of a group interaction. We take up the rest of this subsection to clarify this issue in detail. To this end we will use a partially-stochastic matrix  $P = [p_{v,w}]$  arising from the adjacency matrix  $A_E = [A_E(v, w)]_{v,w \in E^0}$  of the graph  $E$ . Namely, we let

$$p_{v,w} := \begin{cases} \frac{A_E(v,w)}{n_w}, & A_E(v, w) \neq 0, \\ 0, & A_E(v, w) = 0, \end{cases} \tag{3.6}$$

where  $A_E(v, w) = |\{e \in E^1 : s(e) = v, r(e) = w\}|$ . By a *partially-stochastic matrix* we mean a non-negative matrix in which each non-zero column sums up to one.

**Proposition 3.5.** *Let  $s$  be the operator given by (3.2) and let  $n \geq 1$ . The following conditions are equivalent:*

- (i)  $(\mathcal{V}^n, \mathcal{H}^n)$  is an interaction over  $\mathcal{F}_E$ ,
- (ii)  $(\phi_E^n, \mathcal{H}^n)$  is a  $C^*$ -dynamical system on  $\mathcal{D}_E$ ,
- (iii) operator  $s^n$  is a partial isometry,
- (iv)  $n$ th power of the matrix  $P = \{p_{v,w}\}_{v,w \in E^0}$  is partially-stochastic,
- (v) for any  $\mu \in E^n$  and  $\nu \in E^k$ ,  $k < n$ , such that  $r(\mu) = r(\nu)$  we have  $s(\nu) \notin E_{sources}^0$ .

*Proof.* (i)  $\Leftrightarrow$  (iii). As  $\mathcal{V}^n(\cdot) = s^n(\cdot)s^{*n}$  and  $\mathcal{H}^n(\cdot) = s^{*n}(\cdot)s^n$  one readily checks that (iii) implies (i), and if we assume (i) then  $s^n$  is a partial isometry because  $\mathcal{H}^n(1)$  is a projection by Lemma 2.5.

(iii)  $\Leftrightarrow$  (iv). Operator  $s^n$  is a partial isometry iff  $\mathcal{H}^n(1)$  is a projection. Since  $\mathcal{H}(p_\nu) = \sum_{w \in E^0} p_{\nu,w} p_w$ , cf. (3.5), we get

$$\mathcal{H}^n(1) = \sum_{v_0, \dots, v_n \in E^0} p_{v_0, v_1} \cdot p_{v_1, v_2} \cdot \dots \cdot p_{v_{n-1}, v_n} p_{v_n} = \sum_{v, w \in E^0} p_{v, w}^{(n)} p_w$$

where  $P^n = \{p_{v, w}^{(n)}\}_{v, w \in E^0}$  stands for the  $n$ th power of  $P$ . By the orthogonality of projections  $p_w$ , it follows that  $\mathcal{H}^n(1)$  is a projection iff  $\sum_{v \in E^0} p_{v, w}^{(n)} \in \{0, 1\}$  for all  $w \in E^0$ , that is iff  $P^n$  is partially-stochastic.

(ii)  $\Leftrightarrow$  (iv). We know that  $\phi_E : \mathcal{D}_E \rightarrow \mathcal{D}_E$  is an endomorphism and  $\mathcal{H}$  is its transfer operator. Moreover, it is a straight forward fact that an iteration of an endomorphism and its transfer operator gives again an endomorphism and its transfer operator. Thus  $(\phi_E^n, \mathcal{H}^n)$  is a  $C^*$ -dynamical system iff the transfer operator  $\mathcal{H}^n$  is regular, that is iff  $\phi_E^n(\mathcal{H}^n(1)) = \phi_E^n(1)$ . However, as

$$\phi_E^n(\mathcal{H}^n(1)) = \sum_{v, w \in E^0} p_{v, w}^{(n)} \phi_E^n(p_w) = \sum_{v \in E^0, \mu \in E^n} p_{v, r(\mu)}^{(n)} s_\mu s_\mu^*$$

and  $\phi_E^n(1) = \sum_{\mu \in E^n} s_\mu s_\mu^*$  we see that  $\phi_E^n(\mathcal{H}^n(1)) = \phi_E^n(1)$  if and only if  $P^n = \{p_{v, w}^{(n)}\}_{v, w \in E^0}$  is partially-stochastic.

(iv)  $\Rightarrow$  (v). Assume that (v) is not true, that is let  $\mu \in E^n$  and  $\nu \in E^k$ ,  $k < n$ , be such that  $r(\mu) = r(\nu)$  and  $s(\nu) \in E_{sources}^0$ . Notice that the condition  $\sum_{v \in E^0} p_{v, w}^{(n)} > 0$  is equivalent to existence of  $\eta \in E^n$  such that  $w = r(\eta)$ . Hence putting  $w := r(\mu) = r(\nu)$  and  $v_0 := s(\nu)$  we have  $\sum_{v \in E^0} p_{v, w}^{(n)} > 0$  and  $p_{v_0, w}^{(k)} > 0$ . Then  $\sum_{v \in E^0} p_{v, v_0}^{(n-k)} = 0$  (because  $v_0 \in E_{sources}^0$ ) and therefore

$$0 < \sum_{v \in E^0} p_{v, w}^{(n)} = \sum_{\substack{v \in E^0, \\ v_{n-k} \in E^0 \setminus \{v_0\}}} p_{v, v_{n-k}}^{(n-k)} p_{v_{n-k}, w}^{(k)} \leq \sum_{v_{n-k} \in E^0 \setminus \{v_0\}} p_{v_{n-k}, w}^{(k)} < 1,$$

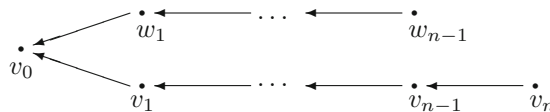
that is  $P^n$  is not partially-stochastic.

(v)  $\Rightarrow$  (iv). Suppose that  $\sum_{v \in E^0} p_{v, w}^{(n)} > 0$ . By our assumption for each  $0 < k < n$  the condition  $p_{v_k, w}^{(n-k)} \neq 0$  implies that  $v_k \notin E_{sources}^0$ . However, relation  $v_k \notin E_{sources}^0$  is equivalent to  $\sum_{v_{k-1} \in E^0} p_{v_{k-1}, v_k}^{(1)} = 1$  (because  $P$  is partially-stochastic). Therefore proceeding inductively for  $k = 1, 2, 3, \dots, n-1$  we get

$$\sum_{v \in E^0} p_{v, w}^{(n)} = \sum_{v_0, v_1 \in E^0} p_{v_0, v_1}^{(1)} p_{v_1, w}^{(n-1)} = \sum_{v_1 \in E^0} p_{v_1, w}^{(n-1)} = \dots = \sum_{v_{n-1} \in E^0} p_{v_{n-1}, w}^{(1)} = 1.$$

□

*Example.* It follows from Proposition 3.5 that if we consider a graph interaction  $(\mathcal{V}, \mathcal{H})$  arising from the following graph



then  $(\mathcal{V}, \mathcal{H})$  has the property that its  $k$ th power  $(\mathcal{V}^k, \mathcal{H}^k)$ , for  $k > 1$ , is an interaction unless  $k = n$ . Hence by considering a disjoint sum of graphs of the above form one can obtain a graph interaction with an arbitrary finite distribution of powers not being interactions.

In our specific situation of graph interactions we may prolong the list of equivalent conditions in Proposition 2.8 as follows.

**Corollary 3.6.** *Let  $(\mathcal{V}, \mathcal{H})$  be the interaction associated to the graph  $E$ . The following conditions are equivalent:*

- (i)  $(\mathcal{V}, \mathcal{H})$  is a  $C^*$ -dynamical system,
- (ii)  $(\mathcal{V}^n, \mathcal{H}^n)$  is an interaction for all  $n \in \mathbb{N}$ ,
- (iii)  $(\phi_E^n, \mathcal{H}^n)$  is a  $C^*$ -dynamical system for all  $n \in \mathbb{N}$ ,
- (iv) operator  $s$  given by (3.2) is a power partial isometry,
- (v) every power of the matrix  $P = \{p_{v,w}\}_{v,w \in E^0}$  is partially-stochastic,
- (vi) every two paths in  $E$  that have the same length and the same ending either both starts in sources or not in sources.

*Proof.* Item (vi) holds if and only if item (v) in Proposition 3.5 holds for all  $n \in \mathbb{N}$ . Hence by Proposition 3.5 we get the equivalence between all the items from (ii) to (vi) in the present assertion. Furthermore, we recall that  $\mathcal{H}(1) = s^*s = \sum_{v \in r(E^1)} p_v$ , and item (i) is equivalent to  $\mathcal{H}(1)$  being a central element in  $\mathcal{F}_E$ , see Proposition 2.8. Hence the equivalence (i)  $\Leftrightarrow$  (vi) follows from the relations

$$\mathcal{H}(1)s_\mu s_\nu^* = \begin{cases} 0, & \text{if } s(\mu) \notin r(E^1) \\ s_\mu s_\nu^*, & \text{otherwise} \end{cases}, \quad s_\mu s_\nu^* \mathcal{H}(1) = \begin{cases} 0, & \text{if } s(\nu) \notin r(E^1) \\ s_\mu s_\nu^*, & \text{otherwise} \end{cases},$$

which hold for all  $\mu, \nu \in E^n, n \in \mathbb{N}$ . □

A natural question to ask is when  $\mathcal{H}$  is multiplicative. We rush to say that it is hardly the case.

**Proposition 3.7.** *The pair  $(\mathcal{H}, \mathcal{V})$ , where  $(\mathcal{V}, \mathcal{H})$  is the interaction of  $E$ , is a  $C^*$ -dynamical system if and only if the mapping  $r : E^1 \rightarrow E^0$  is injective.*

*Proof.* By Proposition 2.8 multiplicativity of  $\mathcal{H}$  is equivalent to  $\mathcal{V}(1)$  being a central element in  $\mathcal{F}_E$ . If  $r : E^1 \rightarrow E^0$  is injective, then  $\mathcal{F}_E = \mathcal{D}_E$  is commutative and  $(\mathcal{H}, \mathcal{V})$  is a  $C^*$ -dynamical system because  $\mathcal{V}(1) \in \mathcal{F}_E$ . Conversely, let us assume that the projection  $\mathcal{V}(1) = ss^* = \sum_{v \in r(E^1)} \frac{1}{n_v} \sum_{e,f \in r^{-1}(v)} s_e s_f^*$  is central in  $\mathcal{F}_E$  and let  $g, h \in E^1$  be such that  $r(g) = r(h) = v$ . Since

$$\mathcal{V}(1)s_g s_h^* = \frac{1}{n_v} \sum_{e \in r^{-1}(v)} s_e s_h^*, \quad \text{and} \quad s_g s_h^* \mathcal{V}(1) = \frac{1}{n_v} \sum_{f \in r^{-1}(v)} s_g s_f^*$$

we have  $\sum_{e \in r^{-1}(v)} s_e s_h^* = \sum_{f \in r^{-1}(v)} s_g s_f^*$ , which implies  $g = h$ . Hence  $r : E^1 \rightarrow E^0$  is injective. □

### 3.3. Dynamical Systems Dual to Graph Interactions

Let  $(\mathcal{V}, \mathcal{H})$  be the interaction of the graph  $E$ . We obtain a satisfactory picture of the system dual to  $(\mathcal{V}, \mathcal{H})$  using a Markov shift  $(\Omega_E, \sigma_E)$  dual to the commutative system  $(\mathcal{D}_E, \phi_E)$ . Namely, we put  $\Omega_E = \bigcup_{N=0}^\infty E_{sinks}^N \cup E^\infty$  and let  $\sigma_E : \Omega_E \setminus E_{sinks}^0 \rightarrow \{\mu \in \Omega_E : s(\mu) \notin E_{sources}^0\}$  be the shift defined by the formula

$$\sigma_E(\mu) = \begin{cases} \mu_2 \mu_3 \dots & \text{if } \mu = \mu_1 \mu_2 \dots \in \bigcup_{N=2}^\infty E_{sinks}^N \cup E^\infty \\ r(\mu) & \text{if } \mu \in E_{sinks}^1. \end{cases}$$



There is a natural ‘product’ topology on  $\Omega_E$  with the basis formed by the cylinder sets  $U_\nu = \{\nu\mu : \nu\mu \in \Omega_E\}$ ,  $\nu \in E^n, n \in \mathbb{N}$ . Equipped with this topology  $\Omega_E$  is a compact Hausdorff space and  $\sigma_E$  is a local homeomorphism whose both domain and codomain are clopen. Moreover, the standard argument, cf. e.g. [20, Lemma 3.2], shows that  $s_\nu s_\nu^* \mapsto \chi_{U_\nu}$ ,  $\nu \in E^n, n \in \mathbb{N}$ , establishes an isomorphism  $\mathcal{D}_E \cong C(\Omega_E)$  which intertwines  $\phi_E : \mathcal{D}_E \rightarrow \mathcal{D}_E$  with the operator of composition with  $\sigma_E$ .

Let us consider the relation of ‘eventual equality’ defined on  $\Omega_E$  as follows:

$$\mu \sim \nu \stackrel{def}{\iff} \begin{cases} \nu, \mu \in E^\infty \text{ and } \mu_N \mu_{N+1} \dots = \nu_N \nu_{N+1} \dots & \text{for some } N \in \mathbb{N}, \\ \nu, \mu \in E_{sinks}^N & \text{for some } N \in \mathbb{N} \text{ and } r(\mu_N) = r(\nu_N). \end{cases}$$

Plainly,  $\sim$  is an equivalence relation. We denote by  $[\mu]$  the equivalence class of  $\mu \in \Omega_E$ , and view  $\Omega_E / \sim$  as a topological space equipped with the quotient topology.

**Lemma 3.8.** *The quotient map  $q : \Omega_E \mapsto \Omega_E / \sim$  is open and the sets*

$$U_{v,n} := \{[\mu] : \exists \eta \in E^k, k \in \mathbb{N} \ s(\eta) = v, r(\eta) = \mu_{n+k}\}, \quad v \in r(E^n), n \in \mathbb{N}, \tag{3.7}$$

*form a basis for the quotient topology of  $\Omega_E / \sim$ . Moreover, the formula*

$$[\sigma_E][\mu] := [\sigma_E(\mu)] \tag{3.8}$$

*defines a partial homeomorphism of  $\Omega_E / \sim$  with natural domain and codomain:*

$$\begin{aligned} \{[\mu] : \mu \in \Omega_E \setminus E_{sinks}^0\} &= \bigcup_{v \in E^0 \setminus E_{sinks}^0} U_{v,0}, \\ \{[\mu] \in \Omega_E : s(\mu) \notin E_{sources}^0\} &= \bigcup_{v \in E^0 \setminus E_{sources}^0} U_{v,0}. \end{aligned}$$

*Proof.* A moment of thought yields that if  $\nu \in E^n$  is such that  $r(\nu) = v$ , then  $q(U_\nu) = U_{v,n}$ . In particular, one sees that

$$\begin{aligned} q^{-1}(U_{v,n}) &= \{\mu \in \Omega_E : \exists \eta \in E^k, k \in \mathbb{N} \ s(\eta) = v, r(\eta) = \mu_{n+k}\} \\ &= \bigcup_{k \in \mathbb{N}} \bigcup_{\substack{\eta \in E^k, \\ s(\eta) = v}} \bigcup_{\substack{\nu \in E^{n+k} \\ r(\nu) = r(\eta)}} U_\nu, \end{aligned}$$

which means that  $U_{v,n}$  is open in  $\Omega_E / \sim$ . We conclude that (3.7) defines a basis for the topology of  $\Omega_E / \sim$  and  $q$  is an open map.

Now, it is straightforward to check that (3.8) gives a well defined mapping whose domain and codomain are open sets of the form described in the assertion. The map  $[\sigma_E]$  is invertible as for  $\mu \in \Omega_E$  such that  $s(\mu) \notin E_{sources}^0$  its inverse can be described by the formula

$$[\sigma_E]^{-1}[\mu] = [e\mu] \quad \text{for an arbitrary edge } e \in E^1 \text{ such that } r(e) = s(\mu),$$

where  $e\mu := e$  when  $\mu \in E_{sinks}^0$  is a vertex. Since  $[\sigma_E](U_{v,n+1}) = U_{v,n}$  and  $[\sigma_E]^{-1}(U_{v,n}) = U_{v,n+1}$  for  $v \in E^n, n \in \mathbb{N}$ , we see that  $[\sigma_E]$  is a partial homeomorphism. □

We show that the quotient partial reversible dynamical system  $(\Omega_E / \sim, [\sigma_E])$  embeds as a dense subsystem into  $(\widehat{\mathcal{F}}_E, \widehat{\mathcal{V}})$ . Under this embedding the relation  $\sim$  coincides with the unitary equivalence of GNS-representations associated to pure extensions of the pure states of  $\mathcal{D}_E = C(\Omega_E)$ . More precisely, for any path  $\mu \in \Omega_E$  the formula

$$\omega_\mu(s_\nu s_\eta^*) = \begin{cases} 1 & \nu = \eta = \mu_1 \cdots \mu_n, \text{ for } \nu, \eta \in E^n, n \in \mathbb{N}, \\ 0 & \text{otherwise} \end{cases} \tag{3.9}$$

determines a pure state  $\omega_\mu : \mathcal{F}_E \rightarrow \mathbb{C}$  (a pure extension of the point evaluation  $\delta_\mu$  acting on  $\mathcal{D}_E = C(\Omega_E)$ ). Indeed, the functional  $\omega_\mu$  is a pure state on each  $\mathcal{F}_k, k \in \mathbb{N}$ , and thus it is also a pure state on  $\mathcal{F}_E = \overline{\bigcup_{k \in \mathbb{N}} \mathcal{F}_k}$ , cf. e.g. [5, 4.16]. We denote by  $\pi_\mu$  the GNS-representation associated to  $\omega_\mu$  and take up the rest of the subsection to prove the following

**Theorem 3.9.** (Partial homeomorphism dual to a graph interaction) *Under the above notation  $[\mu] \mapsto \pi_\mu$  is a topological embedding of  $\Omega_E / \sim$  as a dense subset into  $\widehat{\mathcal{F}}_E$ . This embedding intertwines  $[\sigma_E]$  and  $\widehat{\mathcal{V}}$ . Accordingly, the space  $\widehat{\mathcal{F}}_E$  admits the following decomposition into disjoint sets*

$$\widehat{\mathcal{F}}_E = \bigcup_{N=0}^{\infty} \widehat{G}_N \cup \widehat{G}_\infty$$

where the sets  $\widehat{G}_N = \{\pi_\mu : [\mu] \in E_{sinks}^N / \sim\}$  are open discrete and  $\widehat{G}_\infty = \overline{\{\pi_\mu : [\mu] \in E^\infty / \sim\}}$  is a closed subset of  $\widehat{\mathcal{F}}_E$ . The set

$$\Delta = \widehat{\mathcal{F}}_E \setminus \widehat{G}_0$$

is the domain of  $\widehat{\mathcal{V}}$ , and  $\widehat{\mathcal{V}}$  is uniquely determined by the formula

$$\widehat{\mathcal{V}}(\pi_\mu) = \pi_{\sigma_E(\mu)}, \quad \mu \in \Omega_E \setminus E_{sinks}^0.$$

In particular,  $\pi_\mu \in \widehat{\mathcal{V}}(\Delta)$ , for  $\mu \in E_{sinks}^N$ , iff there is  $\nu \in E_{sinks}^{N+1}$  such that  $r(\mu) = r(\nu)$ , and then  $\widehat{\mathcal{H}}(\pi_\mu) = \pi_\nu$ . Similarly,  $\pi_\mu \in \widehat{\mathcal{V}}(\Delta)$ , for  $\mu \in E^\infty$ , iff there is  $\nu \sim \mu$  such that  $s(\nu)$  is not a source, and then for any  $\nu_0 \in E^1$  such that  $\nu_0 \nu_1 \nu_2 \dots \in E^\infty$  we have  $\widehat{\mathcal{H}}(\pi_\mu) = \pi_{\nu_0 \nu_1 \nu_2 \dots}$ .

*Remark 3.10.* One may verify that if we put

$$A := \text{span} (\{p_v : v \in E_{sinks}^0\} \cup \{s_e s_e^* : e \in E^1\}) \cong \mathbb{C}^{|E_{sinks}^0| + |E^1|},$$

then  $\mathcal{H}$  preserves  $A$  and the smallest  $C^*$ -algebra containing  $A$  and invariant under  $\mathcal{V}$  is  $\mathcal{F}_E$ . In this sense  $\mathcal{H} : \mathcal{F}_E \rightarrow \mathcal{F}_E$  is a natural dilation of the positive linear map  $\mathcal{H} : A \rightarrow A$ . This explains the similarity of assertions in Theorem 3.9 and in [24, Theorem 3.5]; both of these results describe dual partial homeomorphisms obtained in the process of dilations. The essential difference is that a dilation of a multiplicative map on a commutative algebra always leads a commutative  $C^*$ -algebra, cf. [24, 27], while a stochastic factor manifested by a lack of multiplicativity of the initial mapping inevitably leads to noncommutative objects after a dilation. Significantly, our dual picture of the graph interaction ‘collapses’ to the non Hausdorff quotient similar to that of Penrose tilings [8, 3.2].

We start by noting that the infinite direct sum  $\bigoplus_{N=0}^{\infty} \bigoplus_{w \in E_{sinks}^0} \mathcal{F}_N(w)$  yields an ideal  $I_{sinks}$  in  $\mathcal{F}_E$  generated by the projections  $p_w, w \in E_{sinks}^0$ . We rewrite it in the form

$$I_{sinks} = \bigoplus_{N \in \mathbb{N}} G_N, \quad \text{where } G_N := \left( \bigoplus_{w \in E_{sinks}^0} \mathcal{F}_N(w) \right).$$

Plainly,  $\mathcal{F}_N(w) \neq \{0\}$  for  $w \in E_{sinks}^0$  iff there is  $\mu \in E_{sinks}^N$  such that  $r(\mu) = w$  and then (since  $\mathcal{F}_N(w)$  is a finite factor)  $\pi_\mu$  is a unique up to unitary equivalence irreducible representation of  $\mathcal{F}_E$  such that  $\ker \pi_\mu \cap \mathcal{F}_N(w) = \{0\}$ . Consequently, we see that

$$\widehat{I}_{sinks} = \bigcup_{N=0}^{\infty} \widehat{G}_N \ni \pi_\mu \longmapsto [\mu] \in \bigcup_{N=0}^{\infty} E_{sinks}^N / \sim$$

establishes a homeomorphism between the corresponding discrete spaces. The complement of  $\widehat{I}_{sinks} = \bigcup_{N=0}^{\infty} \widehat{G}_N$  in  $\widehat{\mathcal{F}_E}$  is a closed set which we identify in a usual way with the spectrum of the quotient algebra

$$G_\infty := \mathcal{F}_E / I_{sinks}.$$

We will describe a dense subset of  $\widehat{G}_\infty$  exploiting the fact that states  $\omega_\mu$  arising from  $\mu \in E^\infty$  can be viewed as analogs of Glimm’s product states for UHF-algebras, cf. e.g. [37, 6.5].

**Lemma 3.11.** *For infinite paths  $\mu, \nu \in E^\infty$  the representations  $\pi_\mu$  and  $\pi_\nu$  are unitarily equivalent if and only if  $\mu \sim \nu$ . In particular,  $[\mu] \mapsto \pi_\mu$  is a well defined embedding of  $E^\infty / \sim$  into  $\widehat{G}_\infty$ .*

*Proof.* We mimic the proof of the corresponding result for UHF-algebras, cf. [37, 6.5.6]. Note that if  $(\mu_{N+1}, \mu_{N+2}, \dots) = (\nu_{N+1}, \nu_{N+2}, \dots)$ , then both  $s_{\mu_1 \dots \mu_N} s_{\mu_1 \dots \mu_N}^*$  and  $s_{\nu_1 \dots \nu_N} s_{\nu_1 \dots \nu_N}^*$  are in  $F_N(v)$  where  $v = r(\mu_N)$  and since  $F_N(v) \cong M_n(\mathbb{C})$  there is a unitary  $u \in F_N(v)$  such that  $\omega_\mu(a) = \omega_\nu(u^* a u)$  for  $a \in F_N(v)$ . Then automatically  $\omega_\mu(a) = \omega_\nu(u^* a u)$  for all  $a \in \mathcal{F}_E$  and hence  $\pi_\mu \cong \pi_\nu$ . Conversely, suppose that  $\pi_\mu \cong \pi_\nu$ , then, cf. [37, 3.13.4], there is a unitary  $u \in \mathcal{F}_E$  such that  $\omega_\mu(a) = \omega_\nu(u^* a u)$  for all  $a \in \mathcal{F}_E$ . For sufficiently large  $n$  there is  $x \in \mathcal{F}_n$  with  $\|u - x\| < \frac{1}{2}$  and  $\|x\| \leq 1$ . To get the contradiction we assume that  $\mu_k \neq \nu_k$  for some  $k > n$ . The element  $a := s_{\mu_1 \dots \mu_k} s_{\mu_1 \dots \mu_k}^* \in \mathcal{F}_k$  commutes with all the elements from  $\mathcal{F}_n$ . Indeed, if  $b = s_{\alpha f} s_{\beta f}^* \in \mathcal{F}_n, \alpha, \beta \in E^n$ , then either  $s_{\alpha f} s_{\beta f}^* = 0$  for all  $f \in E^{k-n}$  and then  $ab = ba = 0$  or  $b = \sum_{f \in E^{k-n}} s_{\alpha f} s_{\beta f}^*$  and then  $ab = s_{\alpha \mu_{n+1} \dots \mu_k} s_{\beta \mu_{n+1} \dots \mu_k}^* = ba$ . From this it also follows that  $\omega_\nu(ab) = 0$  for all  $b \in \mathcal{F}_n$ . Accordingly,  $\omega_\nu(x^* a x) = \omega_\nu(a x^* x) = 0$  and since  $\|(u^* - x^*) a u\| = \|x^* a (u - x)\| < 1/2$  we get

$$1 > \omega_\nu((u^* - x^*) a u) + \omega_\nu(x^* a (u - x)) = \omega_\nu(u^* a u) - \omega_\nu(x^* a x) = \omega_\mu(a) = 1,$$

an absurd. □

*Remark 3.12.* The  $C^*$ -algebra  $G_\infty$  is a graph algebra arising from a graph which has no sinks. Indeed, the saturation  $\overline{E^0}_{sinks}$  of  $E^0_{sinks}$  (the minimal

saturated set containing  $E^0_{sinks}$ ) is the hereditary and saturated set corresponding to the ideal  $I_{sinks}$  in  $\mathcal{F}_E$ . Hence  $I_{sinks} = \mathcal{F}_{\overline{E^0_{sinks}}}$  and

$$G_\infty \cong \mathcal{F}_{E^0_{sinkless}} \quad \text{where } E^0_{sinkless} := E^0 \setminus \overline{E^0_{sinks}}.$$

Let us now treat  $\mu \in E^\infty$  as the full subdiagram of the Bratteli diagram  $\Lambda(E)$  where the only vertex on the  $n$ th level is  $r(\mu_n)$ . Similarly, we treat  $\mu \in E^N_{sinks}$  as the full subdiagram of  $\Lambda(E)$  where on the  $n$ th level for  $n \leq N$  is  $r(\mu_n)$  and for  $n > N$  is  $r(\mu)^{(N)}$ , cf. notation in Sect. 3.1. For any  $\mu \in \Omega_E$  we denote by  $W(\mu)$  the full subdiagram of  $\Lambda(E)$  consisting of all ancestors of the base vertices of  $\mu \subset \Lambda(E)$ .

**Lemma 3.13.** *For any  $\mu \in \Omega_E$  the Bratteli subdiagram  $\Lambda(\ker \pi_\mu)$  of  $\Lambda(E)$  corresponding to  $\ker \pi_\mu$  is  $\Lambda(E) \setminus W(\mu)$ .*

*Proof.* The assertion follows immediately from the form of primitive ideal subdiagrams, see [5, 3.8], definition (3.9) of  $\omega_\mu$  and the fact that  $\ker \pi_\mu$  is the largest ideal contained in  $\ker \omega_\mu$ . □

**Lemma 3.14.** *The mapping  $[\mu] \mapsto \pi_\mu \in \widehat{\mathcal{F}}_E$  is a homeomorphism from  $\Omega_E / \sim$  onto its image.*

*Proof.* We already know that  $[\mu] \mapsto \pi_\mu$  is injective and restricts to homeomorphism between discrete spaces  $(\Omega_E \setminus E^\infty) / \sim$  and  $\widehat{\mathcal{F}}_E \setminus \widehat{G}_\infty$ . Hence it suffices to prove that  $[\mu] \mapsto \pi_\mu$  is continuous and open when considered as a mapping from  $E^\infty / \sim$  onto  $\{\pi_\mu : [\mu] \in E^\infty / \sim\} \subset \widehat{G}_\infty$ . To this end, we may assume that  $E$  has no sinks, cf. Remark 3.12. Suppose then that  $E$  has no sinks.

Any open set in  $\widehat{\mathcal{F}}_E$  is of the form  $\widehat{J} = \{\pi \in \widehat{\mathcal{F}}_E : \ker \pi \not\supseteq J\} = \{\pi \in \widehat{\mathcal{F}}_E : \Lambda(J) \setminus \Lambda(\ker \pi) \neq \emptyset\}$  where  $J$  is an ideal in  $\mathcal{F}_E$  or equivalently  $\Lambda(J)$  is a hereditary and saturated subdiagram of  $\Lambda(E)$ . It follows that if we denote by  $\Lambda_{v,n}$  the smallest hereditary and saturated subdiagram of  $\Lambda(E)$  which on the  $n$ th level contains vertex  $v$ , then the sets

$$\widehat{J}_{v,n} := \left\{ \pi \in \widehat{\mathcal{F}}_E : \Lambda_{v,n} \setminus \Lambda(\ker \pi) \neq \emptyset \right\}, \quad v \in E^0, n \in \mathbb{N},$$

form a basis for the topology of  $\widehat{\mathcal{F}}_E$ . Moreover, in view of Lemma 3.13, definitions of  $\Lambda_{v,n}$ ,  $W(\mu)$  and form of  $U_{v,n}$ , see (3.7), the preimage of  $\widehat{J}_{v,n}$  under the map  $[\mu] \mapsto \pi_\mu$  is  $\{[\mu] \in \Omega_E / \sim : \Lambda_{v,n} \cap W(\mu) \neq \emptyset\} = \{[\mu] \in \Omega_E : \exists_{\nu \in E^k, k \in \mathbb{N}} s(\nu) = v, r(\nu) = \mu_{n+k}\} = U_{v,n}$ .

Thus, in view of Lemma 3.8, we see that  $[\mu] \mapsto \pi_\mu$  establishes one-to-one correspondence between the topological bases for its domain and codomain and hence is a homeomorphism onto codomain. □

Now, to obtain Theorem 3.9 we only need the following

**Lemma 3.15.** *The mapping  $[\mu] \mapsto \pi_\mu \in \widehat{\mathcal{F}}_E$  intertwines  $[\sigma_E]$  and  $\widehat{\mathcal{V}}$ .*

*Proof.* To see that  $\widehat{\mathcal{V}}(\widehat{\mathcal{F}}_E) = \{\pi \in \widehat{\mathcal{F}}_E : \pi(\mathcal{V}(1)) \neq 0\}$  coincides with  $\Delta = \widehat{\mathcal{F}}_E \setminus \widehat{G}_0$  let  $\pi \in \widehat{\mathcal{F}}_E$  and note that

$$\pi(\mathcal{V}(1)) = 0 \iff \forall_{v \in s(E^1)} \pi(p_v) = 0 \iff \exists_{w \in E^0_{sinks}} \pi \cong \pi_w.$$

Furthermore, by (3.4) and (3.5), we have

$$\mathcal{V}(\mathcal{F}_N(v)) = \mathcal{V}(1)\mathcal{F}_{N+1}(v)\mathcal{V}(1), \quad \mathcal{H}(\mathcal{F}_{N+1}(v)) = \mathcal{F}_N(v), \quad N \in \mathbb{N}, \quad (3.10)$$

and  $\mathcal{H}(\mathcal{F}_0(v)) \subset \sum_{w \in r(s^{-1}(v))} \mathcal{F}_0(w)$ . In particular, for  $\mu \in E_{sinks}^N$ ,  $N > 0$ , we have  $\pi_\mu \in \Delta$  and

$$(\pi_\mu \circ \mathcal{V})(\mathcal{F}_{N-1}(w)) = \pi_\mu(\mathcal{V}(1)\mathcal{F}_N(w)\mathcal{V}(1)) \neq 0.$$

Hence  $\widehat{\mathcal{V}}(\pi_\mu) \cong \pi_{\sigma_E(\mu)}$ . Let us now fix  $\mu = \mu_1\mu_2\mu_3 \dots \in E^\infty$ . Let  $H_\mu$  be the Hilbert space and  $\xi_\mu \in H_\mu$  the cyclic vector associated to the pure state  $\omega_\mu$  via GNS-construction. For  $\nu, \eta \in E^n$ , using (3.4) and (3.9), we get

$$\begin{aligned} \omega_\mu(\mathcal{V}(s_\nu s_\eta^*)) &= \begin{cases} \frac{1}{\sqrt{n_{s(\nu)}n_{s(\eta)}}} \sum_{e,f \in E^1} \omega_\mu(s_{e\nu} s_{f\eta}^*), & n_{s(\nu)}n_{s(\eta)} \neq 0, \\ 0, & n_{s(\nu)}n_{s(\eta)} = 0, \end{cases} \\ &= \begin{cases} \frac{1}{n_{r(\mu_1)}}, & \nu = \eta = \mu_2 \dots \mu_{n+1} \\ 0 & \text{otherwise} \end{cases} \\ &= \frac{1}{n_{r(\mu_1)}} \omega_{\sigma_E(\mu)}(s_\nu s_\eta^*). \end{aligned}$$

It follows that  $\omega_\mu \circ \mathcal{V} = \frac{1}{n_{r(\mu_1)}} \omega_{\sigma_E(\mu)}$  and therefore  $\widehat{\mathcal{V}}(\pi_\mu) \cong \pi_{\sigma_E(\mu)}$ , cf. [37, Corollary 3.3.8]. □

### 3.4. Topological Freeness of Graph Interactions

We will now use Theorem 3.9 to identify the relevant properties of the partial homeomorphism  $\widehat{\mathcal{V}}$  dual to the graph interaction  $(\mathcal{V}, \mathcal{H})$ . We recall that the condition (L) introduced in [29] requires that every loop in  $E$  has an exit. For convenience, by loops we will mean simple loops, that is paths  $\mu = \mu_1 \dots \mu_n$  such that  $s(\mu_1) = r(\mu_n)$  and  $s(\mu_1) \neq r(\mu_k)$  for  $k = 1, \dots, n - 1$ . A loop  $\mu$  is said to have an exit if there is an edge  $e$  such that  $s(e) = s(\mu_i)$  and  $e \neq \mu_i$  for some  $i = 1, \dots, n$ .

**Proposition 3.16.** *Suppose that every loop in  $E$  has an exit. Then every open set intersecting  $\widehat{G}_\infty$  contains infinitely many non-periodic points for  $\widehat{\mathcal{V}}$  and if  $E$  has no sinks the number of this non-periodic points is uncountable. In particular,  $\widehat{\mathcal{V}}$  is topologically free.*

*Proof.* By Theorem 3.9 and Lemma 3.8 it suffices to consider the dynamical system  $(\Omega_E / \sim, [\sigma_E])$  and an open set of the form

$$U_{n,v} = \{[\mu] : \exists_{\eta \in E^k, k \in \mathbb{N}} s(\eta) = v, r(\eta) = \mu_{n+k}\}$$

which contains  $[\mu]$  for  $\mu = \mu_1\mu_2 \dots \in E^\infty$ . Since  $E$  is finite there must be a vertex  $v$  which appears as a base point of  $\mu$  infinitely many times. Namely, there exists an increasing sequence  $\{n_k\}_{k \in \mathbb{N}} \subset \mathbb{N}$  such that  $r(\mu_{n_k}) = v$  for all  $k \in \mathbb{N}$ . Moreover, since every loop in  $E$  has an exit, the vertex  $v$  has to be connected either to a sink or to a vertex lying on two different loops. Let us consider these two cases:

1. Suppose  $\nu$  is a finite path such that  $v = s(\nu)$  and  $w := r(\nu) \in E_{\text{sink}s}^0$ . Consider the family of finite, and hence non-periodic, paths

$$\mu^{(n_k)} := \mu_1 \dots \mu_{n_k} \nu \in E_{\text{sink}s}^{n+|\nu|}, \quad k \in \mathbb{N}.$$

Plainly, all except finitely many of elements  $[\mu^{(n_k)}]$  belong to  $U_{n,v}$  (and they are all different).

2. Suppose  $\nu$  is a finite path such that  $v = s(\nu)$  and the vertex  $w := r(\nu)$  is a base point for two different loops  $\mu^0$  and  $\mu^1$ . We put  $\mu^\epsilon = \mu^{\epsilon_1} \mu^{\epsilon_2} \mu^{\epsilon_3} \dots \in E^\infty$  for  $\epsilon = \{\epsilon_i\}_{i=1}^\infty \in \{0, 1\}^{\mathbb{N} \setminus \{0\}}$ . Since there is an uncountable number of non-periodic sequences in  $\{0, 1\}^{\mathbb{N} \setminus \{0\}}$  which pairwise do not eventually coincide the paths  $\mu^\epsilon$  corresponding to these sequences give rise to the uncountable family of non-periodic elements  $[\mu^\epsilon]$  in  $\Omega_E / \sim$ . Moreover, one readily sees that for sufficiently large  $n_k$  all the equivalent classes of paths

$$\mu^{(\epsilon)} := \mu_1 \dots \mu_{n_k} \nu \mu^\epsilon \in E^\infty, \quad \epsilon = \{\epsilon_i\}_{i=1}^\infty \in \{0, 1\}^{\mathbb{N} \setminus \{0\}}$$

belong to  $U_{n,v}$ . This proves our assertion. □

*Example.* In the case  $C^*(E) = \mathcal{O}_n$  is the Cuntz algebra, that is when  $E$  is the graph with a single vertex and  $n$  edges,  $n \geq 2$ , then  $\mathcal{F}_E$  is an UHF-algebra and the states  $\omega_\mu$  are simply Glimm’s product states. In particular, it is well known that  $\text{Prim}(\mathcal{F}_E) = \{0\}$  and  $\widehat{\mathcal{F}}_E$  is uncountable, cf. [37, 6.5.6]. Hence, on one hand the Rieffel homeomorphism given by the imprimitivity  $\mathcal{F}_E$ -bimodule  $X = \mathcal{F}_E s \mathcal{F}_E$  associated with the graph interaction  $(\mathcal{V}, \mathcal{H})$  is the identity on  $\text{Prim}(\mathcal{F}_E)$ . Thereby it is not topologically free ([28, Theorem 6.5] can not be applied). On the other hand, we have just shown that  $\widehat{\mathcal{F}}_E$  contains uncountably many non-periodic points for  $X\text{-Ind} = \widehat{\mathcal{V}}^{-1}$ , cf. Proposition 2.19, and hence it is topologically free.

Suppose now that  $\mu$  is a loop in  $E$ . Let  $\mu^\infty \in E^\infty$  be the path obtained by the infinite concatenation of  $\mu$ . Then  $\Lambda(E) \setminus W(\mu^\infty)$  is a Bratteli diagram for a primitive ideal in  $\mathcal{F}_E$ , which we denote by  $I_\mu$ . In other words, see Lemma 3.13, we have

$$I_\mu = \ker \pi_{\mu^\infty}$$

where  $\pi_{\mu^\infty}$  is the irreducible representation associated to  $\mu^\infty$ .

**Proposition 3.17.** *If the loop  $\mu$  has no exits, then up to unitary equivalence  $\pi_{\mu^\infty}$  is the only representation of  $\mathcal{F}_E$  whose kernel is  $I_\mu$  and the singleton  $\{\pi_{\mu^\infty}\}$  is an open set in  $\widehat{\mathcal{F}}_E$ .*

*Proof.* The quotient  $\mathcal{F}_E / I_\mu$  is an AF-algebra with the diagram  $W(\mu^\infty)$ . The path  $\mu^\infty$  treated as a subdiagram of  $W(\mu^\infty)$  is hereditary and its saturation  $\overline{\mu^\infty}$  yields an ideal  $\mathcal{K}$  in  $\mathcal{F}_E / I_\mu$ . Since  $\mu^\infty$  has no exits,  $\mathcal{K}$  is isomorphic to the ideal of compact operators  $\mathcal{K}(H)$  on a separable Hilbert space  $H$  (finite or infinite dimensional). Therefore every faithful irreducible representation of  $\mathcal{F}_E / I_\mu$  is unitarily equivalent to the representation given by the isomorphism  $\mathcal{K} \cong \mathcal{K}(H) \subset \mathcal{B}(H)$ . This shows that  $\pi_{\mu^\infty}$  is determined by its kernel. Moreover, since  $W(\mu^\infty)$  contains all its ancestors, the subdiagram  $\overline{\mu^\infty}$  is hereditary

and saturated not only in  $W(\mu^\infty)$  but also in  $\Lambda(E)$ . Therefore we let now  $\mathcal{K}$  stand for the ideal in  $\mathcal{F}_E$ , corresponding to  $\overline{\mu^\infty}$ . Let  $P \in \text{Prim}(\mathcal{F}_E)$ . As  $\mathcal{K}$  is simple  $P \not\supseteq \mathcal{K}$  implies  $\mathcal{K} \cap P = \{0\}$ . By the form of  $W(\mu^\infty)$  and hereditariness of  $\Lambda(P)$ ,  $\mathcal{K} \cap P = \{0\}$  implies  $\Lambda(P) \subset \Lambda(\mathcal{F}_E) \setminus W(\mu^\infty) = \Lambda(I_\mu)$ . However, if  $P \subset I_\mu$ , we must have  $P = I_\mu$  because no part of  $\Lambda(I_\mu)$  is not connected to  $W(\mu^\infty)$  (consult the form of diagrams of primitive ideals [5, 3.8]). Concluding, we get

$$\{P \in \text{Prim}(\mathcal{F}_E) : P \not\supseteq \mathcal{K}\} = \{P \in \text{Prim}(\mathcal{F}_E) : \mathcal{K} \cap P = \{0\}\} = \{I_\mu\},$$

which means that  $\{I_\mu\}$  is open in  $\text{Prim}(\mathcal{F}_E)$ . Accordingly,  $\{\pi_{\mu^\infty}\}$  is open in  $\widehat{\mathcal{F}}_E$ . □

We have the following characterization of minimality of  $\widehat{\mathcal{V}}$ .

**Proposition 3.18.** *The map  $V \mapsto \widehat{\mathcal{F}}_{\Lambda(V)}$  is a one-to-one correspondence between the hereditary saturated subsets of  $E^0$  and  $\widehat{\mathcal{V}}$ -invariant open subsets of  $\widehat{\mathcal{F}}_E$ . In particular,  $\widehat{\mathcal{V}}$  is minimal if and only if there are no nontrivial hereditary saturated subsets of  $E^0$ .*

*Proof.* Recall that for a hereditary and saturated subset  $V$  of  $E^0$  we treat  $\Lambda(V)$  as a subdiagram of  $\Lambda(E)$  where on the  $n$ th level we have  $(r(E^n) \cap V) \cup \bigcup_{k=0}^{n-1} \{v^{(k)} : v \in r(E^k_{\text{sinks}}) \cap V\}$ . Now, using condition (iii) of Lemma 2.22 and relations (3.10) one readily see that the open set  $\widehat{I}$  for an ideal  $I$  in  $\mathcal{F}_E$  is  $\widehat{\mathcal{V}}$ -invariant if and only if the corresponding Bratteli diagram for  $I$  is of the form  $\Lambda(V)$  where  $V \subset E^0$  is hereditary and saturated. □

Combining the above results we not only characterize freeness and topological freeness of  $(\widehat{\mathcal{F}}_E, \widehat{\mathcal{V}})$  but also spot out an interesting dichotomy concerning its core subsystem  $(\widehat{G}_\infty, \widehat{\mathcal{V}})$ , cf. Remark 3.20 below.

**Theorem 3.19.** *Let  $(\widehat{\mathcal{F}}_E, \widehat{\mathcal{V}})$  be a partial homeomorphism dual to the graph interaction  $(\mathcal{V}, \mathcal{H})$ . We have the following dynamical dichotomy:*

- (a) *either every open set intersecting  $\widehat{G}_\infty$  contains infinitely many nonperiodic points for  $\widehat{\mathcal{V}}$ ; this holds if every loop in  $E$  has an exit, or*
- (b) *there are  $\widehat{\mathcal{V}}$ -periodic orbits  $O = \{\pi_{\mu^\infty}, \pi_{\sigma_E(\mu^\infty)} \dots, \pi_{\sigma_E^{n-1}(\mu^\infty)}\}$  in  $\widehat{G}_\infty$  forming open discrete sets in  $\widehat{\mathcal{F}}_E$ ; they correspond to loops without exits  $\mu$ .*

*In particular,*

- (I)  *$\widehat{\mathcal{V}}$  is topologically free if and only if every loop in  $E$  has an exit (satisfies condition (L)),*
- (II)  *$\widehat{\mathcal{V}}$  is free if and only if every loop has an exit connected to this loop (satisfies the so-called condition (K) introduced in [30], see also [4]).*

*Proof.* In view of Propositions 3.16, 3.17 only item (II) requires a comment. By Proposition 3.18 every closed  $\widehat{\mathcal{V}}$ -invariant set is of the form  $\widehat{\mathcal{F}}_E \setminus \widehat{\mathcal{F}}_V = \widehat{\mathcal{F}}_{E \setminus V}$  for a hereditary and saturated subset  $V \subset E^0$ . Hence  $\widehat{\mathcal{V}}$  is free if and only if every loop outside a hereditary saturated set  $V$  has an exit outside

$V$ . The latter condition is clearly equivalent to the condition that every loop has an exit connected to this loop, cf. [4, page 318].  $\square$

*Remark 3.20.* Since  $E$  is finite, by [29, Theorem 3.9],  $C^*(E)$  is purely infinite in the sens of [29,31] if and only if  $E$  has no sinks and every loop in  $E$  has an exit. In view of Proposition 3.16 we conclude that  $C^*(E)$  is purely infinite if and only if every nonempty open set in  $\widehat{\mathcal{F}}_E$  contains uncountable number of nonperiodic points for  $\widehat{\mathcal{V}}$ . In particular, every  $\widehat{\mathcal{V}}$ -periodic orbit  $O = \{\pi_{\mu^\infty}, \pi_{\sigma_E(\mu^\infty)} \dots, \pi_{\sigma_E^{n-1}(\mu^\infty)}\}$  yields a gauge invariant ideal  $J_O$  in  $C^*(E)$  (generated by  $\bigcap_{\pi \in \widehat{\mathcal{F}}_E} \pi^{-1}(O) \ker \pi$ ) which is not purely infinite. Indeed, if  $v = s(\mu)$  is the source of a loop  $\mu$  which has no exit, then  $p_v C^*(E) p_v = p_v J_O p_v = C^*(s_\mu) \cong C(\mathbb{T})$  because  $s_\mu$  is a unitary in  $C^*(s_\mu)$  with the full spectrum, cf. [29, proof of Theorem 2.4].

Concluding, we deduce from our general results for corner interactions the following fundamental classic results for graph algebras, cf. [4,29,30,39].

**Corollary 3.21.** *Consider the graph  $C^*$ -algebra  $C^*(E)$  of the finite directed graph  $E$ .*

- (i) *If every loop in  $E$  has an exit, then any Cuntz–Krieger  $E$ -family  $\{P_v : v \in E^0\}$ ,  $\{S_e : e \in E^1\}$  generates a  $C^*$ -algebra isomorphic to  $C^*(E)$ , via  $s_e \mapsto S_e, p_v \mapsto P_v, e \in E^1, v \in E^0$ .*
- (ii) *If every loop in  $E$  has an exit connected to this loop, then there is a lattice isomorphism between hereditary saturated subsets of  $E^0$  and ideals in  $C^*(E)$ , given by  $V \mapsto J_V$ , where  $J_V$  is an ideal generated by  $p_v, v \in V$ .*
- (iii) *If every loop in  $E$  has an exit and  $E$  has no nontrivial hereditary saturated sets, then  $C^*(E)$  is simple.*

*Proof.* Apply Propositions 3.2, 3.18 and Theorems 2.20, 3.19.  $\square$

**3.5.  $K$ -Theory**

We now turn to description of  $K$ -groups for  $C^*(E)$ . As  $K_1$  groups for AF-algebras are trivial, using Pimsner–Voiculescu sequence from Theorem 2.25 applied to the graph interaction  $(\mathcal{V}, \mathcal{H})$  associated to  $E$  we have

$$K_1(C^*(E)) \cong \ker(\iota_* - \mathcal{H}_*),$$

$$K_0(C^*(E)) \cong \text{coker}(\iota_* - \mathcal{H}_*) = K_0(\mathcal{F}_E) / \text{im}(\iota_* - \mathcal{H}_*)$$

where  $(\iota_* - \mathcal{H}_*) : K_0(\mathcal{V}(\mathcal{F}_E)) \rightarrow K_0(\mathcal{F}_E)$ . Hence to calculate the  $K$ -groups for  $C^*(E)$  we need to identify  $\ker(\iota_* - \mathcal{H}_*)$  and  $\text{coker}(\iota_* - \mathcal{H}_*)$ . We do it in two steps.

**Proposition 3.22.** *( $K_0$ -partial automorphism induced by a graph interaction) The group  $K_0(\mathcal{F}_E)$  is the universal abelian group  $\langle V \rangle$  generated by the set  $V := \{v^{(N)} : v \in r(E^N), N \in \mathbb{N}\}$  of ‘endings of finite paths’, subject to relations*

$$v^{(N)} = \sum_{s(e)=v} r(e)^{(N+1)} \quad \text{for all } v \in r(E^N) \setminus E_{sinks}^0. \tag{3.11}$$



In particular, the subgroup generated by  $v^{(N)} \in V$ ,  $v \in E_{sinks}^0$ ,  $N \in \mathbb{N}$ , in  $K_0(\mathcal{F}_E)$  is free abelian. The groups  $K_0(\mathcal{V}(A))$  and  $K_0(\mathcal{H}(A))$  embed into  $K_0(\mathcal{F}_E)$  and we have

$$K_0(\mathcal{V}(\mathcal{F}_E)) = \left\langle V \setminus \left\{ v^{(0)} : v \in E_{sinks}^0 \right\} \right\rangle,$$

$$K_0(\mathcal{H}(\mathcal{F}_E)) = \left\langle \left\{ v^{(N)} \in V : v^{(N+1)} \in V \right\} \right\rangle.$$

The isomorphism  $\mathcal{H}_* : K_0(\mathcal{V}(A)) \rightarrow K_0(\mathcal{H}(A))$  is determined by

$$\mathcal{H}_* \left( v^{(N+1)} \right) = v^{(N)}, \quad N \in \mathbb{N}. \tag{3.12}$$

*Proof.* We identify  $v^{(N)}$  with the  $K_0$ -group element  $[s_\mu s_\mu^*]$  where  $\mu \in E^N$  and  $v = r(\mu)$ . It follows from (3.1) that the group  $K_0(\mathcal{F}_E)$  is the inductive limit  $\varinjlim (K_0(\mathcal{F}_N), i_E^N)$  where

$$K_0(\mathcal{F}_N) \cong \bigoplus_{v \in r(E^N) \setminus E_{sinks}^0} \mathbb{Z}v^{(N)} \oplus \bigoplus_{k=0, \dots, N} \bigoplus_{v \in r(E^k) \cap E_{sinks}^0} \mathbb{Z}v^{(k)}.$$

Under the above isomorphisms, the bonding maps  $i_E^N : K_0(\mathcal{F}_N) \rightarrow K_0(\mathcal{F}_{N+1})$ ,  $N \in \mathbb{N}$ , are given on generators by the formula

$$i_E^N(v^{(N)}) = \begin{cases} \sum_{s(e)=v} r(e)^{(N+1)}, & v \notin E_{sinks}^0 \\ v^{(N)}, & v \in E_{sinks}^0, \end{cases} \quad v \in r(E^N).$$

This immediately implies the first part of the assertion.

Since  $\mathcal{H}(\mathcal{F}_E) = \mathcal{H}(1)\mathcal{F}_E\mathcal{H}(1)$  is the closure of  $\bigcup_{N \in \mathbb{N}} \mathcal{H}(1)\mathcal{F}_N\mathcal{H}(1)$  and the group  $K_0(\mathcal{H}(1)\mathcal{F}_N\mathcal{H}(1))$  embeds into  $K_0(\mathcal{F}_N)$  we see by continuity of  $K_0$  that  $K_0(\mathcal{H}(\mathcal{F}_E))$  embeds into  $K_0(\mathcal{F}_E) = \langle V \rangle$ . Moreover, as  $\mathcal{H}(1) = \sum_{v \in r(E^1)} p_v$  we get

$$\mathcal{H}(1)\mathcal{F}_N(v)\mathcal{H}(1) \neq \{0\} \iff \mathcal{F}_{N+1}(v) \neq \{0\} \iff v^{(N+1)} \in V,$$

whence  $K_0(\mathcal{H}(\mathcal{F}_E))$  identifies with  $\langle \{v^{(N)} \in V : v^{(N+1)} \in V\} \rangle$ .

Similarly, taking into account that  $\mathcal{V}(1) = \sum_{v \in r(E^1)} \frac{1}{n_v} \sum_{e, f \in r^{-1}(v)} s_e s_f^*$  we infer that  $\mathcal{V}(1)\mathcal{F}_N(v)\mathcal{V}(1) \neq \{0\}$  for all  $v \in E^0$  and  $N > 0$ , and  $\mathcal{V}(1)\mathcal{F}_0(v)\mathcal{V}(1) \neq \{0\}$  if and only if  $v \notin E_{sinks}^0$ . Thus we may identify  $K_0(\mathcal{V}(\mathcal{F}_E))$  with  $\langle V \setminus \{v^{(0)} : v \in E_{sinks}^0\} \rangle$ . Now (3.12) follows from (3.5). Note that (3.12) determines  $\mathcal{H}_*$ , as for  $v \in E^0 \setminus E_{sinks}^0$ , using only (3.11) and (3.12) we have  $\mathcal{H}_*(v^{(0)}) = \mathcal{H}_*(\sum_{s(e)=v} r(e)^{(1)}) = \sum_{s(e)=v} r(e)^{(0)}$ .  $\square$

We let  $\mathbb{Z}(E^0 \setminus E_{sinks}^0)$  and  $\mathbb{Z}E^0$  denote the free abelian groups on free generators  $E^0 \setminus E_{sinks}^0$  and  $E^0$ , respectively. We consider the group homomorphism  $\Delta_E : \mathbb{Z}(E^0 \setminus E_{sinks}^0) \rightarrow \mathbb{Z}E^0$  defined on generators as

$$\Delta_E(v) = v - \sum_{s(e)=v} r(e).$$

The following lemma can be viewed as a counterpart of Lemmas 3.3, 3.4 in [40]. Nevertheless, it is a slightly different statement.

**Lemma 3.23.** *We have isomorphisms*

$$\ker(\iota_* - \mathcal{H}_*) \cong \ker(\Delta_E), \quad \text{coker}(\iota_* - \mathcal{H}_*) \cong \text{coker}(\Delta_E),$$

which are given on generators by

$$\ker \Delta_E \ni v \xrightarrow{i^{(0)}} v^{(0)} \in \ker(\iota_* - \mathcal{H}_*),$$

$$\mathbb{Z}E^0 / \text{im}(\Delta_E) \ni [v] \xrightarrow{j^{(0)}} [v^{(0)}] \in K_0(\mathcal{F}_E) / \text{im}(\iota_* - \mathcal{H}_*).$$

*Proof.* Suppose  $a = \sum_{v \in E^0 \setminus E^0_{\text{sinks}}} a_v v \in \mathbb{Z}(E^0 \setminus E^0_{\text{sinks}})$ . Then by (3.11), (3.12) we have

$$\begin{aligned} (\iota_* - \mathcal{H}_*)(i^{(0)}(a)) &= \sum_{v \in E^0 \setminus E^0_{\text{sinks}}} a_v v^{(0)} - \sum_{v \in E^0 \setminus E^0_{\text{sinks}}} a_v \sum_{s(e)=v} r(e)^{(0)} \\ &= i^{(0)}(\Delta_E(a)). \end{aligned}$$

Accordingly,  $a \in \ker \Delta_E$  implies  $i^{(0)}(a) \in \ker(\iota_* - \mathcal{H}_*)$  and hence  $i^{(0)}$  is well defined. Clearly  $i^{(0)}$  is injective. To show that it is surjective note that

$$x = \sum_{v \in r(E^N) \setminus E^0_{\text{sinks}}} x_v v^{(N)} + \sum_{k=1, \dots, N} \sum_{v \in r(E^k) \cap E^0_{\text{sinks}}} x_v^{(k)} v^{(k)} \tag{3.13}$$

is a general form of an element in  $K_0(\mathcal{V}(A))$  and assume  $x$  is in  $\ker(\iota_* - \mathcal{H}_*)$ . The relation  $x = \mathcal{H}_*(x)$  implies that the coefficients corresponding to sinks in the expansion (3.13) are zero. Thus  $x = \mathcal{H}_*^n(x) = \sum_{v \in E^0 \setminus E^0_{\text{sinks}}} x_v v^{(0)} = i^{(0)}(a)$  where  $a := \sum_{v \in E^0 \setminus E^0_{\text{sinks}}} x_v v$  is in  $\ker \Delta_E$  because  $i^{(0)}(a) = x = \mathcal{H}_*(x) = \mathcal{H}_*(i^{(0)}(a)) = i^{(0)}(\Delta_E(a))$ . Hence  $i^{(0)}$  is an isomorphism.

Since  $i^{(0)}$  intertwines  $\Delta_E$  and  $(\iota_* - \mathcal{H}_*)$  we see that  $j^{(0)}$  is well defined. To show that  $j^{(0)}$  is surjective, let  $y = x + \sum_{v \in E^0_{\text{sinks}}} x_v^{(0)} v^{(0)}$  where  $x$  is given by (3.13) [this is a general form of an element in  $K_0(\mathcal{F}_E)$ ]. Observe that as  $x - \mathcal{H}_*(x) \in \text{im}(\iota_* - \mathcal{H}_*)$  the element  $y$  has the same class in  $\text{coker}(\iota_* - \mathcal{H}_*)$  as

$$\mathcal{H}_*(x) + \sum_{v \in E^0_{\text{sinks}}} x_v^{(0)} v^{(0)} = z + \sum_{k=0,1} \sum_{v \in r(E^k) \cap E^0_{\text{sinks}}} x_v^{(k)} v^{(0)}$$

where  $z = \sum_{v \in r(E^N) \setminus E^0_{\text{sinks}}} x_v v^{(N-1)} + \sum_{k=2, \dots, N} \sum_{v \in r(E^k) \cap E^0_{\text{sinks}}} x_v^{(k)} v^{(k-1)}$  is in  $K_0(\mathcal{V}(A))$ . Applying the above argument to  $z$  and proceeding in this way  $N$  times we get that  $y$  is in the same class in  $\text{coker}(\iota_* - \mathcal{H}_*)$  as

$$\sum_{v \in r(E^N) \setminus E^0_{\text{sinks}}} x_v v^{(0)} + \sum_{k=0,1, \dots, N} \sum_{v \in r(E^k) \cap E^0_{\text{sinks}}} x_v^{(k)} v^{(0)}.$$

Hence

$$y = j^{(0)} \left( \sum_{v \in r(E^N) \setminus E^0_{\text{sinks}}} x_v [v] + \sum_{k=0,1, \dots, N} \sum_{v \in r(E^k) \cap E^0_{\text{sinks}}} x_v^{(k)} [v] \right).$$

The proof of injectivity of  $j^{(0)}$  is slightly more complicated. Let us consider  $a = \sum_{v \in E^0} a_v v \in \mathbb{Z}E^0$  such that  $i^{(0)}(a) \in \text{im}(\iota_* - \mathcal{H}_*)$ . Then  $i^{(0)}(a) = x - \mathcal{H}_*(x)$

for an element  $x$  of the form (3.13), and hence

$$i^0(a) = \sum_{v \in r(E^N) \setminus E^0_{sinks}} x_v \left( v^{(N)} - \sum_{s(e)=v} r(e)^{(N)} \right) + \sum_{k=1, \dots, N} \sum_{v \in r(E^k) \cap E^0_{sinks}} x_v^{(k)} \left( v^{(k)} - v^{(k-1)} \right).$$

On the other hand, applying  $N$ -times relation (3.11) to  $i^0(a) = \sum_{v \in E^0} a_v v^{(0)}$  we get

$$i^0(a) = \sum_{\mu \in E^N} a_{s(\mu)} v_{r(\mu)}^{(N)} + \sum_{k=0, \dots, N} \sum_{\mu \in E^k_{sinks}} a_{s(\mu)} v_{r(\mu)}^{(k)}.$$

Comparing coefficients in the above two formulas one can see that

$$a_v = \sum_{r(e)=v} x_{s(e)} + \sum_{k=1, \dots, N} \sum_{\mu \in E^k, r(\mu)=v} a_{s(\mu)} \quad \text{for } v \in E^0_{sinks} \tag{3.14}$$

(in particular  $a_v = 0$  for  $v \in E^0_{sinks} \setminus r(E^1)$ ), and

$$\sum_{\mu \in E^N, r(\mu)=v} a_{s(\mu)} = x_v - \sum_{r(e)=v} x_{s(e)}, \quad \text{for } v \in r(E^N) \setminus E^0_{sinks}. \tag{3.15}$$

We define an element of  $\mathbb{Z}(E^0 \setminus E^0_{sinks})$  by

$$b := \sum_{v \in r(E^N) \setminus E^0_{sinks}} x_v v + \sum_{k=0, \dots, N-1} \sum_{\mu \in E^k \setminus E^k_{sinks}} a_{s(\mu)} r(\mu).$$

Using (3.14) and (3.15), in the third equality below, we obtain

$$\begin{aligned} \Delta_E b &= b - \sum_{v \in r(E^N)} \left( \sum_{r(e)=v} x_{s(e)} \right) v - \sum_{\mu \in E^k, k=1, \dots, N} a_{s(\mu)} r(\mu) \\ &= \sum_{v \in r(E^N) \setminus E^0_{sinks}} \left( x_v - \sum_{r(e)=v} x_{s(e)} - \sum_{\mu \in E^N, r(\mu)=v} a_{s(\mu)} \right) v \\ &\quad + \sum_{v \in E^0 \setminus E^0_{sinks}} a_v - \sum_{v \in E^0_{sinks}} \left( \sum_{r(e)=v} x_{s(e)} + \sum_{k=1, \dots, N} \sum_{\mu \in E^k, r(\mu)=v} a_{s(\mu)} \right) v \\ &= 0 + \sum_{v \in E^0 \setminus E^0_{sinks}} a_v v + \sum_{v \in E^0_{sinks}} a_v v = a. \quad \square \end{aligned}$$

**Corollary 3.24.** (cf. Theorem 3.2 in [40]) *We have isomorphisms*

$$K_0(C^*(E)) \cong \ker(\Delta_E), \quad K_1(C^*(E)) \cong \text{coker}(\Delta_E).$$

**Open Access.** This article is distributed under the terms of the Creative Commons Attribution License which permits any use, distribution, and reproduction in any medium, provided the original author(s) and the source are credited.

## References

- [1] Abadie, B., Eilers, S., Exel, R.: Morita equivalence for crossed products by Hilbert  $C^*$ -bimodules. *Trans. Am. Math. Soc.* **350**(8), 3043–3054 (1998)
- [2] Antonevich, A.B., Bakhtin, V.I., Lebedev, A.V.: Crossed product of  $C^*$ -algebra by an endomorphism, coefficient algebras and transfer operators. *Math. Sb.* **202**(9), 1253–1283 (2011)
- [3] Ara, P., Exel, R., Katsura, T.: Dynamical systems of type  $(m,n)$  and their  $C^*$ -algebras. *Ergod. Theory Dyn. Syst.* [arXiv:1109.4093](https://arxiv.org/abs/1109.4093)
- [4] Bates, T., Pask, D., Raeburn, I., Szymański, W.: The  $C^*$ -algebras of row-finite graphs. *N. Y. J. Math.* **6**, 307–324 (2000)
- [5] Bratteli, O.: Inductive limits of finite dimensional  $C^*$ -algebras. *Trans. Am. Math. Soc.* **171**, 195–234 (1972)
- [6] Brown, L.G., Mingo, J., Shen, N.: Quasi-multipliers and embeddings of Hilbert  $C^*$ -modules. *Canad. J. Math.* **71**, 1150–1174 (1994)
- [7] Brownlowe, N., Raeburn, I., Vittadello, S.: Exel’s crossed product for non-unital  $C^*$ -algebras. *Math. Proc. Camb. Philos. Soc.* **149**, 423–444 (2010)
- [8] Connes, A.: *Noncommutative Geometry*. Academic Press, San Diego (1994)
- [9] Cuntz, J.: Simple  $C^*$ -algebras generated by isometries. *Commun. Math. Phys.* **57**(2), 173–185 (1977)
- [10] Cuntz, J., Krieger, W.: A class of  $C^*$ -algebras and topological Markov chains. *Invent. Math.* **56**, 256–268 (1980)
- [11] Exel, R.: A new look at the crossed-product of a  $C^*$ -algebra by an endomorphism. *Ergod. Theory Dyn. Syst.* **23**, 1733–1750 (2003)
- [12] Exel, R.: Interactions. *J. Funct. Anal.* **244**, 26–62 (2007)
- [13] Exel, R.: A new look at the crossed-product of a  $C^*$ -algebra by a semigroup of endomorphisms.. *Ergod. Theory Dyn. Syst.* **28**, 749–789 (2008)
- [14] Exel, R.: Interactions and dynamical systems of type  $(n, m)$ —a case study. [arXiv:1212.5963](https://arxiv.org/abs/1212.5963)
- [15] Exel, R., Laca, M.: Cuntz–Krieger algebras for infinite matrices. *J. Reine Angew. Math.* **512**, 119–172 (1999)
- [16] Exel, R., Laca, M.: The  $K$ -Theory of Cuntz–Krieger algebras for infinite matrices. *K-Theory* **19**, 251–268 (2000)
- [17] Exel, R., Renault, J.: Semigroups of local homeomorphisms and interaction groups. *Ergod. Theory Dyn. Syst.* **27**, 1737–1771 (2007)
- [18] Exel, R., Vershik, A.:  $C^*$ -algebras of irreversible dynamical systems. *Can. J. Math.* **58**, 39–63 (2006)
- [19] an Huef, A., Raeburn, I.: Stacey crossed products associated to Exel systems. *Integr. Equ. Oper. Theory* **72**, 537–561 (2012)
- [20] Jeong, Ja A., Hyun Park, Gi: Topological entropy for the canonical completely positive maps on graph  $C^*$ -algebras. *Bull. Aust. Math. Soc.* **70**(1), 101–116 (2004)
- [21] Kajiwara, T., Watatani, Y.: Ideals of the core of  $C^*$ -algebras associated with self-similar maps. [arXiv:1306.1878](https://arxiv.org/abs/1306.1878)
- [22] Katsura, T.: On  $C^*$ -algebras associated with  $C^*$ -correspondences. *J. Funct. Anal.* **217**(2), 366–401 (2004)

- [23] Kwaśniewski, B.K.: On transfer operators for  $C^*$ -dynamical systems. *Rocky J. Math.* **42**(3), 919–938 (2012)
- [24] Kwaśniewski, B.K.:  $C^*$ -algebras associated with reversible extensions of logistic maps. *Mat. Sb.* **203**(10), 1448–1489 (2012)
- [25] Kwaśniewski, B.K.: Topological freeness for Hilbert bimodules. *Isr. J. Math.* doi:[10.1007/s11856-013-0057-0](https://doi.org/10.1007/s11856-013-0057-0)
- [26] Kwaśniewski, B.K.: Exel’s crossed products and crossed products by completely positive maps. [arXiv:1404.4929](https://arxiv.org/abs/1404.4929)
- [27] Kwaśniewski, B.K.: Extensions of  $C^*$ -dynamical systems to systems with complete transfer operators. [arXiv:math/0703800](https://arxiv.org/abs/math/0703800)
- [28] Kwaśniewski, B.K.: Crossed product of a  $C^*$ -algebra by a semigroup of interactions. *Demonstr. Math.* **47**(2) (2014)
- [29] Kumjian, A., Pask, D., Raeburn, I.: Cuntz–Krieger algebras of directed graphs.. *Pac. J. Math.* **184**, 161–174 (1998)
- [30] Kumjian, A., Pask, D., Raeburn, I., Renault, J.: Graphs, groupoids and Cuntz–Krieger algebras. *J. Funct. Anal.* **144**, 505–541 (1997)
- [31] Laca, M., Spielberg, J.: Purely infinite  $C^*$ -algebras from boundary actions of discrete groups. *J. Reine. Angew. Math.* **480**, 125–139 (1996)
- [32] O’Donovan, D.P.: Weighted shifts and covariance algebras. *Trans. Am. Math. Soc.* **208**, 1–25 (1975)
- [33] Olesen, D., Pedersen, G.K.: Applications of the Connes spectrum to  $C^*$ -dynamical systems II. *J. Funct. Anal.* **36**, 18–32 (1980)
- [34] Olesen, D., Pedersen, G.K.: Applications of the Connes spectrum to  $C^*$ -dynamical systems III. *J. Funct. Anal.* **45**, 357–390 (1982)
- [35] Paschke, W.L.: The crossed product of a  $C^*$ -algebra by an endomorphism.. *Proc. Am. Math. Soc.* **80**, 113–118 (1980)
- [36] Paschke, W.L.:  $K$ -theory for actions of the circle group on  $C^*$ -algebras. *J. Oper. Theory* **6**, 125–133 (1981)
- [37] Pedersen, G.K.:  $C^*$ -algebras and their automorphism groups. Academic Press, London (1979)
- [38] Pimsner, M.V.: A class of  $C^*$ -algebras generalizing both Cuntz–Krieger algebras and crossed products by  $\mathbb{Z}$ . *Fields Inst. Commun.* **12**, 189–212 (1997)
- [39] Raeburn, I.: Graph Algebras. BMS Regional Conference Series in Mathematics, vol. 103. American Mathematical Society, Providence (2005)
- [40] Raeburn, I., Szymański, W.: Cuntz–Krieger algebras of infinite graphs and matrices. *Trans. Am. Math. Soc.* **356**, 39–59 (2004)
- [41] Raeburn, I., Williams, D.P.: Morita Equivalence and Continuous-Trace  $C^*$ -Algebras. American Mathematical Society, Providence (1998)
- [42] Rørdam, M.: Classification of certain infinite simple  $C^*$ -algebras. *J. Funct. Anal.* **131**, 415–458 (1995)
- [43] Schweizer, J.: Dilations of  $C^*$ -correspondences and the simplicity of Cuntz–Pimsner algebras. *J. Funct. Anal.* **180**, 404–425 (2001)

B. K. Kwaśniewski  
Institute of Mathematics  
Polish Academy of Science  
ul. Śniadeckich 8  
00-956 Warsaw  
Poland

B. K. Kwaśniewski (✉)  
Institute of Mathematics  
University of Białystok  
ul. Akademicka 2  
15-267 Białystok  
Poland  
e-mail: bartoszk@math.uwb.edu.pl

Received: November 19, 2013.

Revised: May 15, 2014.