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# On form factors and correlation functions in twistor space

Laura Koster,<sup>a</sup> Vladimir Mitev,<sup>b</sup> Matthias Staudacher<sup>a</sup> and Matthias Wilhelm<sup>c</sup>

<sup>a</sup>*Institut für Mathematik, Institut für Physik und IRIS Adlershof,  
Humboldt-Universität zu Berlin,  
Zum Großen Windkanal 6, 12489 Berlin, Germany*

<sup>b</sup>*PRISMA Cluster of Excellence, Institut für Physik, WA THEP,  
Johannes Gutenberg-Universität Mainz,  
Staudingerweg 7, 55128 Mainz, Germany*

<sup>c</sup>*Niels Bohr Institute, Copenhagen University,  
Blegdamsvej 17, 2100 Copenhagen Ø, Denmark*

*E-mail:* [laurakoster@physik.hu-berlin.de](mailto:laurakoster@physik.hu-berlin.de), [vmitev@uni-mainz.de](mailto:vmitev@uni-mainz.de),  
[matthias@math.hu-berlin.de](mailto:matthias@math.hu-berlin.de), [matthias.wilhelm@nbi.ku.dk](mailto:matthias.wilhelm@nbi.ku.dk)

**ABSTRACT:** In this paper, we continue our study of form factors and correlation functions of gauge-invariant local composite operators in the twistor-space formulation of  $\mathcal{N} = 4$  super Yang-Mills theory. Using the vertices for these operators obtained in our recent papers [1, 2], we show how to calculate the twistor-space diagrams for general  $N^k$ MHV form factors via the inverse soft limit, in analogy to the amplitude case. For general operators without  $\dot{\alpha}$  indices, we then reexpress the NMHV form factors from the position-twistor calculation in terms of momentum twistors, deriving and expanding on a relation between the two twistor formalisms previously observed in the case of amplitudes. Furthermore, we discuss the calculation of generalized form factors and correlation functions as well as the extension to loop level, in particular providing an argument promised in [3].

**KEYWORDS:** AdS-CFT Correspondence, Scattering Amplitudes, Supersymmetric gauge theory, Wilson, 't Hooft and Polyakov loops

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**1 Introduction**

The maximally supersymmetric Yang-Mills theory in four dimensions ( $\mathcal{N} = 4$  SYM theory) has many special properties; among others, it has a dual string theory description and is integrable in the planar limit. In addition, the theory can be formulated in twistor space [4, 5]. This alternative formulation has proven to be very efficient for describing on-shell quantities such as  $N^k$ MHV amplitudes [6–11] and in showing the duality between planar scattering amplitudes and the planar integrands of supersymmetric Wilson loops in

twistor space [12, 13]. One advantage of this formulation is the fact that the interactions are reorganized in such a way that the difficulty of a calculation scales with the MHV degree but no longer with the number of legs. The next step is to start calculating more general quantities that contain also gauge-invariant local composite operators; see [3, 14] for applications of twistor-space techniques to correlation functions. This however requires a formulation of general composite operators in twistor space, which cannot be directly obtained by translating the corresponding expressions in space-time [1].

In [1, 2], we have given a formulation of general gauge-invariant local composite operators in twistor space by applying derivatives to a light-like polygonal Wilson loop and taking the limit where this loop is shrunk to a line in twistor space, i.e. a point in space-time. The physical intuition behind this construction stems from recalling three facts. First, that the field-strength is the curvature of the gauge connection. Second, that the curvature of a connection can be obtained by parallel propagating around a parallelogram, taking derivatives of it and shrinking it to a point. Third, that all fundamental fields in  $\mathcal{N} = 4$  SYM theory are related to the field strength via (super)symmetry since they all lie in the singleton representation.<sup>1</sup> As a direct application of the Wilson loop construction, the tree-level MHV form factors of general operators, straightforwardly given by the operator vertices in twistor space, were derived in [2]. Form factors are the simplest objects that contain a local composite operator and are a hybrid between a fully on-shell amplitude and a fully off-shell correlation function. Thus, they provide a useful first step for understanding fully off-shell quantities. Moreover, form factors are also interesting on their own; for instance, they are related to the Higgs-to-gluons amplitude in the  $m_{\text{top}} \rightarrow \infty$  limit and have played an essential role in understanding the infrared divergences of scattering amplitudes [19–22]. Although form factors in  $\mathcal{N} = 4$  SYM theory have received increasing attention in recent years, see [1, 2, 17, 23–55] at weak coupling and [56–58] at strong coupling, still less is known for form factors than for amplitudes. For recent reviews, we refer also to [59, 60].

In the present paper, we show how to compute  $N^k$ MHV form factors and correlation functions of  $\mathcal{N} = 4$  SYM theory via the twistor formulation and the techniques presented in [1, 2]. Unlike in [2], we now have to face diagrams containing propagators and multiple vertices. In order to efficiently deal with the problem, we extend the inverse soft limit used in [10] in the case of amplitudes to our twistor-space vertices for composite operators derived from a Wilson loop [1]. A particular stumbling block to overcome is then translating our results from position twistor space to a standard momentum (twistor) space representation, which requires the calculation of a particular type of Fourier integrals. These integrals distinguish whether or not the operator contains  $\dot{\alpha}$  indices, which correspond to space-time derivatives acting on the Wilson loop.

The paper is organized as follows. In section 2, we review twistor space, introduce notation, recall the construction of the composite operator from a Wilson loop and derive an inverse soft limit for the corresponding vertices in twistor space. In section 3, we discuss the calculation of general tree-level  $N^k$ MHV form factors in position twistor space.

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<sup>1</sup>A different formulation of composite operators in the so-called “Lorentz harmonic chiral” (LHC) superspace was independently developed in [15–17] and argued to be related to our twistor-space formulation in [17, 18].

In section 4, we first show how to translate amplitudes from position twistor space to momentum (twistor) space, deriving a relation between the two twistor formalisms previously observed in [10]. We then apply these techniques to the NMHV tree-level form factors of operators without  $\dot{\alpha}$  indices. We conclude section 4 with a computation of the NMHV form factor of an operator with an  $\dot{\alpha}$  index. In section 5, we give a general prescription of how to use the formalism for correlation functions and generalized form factors. We conclude in section 6 with a summary and a brief overview of open questions. Three appendices provide further details. We expand on the calculation of the NMHV form factors and the Fourier-type integrals in appendices A and B, respectively. In appendix C, we revisit the twistor calculation of the one-loop two-point function in the SO(6) sector and provide an argument that was promised in [3].

## 2 Twistor space, Wilson loops and composite operators

In this section, we set the stage — introducing all important concepts that will play a role in the sections to come. These are the twistor action of  $\mathcal{N} = 4$  SYM theory as introduced for the calculation of scattering amplitudes in [5, 6] (subsection 2.1), the construction of composite operators in twistor space from Wilson loops as proposed in our recent papers [1, 2] (subsection 2.2), and the inverse soft limit, following [10], which allows to express the  $n$ -point interaction vertices of the elementary fields and composite operators in terms of  $(n - 1)$ -point vertices (subsection 2.3).

### 2.1 The twistor-space formalism

This section contains a very short introduction to the main ingredients of the twistor-space formulation of  $\mathcal{N} = 4$  SYM theory that serves mainly to set up the notation in use throughout the rest of the article. We refer the reader to [3, 61] and in particular [1, 2] for a more thorough introduction to twistor space, the formalism in use here as well as to our notation.

Each point  $(x, \theta)$  in complexified and conformally compactified super-Minkowski space  $\mathbb{M}^{4|8}$  corresponds to a unique projective line  $(x, \theta) \cong \mathbb{CP}^1$  in supertwistor space  $\mathbb{CP}^{3|4}$ . Any supertwistor  $\mathcal{Z}$  that is incident with  $(x, \theta)$ , i.e. that lies on the corresponding line, can be written as<sup>2</sup>

$$\mathcal{Z} = (\lambda_\alpha, \mu^{\dot{\alpha}}, \chi^a) = (\lambda_\alpha, ix^{\alpha\dot{\alpha}}\lambda_\alpha, i\theta^{a\alpha}\lambda_\alpha), \tag{2.1}$$

for  $\alpha \in \{1, 2\}$ ,  $\dot{\alpha} \in \{\dot{1}, \dot{2}\}$  and  $a \in \{1, 2, 3, 4\}$ . By slight abuse of notation, we will denote lines by  $x$  instead of  $(x, \theta)$  and will write  $\mathcal{Z}_x(\lambda)$  for the supertwistor (2.1) on the line given by  $(x, \theta)$ . Furthermore, we shall refer to the bosonic part  $(\lambda, \mu)$  of a supertwistor  $\mathcal{Z}$  as  $Z$ . Contractions of the indices are denoted by corresponding brackets as<sup>3</sup>

$$\langle \lambda \lambda' \rangle \equiv \lambda^\alpha \lambda'_\alpha, \quad [\mu \mu'] \equiv \mu_{\dot{\alpha}} \mu'^{\dot{\alpha}}, \quad \{ \chi \eta \} \equiv \chi^a \eta_a, \tag{2.2}$$

---

<sup>2</sup>Note that, unlike some texts on the subject, we include an  $i$  in the relation between  $\chi$  and  $\theta$  such that it mirrors the relation between  $\mu$  and  $x$ .

<sup>3</sup>Unlike for the angle bracket  $\langle \lambda \lambda' \rangle$ , where both  $\lambda$  and  $\lambda'$  live in the same space,  $\chi$  and  $\eta$  live in dual SU(4) representations. Nevertheless, the shorthand  $\{ \chi \eta \}$  will turn out to be useful.

with  $\epsilon^{\alpha\beta}\lambda_\alpha = \lambda^\beta$  and  $\epsilon^{12} = \epsilon^{i\dot{2}} = 1 = \epsilon_{21} = \epsilon_{\dot{2}i}$ . In Minkowski space, we use the mostly-minus metric with the definitions  $xy \equiv x_{\alpha\dot{\alpha}}y^{\alpha\dot{\alpha}} = 2x_\mu y^\mu$  and  $x^2 \equiv \frac{1}{2}x_{\alpha\dot{\alpha}}x^{\alpha\dot{\alpha}} = x_\mu x^\mu$ .

We now define a supertwistor connection  $(0, 1)$ -form  $\mathcal{A}(\mathcal{Z})$  on supertwistor space  $\mathbb{CP}^{3|4}$ . When expanded in the Graßmann variables  $\chi$  as in [62], it has the form

$$\mathcal{A}(\mathcal{Z}) = g^+ + \chi^a \bar{\psi}_a + \frac{1}{2}\chi^a \chi^b \phi_{ab} + \frac{1}{3!}\chi^a \chi^b \chi^c \psi^d \epsilon_{abcd} + \chi^1 \chi^2 \chi^3 \chi^4 g^-, \quad (2.3)$$

i.e. it combines all on-shell degrees of freedom of  $\mathcal{N} = 4$  SYM theory. The action of  $\mathcal{N} = 4$  SYM theory  $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2$  can be split into a self-dual part  $\mathcal{S}_1$  and an interaction part  $\mathcal{S}_2$  that contains the non-self-dual contributions. Translated to supertwistor space  $\mathbb{CP}^{3|4}$ , we can write the action in terms of the supertwistor field  $\mathcal{A}(\mathcal{Z})$  as the sum of a holomorphic Chern-Simons action  $\mathcal{S}_1$ , introduced in [4], and an interaction part  $\mathcal{S}_2$  [5] that is given by an integral over the space of lines in  $\mathbb{CP}^{3|4}$ , i.e. over Minkowski space  $\mathbb{M}^{4|8}$ . From the interaction part  $\mathcal{S}_2$ , one can derive the (color-stripped) vertices [10]

$$\mathbf{V}(\mathcal{A}_1, \dots, \mathcal{A}_n) = \int_{\mathbb{M}^{4|8}} \frac{d^4z d^8\theta}{(2\pi)^4} \int_{(\mathbb{CP}^1)^n} D\lambda_1 D\lambda_2 \dots D\lambda_n \frac{\mathcal{A}_1(\lambda_1)\mathcal{A}_2(\lambda_2)\dots\mathcal{A}_n(\lambda_n)}{\langle\lambda_1\lambda_2\rangle\dots\langle\lambda_{n-1}\lambda_n\rangle\langle\lambda_n\lambda_1\rangle}, \quad (2.4)$$

where  $D\lambda \equiv \frac{\langle\lambda d\lambda\rangle}{2\pi i}$  and  $n \geq 2$ . We refer to the action  $\mathcal{S}$  as being the *position twistor space* formulation of  $\mathcal{N} = 4$  SYM theory in order to distinguish the twistor space used here from the *momentum twistor space* introduced in [63] for the study of amplitudes in which twistors are introduced for the dual region variables.<sup>4</sup> For simplicity of notation, we shall from now on assume that  $\int D\lambda$  is integrated over  $\mathbb{CP}^1$  and that  $\int d^4z d^8\theta$  is integrated over  $\mathbb{M}^{4|8}$  and thus omit the space of integration.

Choosing an axial (or light-cone) gauge, the action  $\mathcal{S}_1$  becomes free and we obtain the propagator

$$\Delta(\mathcal{Z}_1, \mathcal{Z}_2) = i(4\pi)^2 \bar{\delta}^{2|4}(\mathcal{Z}_1, \star, \mathcal{Z}_2), \quad (2.5)$$

where  $\star \equiv \mathcal{Z}_\star = (Z_\star, 0) = (0, \zeta^\dot{\alpha}, 0)$  is a reference twistor. The interested reader is referred to [10] for a definition and discussion of the  $\bar{\delta}^{i|4}$  functions in supertwistor space. For us, the important properties are that the  $\bar{\delta}^{i|4}$  are homogeneous of degree zero in any of their arguments and that they are antisymmetric with respect to the exchange of any two arguments.

In order to obtain amplitudes or form factors, we need external states. An external state in position twistor space is [10]

$$\mathcal{A}_{\tilde{\mathcal{Z}}}(\mathcal{Z}) = 2\pi i \bar{\delta}^{3|4}(\mathcal{Z}, \tilde{\mathcal{Z}}), \quad (2.6)$$

where we denote the external supertwistors as  $\tilde{\mathcal{Z}}$ . We are of course interested in obtaining amplitudes and form factors in momentum space, so that we instead need to insert the

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<sup>4</sup>Amplitudes are known to be dual to light-like Wilson loops in the space of region (super)momenta, and the latter can be calculated using the twistor action as well [64]. However, this is not what we are doing here; we are introducing Wilson loops in twistor space for the operator  $\mathcal{O}(x)$  (2.9).

on-shell momentum representation<sup>5</sup>

$$\mathcal{A}_{\mathbf{P}}(\mathcal{Z}) = 2\pi i \int_{\mathbb{C}} \frac{ds}{s} e^{s([\bar{p}\mu] + \{\chi\eta\})} \bar{\delta}^2(s\lambda_{\alpha} - p_{\alpha}), \quad \mathcal{Z} = (\lambda_{\alpha}, \mu^{\dot{\alpha}}, \chi^a); \quad (2.7)$$

see for instance [10]. We denote the on-shell supermomenta in terms of super-spinor-helicity variables as  $\mathbf{P} = (p_{\alpha\dot{\alpha}}, \eta_a) \equiv (p_{\alpha}, \bar{p}_{\dot{\alpha}}, \eta_a)$ , i.e. we write an on-shell momentum as  $p_{\alpha\dot{\alpha}} = p_{\alpha}\bar{p}_{\dot{\alpha}}$ . Inserting the on-shell states (2.7) into the interaction vertices (2.4) immediately yields the tree-level MHV super amplitudes:

$$\mathcal{A}^{\text{MHV}}(\mathbf{P}_1, \dots, \mathbf{P}_n) = \mathbf{V}(\mathcal{A}_{\mathbf{P}_1}, \dots, \mathcal{A}_{\mathbf{P}_n}) = \frac{\delta^{4|8}(\sum_{j=1}^n \mathbf{P}_j)}{\langle p_1 p_2 \rangle \cdots \langle p_n p_1 \rangle}, \quad (2.8)$$

see [6] for details.

## 2.2 Composite operators from Wilson loops

We now recall the construction of composite operators in twistor space as introduced in [2]. The presentation here will be rather brief and we refer the reader to [2] for details.

Let us start by recalling the classification of composite operators in space-time, see e.g. [65, 66] for reviews. In the planar limit, it is sufficient to look at so-called single-trace operators, which are built from traces of products of covariantly transforming fields at a common space-time point  $x$ :

$$\mathcal{O}(x) = \text{Tr} \left( D^{k_1} A_1(x) D^{k_2} A_2(x) \cdots D^{k_L} A_L(x) \right). \quad (2.9)$$

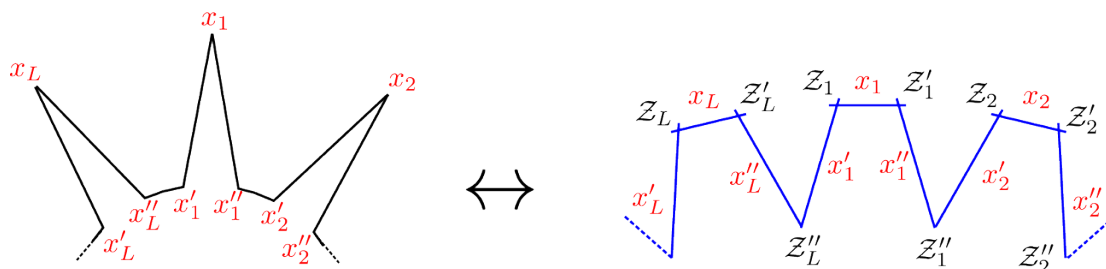
Here,  $D$  denotes a covariant derivative and the  $A_i$  can be scalars  $\phi_{ab}$ , fermions  $(\psi_{bcd})_{\alpha} = \epsilon_{abcd}\psi_{\alpha}^d$ , anti-fermions  $\bar{\psi}_{\alpha\dot{\alpha}}$  and the self-dual and anti-self-dual parts of the field strength  $F_{\alpha\beta}$  and  $\bar{F}_{\dot{\alpha}\dot{\beta}}$ . Using the equations of motions for these fields, the definition of the field strength and the Bianchi identity, covariant derivatives of fields with all combinations of spinor indices can be expressed in terms of irreducible fields, which are completely symmetric in all  $\alpha$  and  $\dot{\alpha}$  indices. They form the irreducible singleton representation  $\mathbb{V}_{\mathfrak{S}}$  of the symmetry algebra  $\mathfrak{psu}(2, 2|4)$ . Using superpolarization vectors  $\mathbb{T}_i = (\tau_i^{\alpha}, \bar{\tau}_i^{\dot{\alpha}}, \xi_i^a)$ , we can write the irreducible fields as

$$\begin{aligned} D^k A \in \{ & + \tau^{\alpha_1} \dots \tau^{\alpha_k} \bar{\tau}^{\dot{\alpha}_1} \dots \bar{\tau}^{\dot{\alpha}_{k+2}} & D_{\alpha_1 \dot{\alpha}_1} \cdots D_{\alpha_k \dot{\alpha}_k} \bar{F}_{\dot{\alpha}_{k+1} \dot{\alpha}_{k+2}}, \\ & + \tau^{\alpha_1} \dots \tau^{\alpha_k} \bar{\tau}^{\dot{\alpha}_1} \dots \bar{\tau}^{\dot{\alpha}_{k+1}} \xi^a & D_{\alpha_1 \dot{\alpha}_1} \cdots D_{\alpha_k \dot{\alpha}_k} \bar{\psi}_{\dot{\alpha}_{k+1} a}, \\ & - \tau^{\alpha_1} \dots \tau^{\alpha_k} \bar{\tau}^{\dot{\alpha}_1} \dots \bar{\tau}^{\dot{\alpha}_k} \xi^a \xi^b & D_{\alpha_1 \dot{\alpha}_1} \cdots D_{\alpha_k \dot{\alpha}_k} \phi_{ab}, \\ & - \tau^{\alpha_1} \dots \tau^{\alpha_{k+1}} \bar{\tau}^{\dot{\alpha}_1} \dots \bar{\tau}^{\dot{\alpha}_k} \xi^a \xi^b \xi^c & D_{\alpha_1 \dot{\alpha}_1} \cdots D_{\alpha_k \dot{\alpha}_k} \psi_{\alpha_{k+1} abc}, \\ & + \tau^{\alpha_1} \dots \tau^{\alpha_{k+2}} \bar{\tau}^{\dot{\alpha}_1} \dots \bar{\tau}^{\dot{\alpha}_k} \xi^a \xi^b \xi^c \xi^d & D_{\alpha_1 \dot{\alpha}_1} \cdots D_{\alpha_k \dot{\alpha}_k} F_{\alpha_{k+1} \alpha_{k+2} abc} \}, \end{aligned} \quad (2.10)$$

where  $F_{\alpha_{k+1} \alpha_{k+2} abc} = \frac{1}{4!} \epsilon_{abcd} F_{\alpha_{k+1} \alpha_{k+2}}$ . The indices are recovered by taking appropriately weighted derivatives with respect to  $\tau$ ,  $\bar{\tau}$  and  $\xi$ .<sup>6</sup>

<sup>5</sup>Note that the  $\lambda_{\alpha}$  are components of the supertwistor  $\mathcal{Z}$  in (2.1) which are integrated over at each vertex (2.4). In particular, they are not spinor-helicity variables parametrizing momenta. When inserting the supermomentum eigenstate (2.7), however, the  $s$  and  $\lambda$  integration in combination with the delta function effectively replace  $\lambda_{\alpha}$  with the spinor-helicity variable  $p_{\alpha}$ .

<sup>6</sup>See [2] for a pedagogical example.



**Figure 1.** This figure illustrates the  $3L$ -gonal “cogwheel” light-like Wilson loop that we use to construct an operator containing  $L$  irreducible fields.

We can construct the twistor-space vertex of a composite operator built from  $L$  irreducible fields by taking derivatives of a polygonal light-like Wilson loop in twistor space and shrinking the Wilson loop to a line in twistor space, i.e. a point in space-time. One possible choice for the geometry of the Wilson loop has  $3L$  vertices in Minkowski space that we label  $x'_1, x_1, x''_1, \dots, x'_L, x_L, x''_L$ .<sup>7</sup> It can be drawn in the shape of a cogwheel, see the left-hand side of figure 1. In twistor space, each pair of neighboring points intersects and we label the intersection points  $\mathcal{Z}_i = x_i \cap x'_i$ ,  $\mathcal{Z}'_i = x_i \cap x''_i$  and  $\mathcal{Z}''_i = x''_i \cap x'_{i+1}$ , see the right-hand side of figure 1. Then, the cogwheel Wilson loop is simply obtained by multiplying parallel propagators  $U$  connecting adjacent twistors as

$$\begin{aligned} \mathcal{W}(x'_1, x_1, x''_1, \dots, x'_L, x_L, x''_L) = \text{Tr} \left[ U_{x'_1}(\mathcal{Z}''_L, \mathcal{Z}_1) U_{x_1}(\mathcal{Z}_1, \mathcal{Z}'_1) U_{x''_1}(\mathcal{Z}'_1, \mathcal{Z}''_1) \right. \\ U_{x'_2}(\mathcal{Z}''_1, \mathcal{Z}_2) U_{x_2}(\mathcal{Z}_2, \mathcal{Z}'_2) U_{x''_2}(\mathcal{Z}'_2, \mathcal{Z}''_2) \cdots \\ \left. \cdots U_{x'_L}(\mathcal{Z}''_{L-1}, \mathcal{Z}_L) U_{x_L}(\mathcal{Z}_L, \mathcal{Z}'_L) U_{x''_L}(\mathcal{Z}'_L, \mathcal{Z}''_L) \right], \end{aligned} \quad (2.11)$$

where the parallel propagator is [13]

$$U_x(\mathcal{Z}_1, \mathcal{Z}_2) = 1 + \sum_{m=1}^{\infty} \int \frac{\langle \lambda_1 \lambda_2 \rangle D\tilde{\lambda}_1 \cdots D\tilde{\lambda}_m}{\langle \lambda_1 \tilde{\lambda}_1 \rangle \langle \tilde{\lambda}_1 \tilde{\lambda}_2 \rangle \cdots \langle \tilde{\lambda}_m \lambda_2 \rangle} \mathcal{A}(\mathcal{Z}_x(\tilde{\lambda}_1)) \cdots \mathcal{A}(\mathcal{Z}_x(\tilde{\lambda}_m)). \quad (2.12)$$

In order to obtain vertices for the composite operator  $\mathcal{O}$  (2.9), we act on  $\mathcal{W}$  with a differential operator, which we call the *forming operator*. It is the product of respective forming operators for the irreducible fields  $D^{k_i} A_i$  out of which  $\mathcal{O}$  is built:

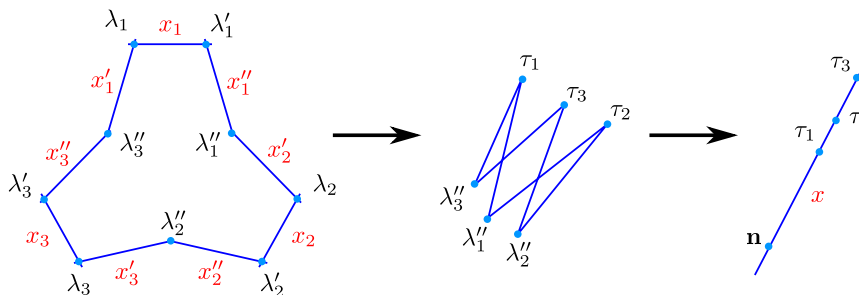
$$\mathbf{F}_{\mathcal{O}} = \prod_{i=1}^L \mathbf{F}_{D^{k_i} A_i}. \quad (2.13)$$

The forming operators of the irreducible fields (2.10) are

$$\mathbf{F}_{D^k A} = -\frac{\lambda^\alpha \lambda'^\beta}{\langle \lambda \lambda' \rangle} \left( -i\tau^\gamma \bar{\tau}^{\dot{\gamma}} \partial_{\gamma\dot{\gamma}} \right)^k \begin{cases} (-i\bar{\tau}^{\dot{\alpha}} \partial_{\alpha\dot{\alpha}})(-i\bar{\tau}^{\dot{\beta}} \partial_{\beta\dot{\beta}}) & \text{for } A = \bar{F}, \\ (-i\bar{\tau}^{\dot{\alpha}} \partial_{\alpha\dot{\alpha}})(-i\xi^a \partial_{\beta a}) & \text{for } A = \bar{\psi}, \\ (-i\xi^a \partial_{\alpha a})(-i\xi^b \partial_{\beta b}) & \text{for } A = \phi, \\ (-i\xi^a \partial_{\alpha a})(-i\xi^b \partial_{\beta b})(-i\tau^\gamma \xi^c \partial_{\gamma c}) & \text{for } A = \psi, \\ (-i\xi^a \partial_{\alpha a})(-i\xi^b \partial_{\beta b})(-i\tau^\gamma \xi^c \partial_{\gamma c})^2 & \text{for } A = F, \end{cases} \quad (2.14)$$

<sup>7</sup>There is large freedom in the choice of the geometry of the Wilson loop provided that certain minimal requirements are satisfied, see [2]. Once the Wilson loop is properly shrunk to a line in twistor space, the operator vertex is blind to that choice of geometry.





**Figure 2.** This figure illustrates the operator limit for  $L = 3$ . To visualize the process a bit better, as a first step, we set  $\lambda_i = \lambda'_i = \tau_i$  while bringing the  $x_i$  closer to each other. The second step then just sends all  $x_i$  to  $x$  and  $\lambda''_i \rightarrow \mathbf{n}$ .

where we have used the abbreviations

$$\partial_{\alpha\dot{\alpha}} \equiv \frac{\partial}{\partial x^{\alpha\dot{\alpha}}}, \quad \partial_{\alpha a} \equiv \frac{\partial}{\partial \theta^{\alpha a}} \quad (2.15)$$

and suppressed the index  $i$ . The forming operator  $\mathbf{F}_{D^{k_i} A_i}$  is then simply defined as (2.14) acting on the edge  $(x_i, \theta_i)$  of the Wilson loop of figure 1.

Finally, we take the operator limit, which we denote by  $\circlearrowleft \rightarrow \cdot$  or by  $\circlearrowleft \rightarrow \cdot_x$  if we want to emphasize the point to which the loop shrinks. We refer to appendix A of [2] for more details on how to take the limit, for now it is only important that the operator limit sets

$$x_i, x'_i, x''_i \rightarrow x, \quad \lambda_i, \lambda'_i \rightarrow \tau_i, \quad \lambda''_i \rightarrow \mathbf{n}; \quad (2.16)$$

see figure 2. The reference spinor  $\mathbf{n}$  has to drop out of any physical quantity. Moreover, we set  $\theta = 0$  to obtain only the component operator  $\mathcal{O}$  and not part of its supermultiplet.

Combining everything, we obtain the following expression for the operator vertices:

$$\mathbf{W}_{\mathcal{O}(x)} = \lim_{\circlearrowleft \rightarrow \cdot} \mathbf{F}_{\mathcal{O}} \mathcal{W}|_{\theta=0}. \quad (2.17)$$

Similar to the case of amplitudes, inserting momentum eigenstates (2.7) into the operator vertex (2.17) immediately yields all MHV form factors, cf. [2].

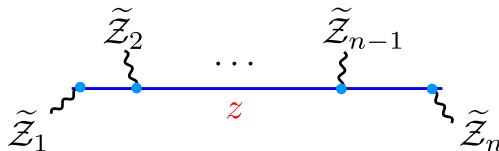
### 2.3 The inverse soft limit

Via the so-called inverse soft limit, the  $n$ -point twistor-space vertices of the elementary interactions can be expressed in terms of  $(n - 1)$ -point vertices. This procedure played a crucial role in the calculation of amplitudes beyond MHV level in position twistor space [10]. Here, we find a similar recursion for the Wilson loop vertices, which will play an equally important role in the calculation of form factors beyond MHV level.

We first review the case of the interaction vertices. The  $n$ -point vertices

$$\mathbf{V}(\tilde{\mathcal{Z}}_1, \dots, \tilde{\mathcal{Z}}_n) \equiv \mathbf{V}(\mathcal{A}_{\tilde{\mathcal{Z}}_1}, \dots, \mathcal{A}_{\tilde{\mathcal{Z}}_n}) \quad (2.18)$$

of (2.4), see figure 3, can be reduced to  $(n - 1)$ -point vertices via recursion relations [10] that allow us to reduce them all the way down to the two-point vertices. Specifically, the



**Figure 3.** The  $n$ -point vertex (2.4) from the action  $\mathcal{S}_2$ . Throughout this paper, labels of the vertices are read off clockwise.

inverse soft limit is derived by first parametrizing the  $\lambda_{j\alpha}$  as  $(1, \sigma_j)_\alpha$  so that  $D\lambda_j = \frac{d\sigma_j}{2\pi i}$  and  $\langle \lambda_j \lambda_{j+1} \rangle = \sigma_{j+1} - \sigma_j$ . We now choose a  $k \in \{1, \dots, n\}$  and replace  $\sigma_k$  by  $s = \frac{\sigma_k - \sigma_{k+1}}{\sigma_{k-1} - \sigma_k}$  such that  $\frac{ds}{s} = -\frac{(\sigma_{k+1} - \sigma_{k-1})d\sigma_k}{(\sigma_{k+1} - \sigma_k)(\sigma_k - \sigma_{k-1})}$  and  $\mathcal{Z}_z(\sigma_k) = \frac{1}{1+s} [s\mathcal{Z}_z(\sigma_{k-1}) + \mathcal{Z}_z(\sigma_{k+1})]$ . Employing  $\int \frac{ds}{s} \bar{\delta}^{3|4}(s\mathcal{Z}_z(\sigma_{k-1}) + \mathcal{Z}_z(\sigma_{k+1}), \tilde{\mathcal{Z}}_k) = -\bar{\delta}^{2|4}(\tilde{\mathcal{Z}}_{k-1}, \tilde{\mathcal{Z}}_k, \tilde{\mathcal{Z}}_{k+1})$ , where we have used the  $\bar{\delta}^{3|4}$  to replace the  $\mathcal{Z}_z(\sigma_{k\pm 1})$  by  $\tilde{\mathcal{Z}}_{k\pm 1}$ , leads to

$$\mathbf{V}(\tilde{\mathcal{Z}}_1, \dots, \tilde{\mathcal{Z}}_n) = \mathbf{V}(\tilde{\mathcal{Z}}_1, \dots, \tilde{\mathcal{Z}}_{k-1}, \tilde{\mathcal{Z}}_{k+1}, \dots, \tilde{\mathcal{Z}}_n) \bar{\delta}^{2|4}(\tilde{\mathcal{Z}}_{k-1}, \tilde{\mathcal{Z}}_k, \tilde{\mathcal{Z}}_{k+1}). \quad (2.19)$$

Equation (2.19) implies that the vertex can be written as the product

$$\mathbf{V}(\tilde{\mathcal{Z}}_1, \dots, \tilde{\mathcal{Z}}_n) = \mathbf{V}(\tilde{\mathcal{Z}}_1, \tilde{\mathcal{Z}}_2) \prod_{i=3}^n \bar{\delta}^{2|4}(\tilde{\mathcal{Z}}_1, \tilde{\mathcal{Z}}_{i-1}, \tilde{\mathcal{Z}}_i). \quad (2.20)$$

There are different ways of writing the  $n$ -point amplitude that are related by identities of the kind  $\mathbf{V}(\tilde{\mathcal{Z}}_1, \tilde{\mathcal{Z}}_2, \tilde{\mathcal{Z}}_3) \bar{\delta}^{2|4}(\tilde{\mathcal{Z}}_1, \tilde{\mathcal{Z}}_3, \tilde{\mathcal{Z}}_4) = \mathbf{V}(\tilde{\mathcal{Z}}_2, \tilde{\mathcal{Z}}_3, \tilde{\mathcal{Z}}_4) \bar{\delta}^{2|4}(\tilde{\mathcal{Z}}_2, \tilde{\mathcal{Z}}_4, \tilde{\mathcal{Z}}_1)$ , which come from the total antisymmetry of the  $\bar{\delta}^{2|4}$  and from the different choices of points  $k$  to remove using (2.19).

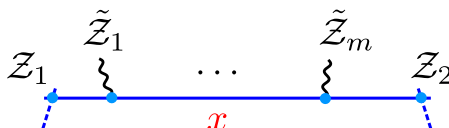
In space-time, MHV form factors can be constructed via inverse soft limits as well [67]. In our twistor-space formulation, this construction is based on the parallel propagator (2.12). Using the external states in position twistor space (2.6), the parallel propagator  $U_x(\mathcal{Z}_1, \mathcal{Z}_2)$  in (2.12) leads to the following Wilson line vertex:

$$\begin{aligned} \mathbf{W}_x(\mathcal{Z}_1, \mathcal{Z}_2; \tilde{\mathcal{Z}}_1, \dots, \tilde{\mathcal{Z}}_m) &\equiv \mathbf{W}_x(\mathcal{Z}_1, \mathcal{Z}_2; \mathcal{A}_{\tilde{\mathcal{Z}}_1}, \dots, \mathcal{A}_{\tilde{\mathcal{Z}}_m}) \\ &= \int \frac{\langle \lambda_1 \lambda_2 \rangle \prod_{j=1}^m \mathcal{A}_{\tilde{\mathcal{Z}}_j}(\mathcal{Z}_x(\tilde{\lambda}_j)) D\tilde{\lambda}_j}{\langle \lambda_1 \tilde{\lambda}_1 \rangle \langle \tilde{\lambda}_1 \tilde{\lambda}_2 \rangle \cdots \langle \tilde{\lambda}_m \lambda_2 \rangle}, \end{aligned} \quad (2.21)$$

see figure 4. In particular, for  $m = 0$  we set  $\mathbf{W}_x(\mathcal{Z}_1, \mathcal{Z}_2; ) \equiv 1$ . We keep in mind the Wilson loop origin of the vertices  $\mathbf{W}$  and hence denote by  $\mathcal{Z}_1$  and  $\mathcal{Z}_2$  the two twistors at which the line  $x$  intersects with the neighboring lines in the Wilson loop, see figure 1. Furthermore, we write  $\tilde{\mathcal{Z}}_i$  for the twistors on the line  $x$  where on-shell states or propagators are to be attached. In particular, the smallest non-trivial vertex is

$$\begin{aligned} \mathbf{W}_x(\mathcal{Z}_1, \mathcal{Z}_2; \tilde{\mathcal{Z}}) &= \int D\tilde{\lambda} \frac{\langle \lambda_1 \lambda_2 \rangle}{\langle \lambda_1 \tilde{\lambda} \rangle \langle \tilde{\lambda} \lambda_2 \rangle} 2\pi i \bar{\delta}^{3|4}(\tilde{\mathcal{Z}}, \mathcal{Z}_x(\tilde{\lambda})) \\ &= \int_{\mathbb{C}} \frac{du}{u} \bar{\delta}^{3|4}(\mathcal{Z}_1 + u\mathcal{Z}_2, \tilde{\mathcal{Z}}) = \bar{\delta}^{2|4}(\mathcal{Z}_1, \tilde{\mathcal{Z}}, \mathcal{Z}_2), \end{aligned} \quad (2.22)$$

where in the second line we used the parametrization of the line  $x$  given by  $\mathcal{Z}_x(\tilde{\lambda}) = \mathcal{Z}_1 + u\mathcal{Z}_2$ , which implies that  $\langle \lambda_1 \tilde{\lambda} \rangle = u\langle \lambda_1 \lambda_2 \rangle$ ,  $\langle \tilde{\lambda} \lambda_2 \rangle = \langle \lambda_1 \lambda_2 \rangle$  and  $D\tilde{\lambda} = \langle \lambda_1 \lambda_2 \rangle \frac{du}{2\pi i}$ .



**Figure 4.** The  $m$ -point vertex (2.21) from the parallel propagator  $U_x(\mathcal{Z}_1, \mathcal{Z}_2)$ .

Using now the same method as for the vertices  $\mathbf{V}$ , we arrive at the identity

$$\begin{aligned} \mathbf{W}_x(\mathcal{Z}_1, \mathcal{Z}_2; \tilde{\mathcal{Z}}_1, \dots, \tilde{\mathcal{Z}}_m) \\ = \mathbf{W}_x(\mathcal{Z}_1, \mathcal{Z}_2; \tilde{\mathcal{Z}}_1, \dots, \tilde{\mathcal{Z}}_{k-1}, \tilde{\mathcal{Z}}_{k+1}, \dots, \tilde{\mathcal{Z}}_m) \delta^{2|4}(\tilde{\mathcal{Z}}_{k-1}, \tilde{\mathcal{Z}}_k, \tilde{\mathcal{Z}}_{k+1}), \end{aligned} \quad (2.23)$$

where  $k \in \{1, \dots, m\}$  and it is understood that  $\tilde{\mathcal{Z}}_0 \equiv \mathcal{Z}_1$  and  $\tilde{\mathcal{Z}}_{m+1} \equiv \mathcal{Z}_2$ . Using the relation (2.23), we can reduce every Wilson line vertex to the minimal Wilson line vertex (2.22):

$$\mathbf{W}_x(\mathcal{Z}_1, \mathcal{Z}_2; \tilde{\mathcal{Z}}_1, \dots, \tilde{\mathcal{Z}}_m) = \prod_{j=1}^m \delta^{2|4}(\tilde{\mathcal{Z}}_{j-1}, \tilde{\mathcal{Z}}_j, \mathcal{Z}_2). \quad (2.24)$$

The expression (2.24) has a very geometric interpretation, which makes its origin obvious and goes as follows. The vertices  $\mathbf{W}_x(\mathcal{Z}_1, \mathcal{Z}_2; \tilde{\mathcal{Z}}_1, \dots, \tilde{\mathcal{Z}}_m)$  in (2.21) describe a Wilson line in position twistor space. Hence, all position twistors  $\mathcal{Z}_1, \mathcal{Z}_2, \tilde{\mathcal{Z}}_1, \dots, \tilde{\mathcal{Z}}_m$  have to lie on one line. This is expressed in (2.24) by enforcing each of the first  $m$  pairs of neighboring position twistors to be collinear with the last position twistor. However, a lot of equivalent ways exist to express the collinearity of all  $m + 2$  position twistors by forcing  $m$  triplets of them to be collinear.

### 3 $N^k$ MHV form factors in position twistor space

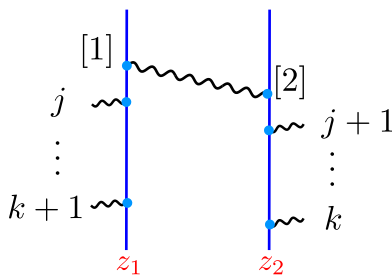
Using the building blocks set up in the previous section, we can now calculate the diagrams for  $N^k$ MHV tree-level form factors of general operators in position twistor space. We start discussing the case of NMHV form factors and then generalize to arbitrary  $k$ . This calculation is similar to the one for amplitudes [10], and we will frequently refer to this case for intuition.

#### 3.1 NMHV amplitudes and form factors

In the twistor-space formalism, the MHV degree  $k$  directly translates to twistor-space diagrams involving  $k$  propagators at tree level. At the NMHV level, we therefore have exactly  $k = 1$  propagator.

**Amplitudes.** Let us first review the case of amplitudes [10]. In this case, the single propagator has to connect two interaction vertices (2.18). The most general such diagram is shown in figure 5.<sup>8</sup> The interaction vertex  $\mathbf{V}$  can involve two or more fields. However, the two-point vertex yields zero when it is directly connected to an external field, see [10].

<sup>8</sup>From now on, in figures we shall refrain from writing  $\tilde{\mathcal{Z}}_j$  for the supertwistors of the external on-shell states and just label them by their index  $j$ .



**Figure 5.** The most general diagram for amplitudes at NMHV level in position twistor space. The points  $z_1$  and  $z_2$  are integrated over as required by the definition of the vertex  $\mathbf{V}$ .

Hence, we can restrict ourselves to the case of at least three fields in  $\mathbf{V}$ , which allows us to use the inverse soft limit. Using (2.19), we find

$$\begin{aligned}
 & \int D^{3|4} \mathcal{Z}_{[1]} D^{3|4} \mathcal{Z}_{[2]} \mathbf{V}(\dots, \tilde{\mathcal{Z}}_j, \mathcal{Z}_{[1]}, \tilde{\mathcal{Z}}_{k+1}, \dots) \Delta(\mathcal{Z}_{[1]}, \mathcal{Z}_{[2]}) \mathbf{V}(\dots, \tilde{\mathcal{Z}}_k, \mathcal{Z}_{[2]}, \tilde{\mathcal{Z}}_{j+1}, \dots) \\
 &= i(4\pi)^2 \mathbf{V}(\dots, \tilde{\mathcal{Z}}_j, \tilde{\mathcal{Z}}_{k+1}, \dots) \mathbf{V}(\dots, \tilde{\mathcal{Z}}_k, \tilde{\mathcal{Z}}_{j+1}, \dots) \\
 & \quad \times \int D^{3|4} \mathcal{Z}_{[1]} D^{3|4} \mathcal{Z}_{[2]} \delta^{2|4}(\tilde{\mathcal{Z}}_j, \mathcal{Z}_{[1]}, \tilde{\mathcal{Z}}_{k+1}) \delta^{2|4}(\mathcal{Z}_{[1]}, \star, \mathcal{Z}_{[2]}) \delta^{2|4}(\tilde{\mathcal{Z}}_k, \mathcal{Z}_{[2]}, \tilde{\mathcal{Z}}_{j+1}) \\
 &= i(4\pi)^2 \mathbf{V}(\dots, \tilde{\mathcal{Z}}_j, \tilde{\mathcal{Z}}_{k+1}, \dots) \mathbf{V}(\dots, \tilde{\mathcal{Z}}_k, \tilde{\mathcal{Z}}_{j+1}, \dots) [\tilde{\mathcal{Z}}_{k+1}, \tilde{\mathcal{Z}}_j, \star, \tilde{\mathcal{Z}}_{j+1}, \tilde{\mathcal{Z}}_k],
 \end{aligned} \tag{3.1}$$

where the five-bracket is defined as

$$[\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4, \mathcal{Z}_5] \equiv \delta^{0|4}(\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4, \mathcal{Z}_5) = \frac{\prod_{a=1}^4 (\chi_1(2345) + \text{cyclic})^a}{(1234)(2345)(3451)(4512)(5123)} \tag{3.2}$$

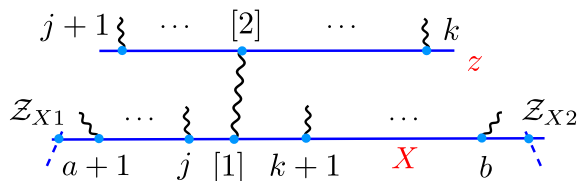
with the four-bracket  $(ijkl) \equiv \det(Z_i, Z_j, Z_k, Z_l)$  given by the determinant of four bosonic twistors. We recall that the five-bracket, like  $\delta^{i|4}$ , is totally antisymmetric in its arguments. Summing over all diagrams of this type yields the NMHV amplitude

$$\begin{aligned}
 \mathbb{A}^{\text{NMHV}} &= -i(4\pi)^2 \sum_{1 \leq j < k \leq n} \mathbf{V}(\dots, \tilde{\mathcal{Z}}_j, \tilde{\mathcal{Z}}_{k+1}, \dots) \\
 & \quad \times [\tilde{\mathcal{Z}}_j, \tilde{\mathcal{Z}}_{j+1}, \star, \tilde{\mathcal{Z}}_k, \tilde{\mathcal{Z}}_{k+1}] \mathbf{V}(\dots, \tilde{\mathcal{Z}}_k, \tilde{\mathcal{Z}}_{j+1}, \dots),
 \end{aligned} \tag{3.3}$$

where we have used said antisymmetry. In section 4.1, we shall derive the usual expression for the NMHV amplitudes in momentum twistor space from (3.3) by inserting the momentum eigenstates and computing the integrals that are left implicit in  $\mathbf{V}$ .

**Form factors.** Let us now return to form factors and calculate the NMHV form factors of Wilson loops as well as of the corresponding general composite operators. The propagator in this case has to connect a Wilson line vertex (2.21) corresponding to an edge of the Wilson loop (2.11) with an interaction vertex (2.18). In the end, we have to sum over all such edges and over all possible combinations to distribute the external fields on the interaction vertex  $\mathbf{V}$  and the Wilson line vertices  $\mathbf{W}$  of the different edges.

To simplify the notation and keep the presentation transparent, we will consider a particular edge  $X$  with vertex  $\mathbf{W}_X(\mathcal{Z}_{X1}, \mathcal{Z}_{X2}; \tilde{\mathcal{Z}}_{a+1}, \dots, \tilde{\mathcal{Z}}_b)$ , where  $\mathcal{Z}_{X1}$  ( $\mathcal{Z}_{X2}$ ) denotes



**Figure 6.** A generic NMHV diagram in which an interaction vertex  $\mathbf{V}$  is connected to a  $\mathbf{W}_X$  one. The line  $X$  in twistor space contains the two twistors  $\mathcal{Z}_{X1}$  and  $\mathcal{Z}_{X2}$  at which it intersects the lines of the cogwheel Wilson loop adjacent to it.

the twistor corresponding to the intersection with the previous (next) edge. Concretely,  $(X, \mathcal{Z}_{X1}, \mathcal{Z}_{X2})$  is an element of the union  $\{(x_i, \mathcal{Z}_i, \mathcal{Z}'_i) | i = 1, \dots, L\} \cup \{(x''_i, \mathcal{Z}'_i, \mathcal{Z}''_i) | i = 1, \dots, L\} \cup \{(x'_i, \mathcal{Z}''_{i-1}, \mathcal{Z}_i) | i = 1, \dots, L\}$ . Moreover, we let  $\tilde{\mathcal{Z}}_{a+1}$  ( $\tilde{\mathcal{Z}}_b$ ) denote the first (last) external field on that edge. For a generic distribution of the external fields, shown in figure 6, we then find

$$\begin{aligned}
 & \int D^{3|4} \mathcal{Z}_{[1]} D^{3|4} \mathcal{Z}_{[2]} \mathbf{W}_X(\mathcal{Z}_{X1}, \mathcal{Z}_{X2}; \tilde{\mathcal{Z}}_{a+1}, \dots, \tilde{\mathcal{Z}}_j, \mathcal{Z}_{[1]}, \tilde{\mathcal{Z}}_{k+1}, \dots, \tilde{\mathcal{Z}}_b) \Delta(\mathcal{Z}_{[1]}, \mathcal{Z}_{[2]}) \\
 & \quad \times \mathbf{V}(\dots, \tilde{\mathcal{Z}}_k, \mathcal{Z}_{[2]}, \tilde{\mathcal{Z}}_{j+1}, \dots) \\
 & = i(4\pi)^2 \mathbf{W}_X(\mathcal{Z}_{X1}, \mathcal{Z}_{X2}; \tilde{\mathcal{Z}}_{a+1}, \dots, \tilde{\mathcal{Z}}_j, \tilde{\mathcal{Z}}_{k+1}, \dots, \tilde{\mathcal{Z}}_b) \mathbf{V}(\dots, \tilde{\mathcal{Z}}_k, \tilde{\mathcal{Z}}_{j+1}, \dots) \\
 & \quad \times \int D^{3|4} \mathcal{Z}_{[1]} D^{3|4} \mathcal{Z}_{[2]} \delta^{2|4}(\tilde{\mathcal{Z}}_j, \mathcal{Z}_{[1]}, \tilde{\mathcal{Z}}_{k+1}) \delta^{2|4}(\mathcal{Z}_{[1]}, \star, \mathcal{Z}_{[2]}) \delta^{2|4}(\tilde{\mathcal{Z}}_k, \mathcal{Z}_{[2]}, \tilde{\mathcal{Z}}_{j+1}) \\
 & = i(4\pi)^2 \mathbf{W}_X(\mathcal{Z}_{X1}, \mathcal{Z}_{X2}; \tilde{\mathcal{Z}}_{a+1}, \dots, \tilde{\mathcal{Z}}_j, \tilde{\mathcal{Z}}_{k+1}, \dots, \tilde{\mathcal{Z}}_b) \mathbf{V}(\dots, \tilde{\mathcal{Z}}_k, \tilde{\mathcal{Z}}_{j+1}, \dots) \\
 & \quad \times [\tilde{\mathcal{Z}}_{k+1}, \tilde{\mathcal{Z}}_j, \star, \tilde{\mathcal{Z}}_{j+1}, \tilde{\mathcal{Z}}_k],
 \end{aligned} \tag{3.4}$$

which is completely analogous to (3.1) except that we have also used the inverse soft limit (2.23) for the Wilson line vertex. We have to consider two special distributions. If  $a = j$ , the external supertwistor  $\tilde{\mathcal{Z}}_j$  is not on the edge  $X$  and we have to use  $\mathcal{Z}_{X1}$  for the inverse soft limit. We then need to replace  $\tilde{\mathcal{Z}}_j \rightarrow \mathcal{Z}_{X1}$  in the five-bracket of the last line of (3.4). Similarly, if  $b = k$ , the external supertwistor  $\tilde{\mathcal{Z}}_{k+1}$  is not on the edge  $X$ , which leads to a replacement  $\tilde{\mathcal{Z}}_{k+1} \rightarrow \mathcal{Z}_{X2}$  in the five-bracket. In order to write the result in a condensed form, we shall denote by

$$\tilde{\mathcal{Z}}_j \xrightarrow{j=a} \mathcal{Z}_{X1} \tag{3.5}$$

the supertwistor that is  $\tilde{\mathcal{Z}}_j$  if  $j \neq a$  and becomes equal to  $\mathcal{Z}_{X1}$  if  $j = a$ .

It follows from the preceding discussion that the total position twistor space form factor is

$$\begin{aligned}
 \mathbb{F}^{\text{NMHV}} & = -i(4\pi)^2 \sum_X \left\{ \sum_{a \leq j < k \leq b} \mathbf{W}_X(\mathcal{Z}_{X1}, \mathcal{Z}_{X2}; \tilde{\mathcal{Z}}_{a+1}, \dots, \tilde{\mathcal{Z}}_j, \tilde{\mathcal{Z}}_{k+1}, \dots, \tilde{\mathcal{Z}}_b) \right. \\
 & \quad \times \mathbf{V}(\dots, \tilde{\mathcal{Z}}_k, \tilde{\mathcal{Z}}_{j+1}, \dots) \left[ \tilde{\mathcal{Z}}_j \xrightarrow{j=a} \mathcal{Z}_{X1}, \tilde{\mathcal{Z}}_{j+1}, \star, \tilde{\mathcal{Z}}_k, \tilde{\mathcal{Z}}_{k+1} \xrightarrow{k=b} \mathcal{Z}_{X2} \right] \left. \right\} \\
 & \quad \times \prod_{X' \neq X} \mathbf{W}_{X'}(\mathcal{Z}_{X1'}, \mathcal{Z}_{X2'}; \dots),
 \end{aligned} \tag{3.6}$$

where we have kept the sum over all distributions of the remaining  $n - b + a$  external fields on the  $3L - 1$  remaining edges implicit.

To compute the NMHV form factor  $\mathbb{F}_{\mathcal{O}}^{\text{NMHV}}$  of a specific operator  $\mathcal{O}$  (2.9), we must now act with the forming factor, do the integral and take the operator limit as in (2.17). We will do this and rephrase the result in momentum twistor space in several cases in section 4.

### 3.2 $N^k$ MHV form factors

Having shown how to compute the NMHV form factors in position twistor space, let us now consider the case of arbitrary high MHV degree. The discussion parallels the one for amplitudes in [10], and hence we shall mostly concern ourselves with highlighting the differences. There are three different kinds of twistor-space diagrams for amplitudes, namely *generic*, *boundary* and *boundary-boundary*, and the same applies to form factors. The types of diagrams differ in the relative positions of the propagators in the (interaction) vertices.

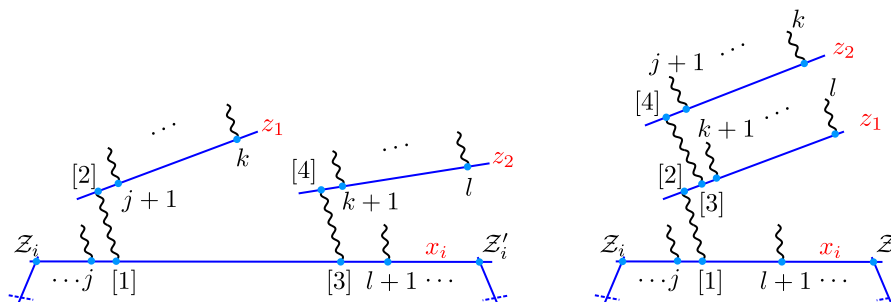
**Generic.** For these diagrams, no adjacent propagators occur at any vertex such that they can be calculated in complete analogy to the NMHV diagrams. The corresponding contribution to the amplitude/form factor is given by products of twistor-space vertices and of R-invariants of the kind  $[\tilde{\mathcal{Z}}_{a_i}, \tilde{\mathcal{Z}}_{a_j}, \star, \tilde{\mathcal{Z}}_{a_k}, \tilde{\mathcal{Z}}_{a_l}]$  with the  $\tilde{\mathcal{Z}}_{a_n}$  indicating some external supertwistors. For some diagrams, there are no external particles to the left (right) of a propagator on the Wilson line vertex  $\mathbf{W}_x$ . In this case, we need to replace the external twistors  $\tilde{\mathcal{Z}}_j$  ( $\tilde{\mathcal{Z}}_{k+1}$ ) by the appropriate twistors fixing the line  $x$ , cf. (3.5).

**Boundary.** In this case, some propagators are inserted next to each other, but each vertex for which that happens is either a Wilson line vertex  $\mathbf{W}$  or it is an interaction vertex  $\mathbf{V}$  but has at least two external particles, see figure 7 for one example of each. This allows us to use the inverse soft limit and the resulting delta functions to do all twistor integrals. The diagram in position twistor space thus evaluates to a product of R-invariants and twistor-space vertices  $\mathbf{V}$  and  $\mathbf{W}$ . However, unlike in the generic case, the supertwistors entering the R-invariants are not the external ones but rather are obtained by simple geometric means, namely intersecting a line and a plane, as explained in the following. Assuming that there is a propagator between the line given by the twistors  $\tilde{\mathcal{Z}}_{a_1}$  and  $\tilde{\mathcal{Z}}_{a_2}$  and another line given by  $\tilde{\mathcal{Z}}_{b_1}$  and  $\tilde{\mathcal{Z}}_{b_2}$ , the contribution is given by the R-invariant

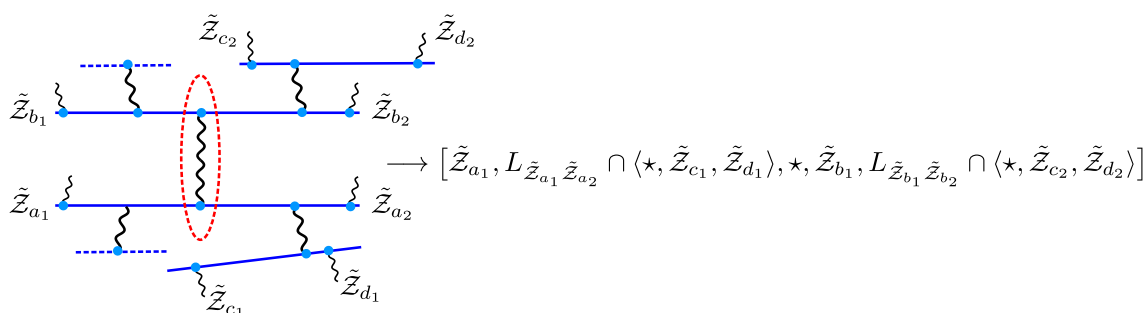
$$\left[ \tilde{\mathcal{Z}}_{a_1}, \hat{\tilde{\mathcal{Z}}}_{a_2}, \star, \tilde{\mathcal{Z}}_{b_1}, \hat{\tilde{\mathcal{Z}}}_{b_2} \right]. \tag{3.7}$$

Here,  $\hat{\tilde{\mathcal{Z}}}_{a_2} = \tilde{\mathcal{Z}}_{a_2}$  if there is no propagator to the right of the insertion twistor on the line  $L_{\tilde{\mathcal{Z}}_{a_1}\tilde{\mathcal{Z}}_{a_2}}$  given by the twistors  $\tilde{\mathcal{Z}}_{a_1}$  and  $\tilde{\mathcal{Z}}_{a_2}$ . Otherwise,  $\hat{\tilde{\mathcal{Z}}}_{a_2}$  is the intersection of the line  $L_{\tilde{\mathcal{Z}}_{a_1}\tilde{\mathcal{Z}}_{a_2}}$  and the plane  $\langle \star, \tilde{\mathcal{Z}}_{c_1}, \tilde{\mathcal{Z}}_{d_1} \rangle$  spanned by the twistors  $\tilde{\mathcal{Z}}_{c_1}$ ,  $\tilde{\mathcal{Z}}_{d_1}$  and the reference twistor  $\star$ . Here,  $\tilde{\mathcal{Z}}_{c_1}$  and  $\tilde{\mathcal{Z}}_{d_1}$  define the line that is connected with a propagator to the twistor on the right of the insertion twistor on the line  $L_{\tilde{\mathcal{Z}}_{a_1}\tilde{\mathcal{Z}}_{a_2}}$ :

$$\hat{\tilde{\mathcal{Z}}}_{a_2} = L_{\tilde{\mathcal{Z}}_{a_1}\tilde{\mathcal{Z}}_{a_2}} \cap \langle \star, \tilde{\mathcal{Z}}_{c_1}, \tilde{\mathcal{Z}}_{d_1} \rangle = (\star, \tilde{\mathcal{Z}}_{c_1}, \tilde{\mathcal{Z}}_{d_1}, \tilde{\mathcal{Z}}_{a_1})\tilde{\mathcal{Z}}_{a_2} - (\star, \tilde{\mathcal{Z}}_{c_1}, \tilde{\mathcal{Z}}_{d_1}, \tilde{\mathcal{Z}}_{a_2})\tilde{\mathcal{Z}}_{a_1}, \tag{3.8}$$



**Figure 7.** This figure depicts two boundary cases. On the left, the propagator insertions at the twistors [1] and [3] are adjacent, but they are on the line  $x_i$  corresponding to an edge of the Wilson loop. On the right, the twistors [2] and [3] are adjacent but there are at least two external particles on the line  $z_1$  if  $l > k + 1$ . We denote the two interaction lines by  $z_1$  and  $z_2$ , respectively.



**Figure 8.** The propagator encircled in red is replaced by the R-invariant (3.7) with two twistors replaced by intersections (3.8) as indicated.

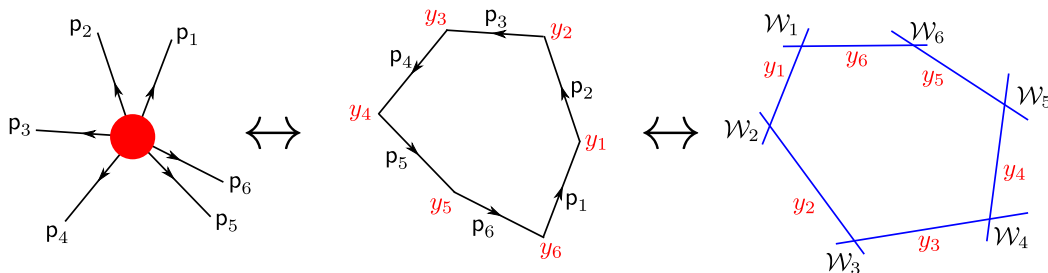
see figure 8. We further remark that in any computation in which some of the external twistors have to be replaced by the twistors on the corners of the Wilson loop as in (3.5), the replacement rule just carries through the computation, i.e. we simply have to replace the respective twistors in (3.7) and (3.8) via (3.5).

**Boundary-boundary.** This case is similar to the boundary one, but differs in that there are now less than two external particles at a vertex  $\mathbf{V}$ . Thus, we cannot use the inverse soft limit to do all twistor integrals, as the inverse soft limit cannot be applied to the two-point vertex. It seems difficult to obtain an expression built only out of R-invariants and MHV vertices for this case, but, as mentioned in [10], the boundary-boundary case is still fully described by the twistor formalism. As no boundary-boundary case can occur at the Wilson line vertices  $\mathbf{W}$ ,<sup>9</sup> the boundary-boundary diagrams for form factors are however no more difficult than for amplitudes.

#### 4 From position twistors to momentum twistors

In the previous section, we have shown how to calculate general tree-level  $N^k$ MHV form factors in position twistor space. For comparisons with the literature as well as for applications e.g. in unitarity calculations, it is however more useful to write the form factors

<sup>9</sup>Recall that we can always use the inverse soft limit for Wilson line vertices.



**Figure 9.** This figure illustrates the relationship between (super)momenta, region momenta and momentum twistors for  $n = 6$ . We suppress the fermionic part.

in momentum space, in terms of spinor-helicity variables or even momentum twistors. In particular, this requires the explicit integration over the positions of all vertices. In subsection 4.1, we will perform this for amplitudes, and then move on to form factors, restricting ourselves to the NMHV case for simplicity. This is easiest for forming operators that contain only fermionic derivatives, which correspond to operators without  $\dot{\alpha}$  indices; they trivially commute with the integrations. We treat this case in subsection 4.2. In subsection 4.3, we demonstrate using a simple example how our formalism continues to work for general operators, where the space-time derivatives that occur for operators with  $\dot{\alpha}$  do not commute with the integration. The evaluation of the occurring Fourier-type integrals is relegated to appendix B.

### 4.1 NMHV amplitudes

In [10], it was remarked that expression (3.3) for the NMHV amplitude  $\mathbb{A}^{\text{NMHV}}$  in position twistor space closely resembles the corresponding expression in momentum twistor space:

$$\mathcal{A}^{\text{NMHV}}(1, \dots, n) = \mathcal{A}^{\text{MHV}}(1, \dots, n) \sum_{1 \leq j < k \leq n} [\mathcal{W}_j, \mathcal{W}_{j+1}, \star, \mathcal{W}_k, \mathcal{W}_{k+1}], \quad (4.1)$$

where  $\mathcal{A}^{\text{MHV}}$  was given in (2.8) and the definition of the momentum twistors  $\mathcal{W}$  is given below. We will now derive this remark starting from formula (3.3) for the NMHV amplitudes in position twistor space.

Let us first briefly review the definition of momentum twistors [63]. Starting with  $n$  supermomenta  $P_i = (\mathbf{p}_i, \eta_i)$ , we define dual, or region, momenta and supermomenta by

$$\mathbf{p}_i^{\alpha\dot{\alpha}} = (y_i - y_{i-1})^{\alpha\dot{\alpha}}, \quad p_i^\alpha \eta_{i,a} = (\Gamma_i - \Gamma_{i-1})^\alpha_a, \quad (4.2)$$

see figure 9 for an illustration in the case  $n = 6$ . The origin in dual (super)momentum space is arbitrary. For example, we can choose

$$y_i^{\alpha\dot{\alpha}} = \sum_{j=1}^i p_j^\alpha \bar{p}_j^{\dot{\alpha}}, \quad (\Gamma_i)^\alpha_a = \sum_{j=1}^i p_j^\alpha \eta_{j,a}. \quad (4.3)$$

The momentum twistors  $\mathcal{W}_j$  are then defined as the intersections of the lines  $(y_{j-1}, \Gamma_{j-1})$  and  $(y_j, \Gamma_j)$  in twistor space:

$$\mathcal{W}_j \equiv (y_{j-1}, \Gamma_{j-1}) \cap (y_j, \Gamma_j) = (p_{j\alpha}, iy_j^{\alpha\dot{\alpha}} p_{j\alpha}, i(\Gamma_j)^\alpha_a p_{j\alpha}) \equiv \mathcal{Z}_{(y_j, \Gamma_j)}(p_j). \quad (4.4)$$

As in the case of position twistors, we will frequently abbreviate  $\mathcal{Z}_{y_j}(p_j) \equiv \mathcal{Z}_{(y_j, \Gamma_j)}(p_j)$ .



Having introduced our notation for the momentum twistors, we can now compute the NMHV amplitudes and rephrase them in momentum twistor space. Inserting the momentum eigenstates (2.7) into (3.1) is effectively accomplished by the following integration

$$\mathcal{A}^{\text{NMHV}}(1, \dots, n) = \int \left[ \prod_{s=1}^n \text{D}\mathcal{Z}_s \frac{\mathcal{A}_{\mathbb{P}_s}(\mathcal{Z}_s)}{2\pi i} \right] \mathbb{A}^{\text{NMHV}}(1, \dots, n), \quad (4.5)$$

cf. (2.6) and (2.7). Now using the methods of for example [10], we find

$$\begin{aligned} \mathcal{A}^{\text{NMHV}}(1, \dots, n) &= i(4\pi)^2 \sum_{1 \leq j < k \leq n} \int \frac{d^4 z_1 d^8 \vartheta_1}{(2\pi)^4} \frac{d^4 z_2 d^8 \vartheta_2}{(2\pi)^4} \frac{1}{\prod_{i=1}^n \langle p_i p_{i+1} \rangle} \frac{\langle p_j p_{j+1} \rangle \langle p_k p_{k+1} \rangle}{\langle p_{k+1} p_j \rangle \langle p_{j+1} p_k \rangle} \\ &\quad \times [\mathcal{Z}_{(z_1, \vartheta_1)}(p_{k+1}), \mathcal{Z}_{(z_1, \vartheta_1)}(p_j), \star, \mathcal{Z}_{(z_2, \vartheta_2)}(p_{j+1}), \mathcal{Z}_{(z_2, \vartheta_2)}(p_k)] \\ &\quad \times e^{i \sum_{s=k+1}^j (z_1 p_s + \vartheta_1 p_s \eta_s)} e^{i \sum_{s=j+1}^k (z_2 p_s + \vartheta_2 p_s \eta_s)}. \end{aligned} \quad (4.6)$$

Using (B.13), we see that the five-bracket only depends on the differences  $z_2 - z_1$  and  $\vartheta_2 - \vartheta_1$ . Thus, we shift the integration variables as  $z_2 \rightarrow z_1 + z_2$  and  $\vartheta_2 \rightarrow \vartheta_1 + \vartheta_2$ . As a result, the five-bracket no longer depends on  $z_1$  and  $\vartheta_1$ . Moreover, considering the explicit form (3.2) of the five-bracket, the replacement allows us to use a Fourier-type integration identity derived in appendix B for  $(z_2, \vartheta_2)$ . In the five-bracket, the formula (B.12) for the  $(z_2, \vartheta_2)$  integration effectively replaces

$$z_2 \rightarrow \sum_{s=j+1}^k p_s \bar{p}_s = y_k - y_j, \quad \vartheta_2 \rightarrow \sum_{s=j+1}^k p_s \eta_s = \Gamma_k - \Gamma_j \quad (4.7)$$

and multiplies by an overall factor of  $\frac{1}{i(4\pi)^2}$ . Hence, (4.6) becomes

$$\begin{aligned} \mathcal{A}^{\text{NMHV}}(1, \dots, n) &= \sum_{1 \leq j < k \leq n} \int \frac{d^4 z_1 d^8 \vartheta_1}{(2\pi)^4} \frac{e^{i(z_1 \sum_{s=1}^n p_s + \vartheta_1 \sum_{s=1}^n p_s \eta_s)} \langle p_j p_{j+1} \rangle \langle p_k p_{k+1} \rangle}{\prod_{i=1}^n \langle p_i p_{i+1} \rangle} \frac{\langle p_j p_{j+1} \rangle \langle p_k p_{k+1} \rangle}{\langle p_{k+1} p_j \rangle \langle p_{j+1} p_k \rangle} \\ &\quad \times [\mathcal{Z}_0(p_{k+1}), \mathcal{Z}_0(p_j), \star, \mathcal{Z}_{y_k - y_j}(p_{j+1}), \mathcal{Z}_{y_k - y_j}(p_k)], \end{aligned} \quad (4.8)$$

where we keep in mind that  $\star = (0, \zeta, 0)$ . The only remaining dependence on the integration variables  $z_1$  and  $\vartheta_1$  is in the exponential factor, which reduces to the momentum- and supermomentum-conserving delta functions. Hence, we find

$$\begin{aligned} &\mathcal{A}^{\text{NMHV}}(1, \dots, n) \\ &= \sum_{1 \leq j < k \leq n} \frac{\delta^{4|8}(\sum_{i=1}^n \mathbb{P}_i)}{\prod_{i=1}^n \langle p_i p_{i+1} \rangle} \frac{\langle p_j p_{j+1} \rangle \langle p_k p_{k+1} \rangle}{\langle p_{k+1} p_j \rangle \langle p_{j+1} p_k \rangle} [\mathcal{Z}_0(p_{k+1}), \mathcal{Z}_0(p_j), \star, \mathcal{Z}_{y_k - y_j}(p_{j+1}), \mathcal{Z}_{y_k - y_j}(p_k)] \\ &= \mathcal{A}^{\text{MHV}}(1, \dots, n) \sum_{1 \leq j < k \leq n} [\mathcal{Z}_0(p_j), \mathcal{Z}_0(p_{j+1}), \star, \mathcal{Z}_{y_k - y_j}(p_k), \mathcal{Z}_{y_k - y_j}(p_{k+1})], \end{aligned} \quad (4.9)$$

where in the last step we made use of (B.13) and the total antisymmetry of the five-bracket. The prefactor in the last step of (4.9) is exactly the MHV amplitude (2.8), while the

five-bracket equals

$$\begin{aligned}
 & [\mathcal{Z}_0(p_j), \mathcal{Z}_0(p_{j+1}), \star, \mathcal{Z}_{y_k-y_j}(p_k), \mathcal{Z}_{y_k-y_j}(p_{k+1})] \\
 &= [\mathcal{Z}_{y_j}(p_j), \mathcal{Z}_{y_j}(p_{j+1}), \star, \mathcal{Z}_{y_k}(p_k), \mathcal{Z}_{y_k}(p_{k+1})] \\
 &= [\mathcal{Z}_{y_j}(p_j), \mathcal{Z}_{y_{j+1}}(p_{j+1}), \star, \mathcal{Z}_{y_k}(p_k), \mathcal{Z}_{y_{k+1}}(p_{k+1})] \\
 &\equiv [\mathcal{W}_j, \mathcal{W}_{j+1}, \star, \mathcal{W}_k, \mathcal{W}_{k+1}].
 \end{aligned} \tag{4.10}$$

In the first step of (4.10), we have used the invariance of the five-bracket under shifts (B.13) and in the second step the fact that, due to the definition (4.3), we have the relations

$$p_{i+1}y_{i+1} = p_{i+1}y_i, \quad p_{i+1}\Gamma_{i+1} = p_{i+1}\Gamma_i. \tag{4.11}$$

Thus, we indeed obtain the desired amplitude in momentum twistor space (4.1).

## 4.2 NMHV form factors for operators without $\dot{\alpha}$ indices

Having understood how to transition from position to momentum twistor space for amplitudes in the previous subsection, we now want to do the same for the NMHV form factors of subsection 3.1. In the case of form factors, the forming operator (2.14) occurs inside of the (super) Fourier-type integrals over the positions of the vertices and the operator. While the fermionic  $\theta_i$  derivatives in the forming operator trivially commute with the fermionic integrals, this is in general not the case for the bosonic  $x_i$  derivatives and the space-time integrals — as we will demonstrate explicitly in appendix B. In the second half of this subsection, we restrict ourselves to operators without  $\dot{\alpha}$  indices, i.e. to forming operators without  $x_i$  derivatives.

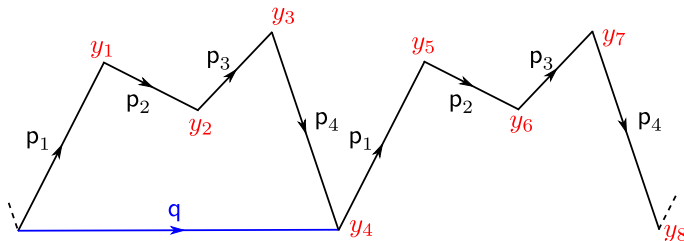
One difference from the amplitude case lies in the definition of the momentum twistors for form factors. In contrast to the amplitude case, for form factors the on-shell momenta  $\mathbf{p}_i$  of the  $n$  external on-shell states do not add up to zero but to the off-shell momentum  $\mathbf{q}$  of the composite operator. Hence, the contour they form in the space of region momenta is periodic instead of closed [56]:

$$y_{i+n} = y_i + \mathbf{q}; \tag{4.12}$$

see figure 10. One way to define momentum twistors in this case is to use two periods of the contour, which results in  $2n$  momentum twistors [26, 37] still defined via (4.4) but with  $i = 1, \dots, 2n$ .<sup>10</sup>

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<sup>10</sup>Alternatively, the momentum twistors can be defined by closing a single period of the contour via two auxiliary on-shell momenta as done in [42]. This results in  $n+2$  momentum twistors but requires to consider the different possible ways to close the contour.



**Figure 10.** An illustration of the region momenta used for form factors. Here, we consider a case with  $n = 4$ .

The computation of the NMHV form factor parallels the calculations done for amplitudes in the preceding subsection. Inserting momentum eigenstates in  $\mathbb{F}^{\text{NMHV}}$  of (3.6), including the forming operator (2.13) and taking the operator limit, we find

$$\begin{aligned}
 \mathcal{F}_{\mathcal{O}}^{\text{NMHV}}(1, \dots, n; \mathbf{q}) &= i(4\pi)^2 \sum_X \sum_{a \leq j < k \leq b} \int \frac{d^4x}{(2\pi)^4} e^{-ix\mathbf{q}} \int \frac{d^4z d^8\vartheta}{(2\pi)^4} \lim_{\mathcal{O} \rightarrow \cdot} \left\{ \mathbf{F}_{\mathcal{O}} \right. \\
 &\times \frac{1}{\prod_{i=a+1}^b \langle p_i p_{i+1} \rangle} \frac{\langle (p_j \xrightarrow{j=a} \lambda_{X1}) p_{j+1} \rangle \langle p_k (p_{k+1} \xrightarrow{k=b} \lambda_{X2}) \rangle}{\langle (p_j \xrightarrow{j=a} \lambda_{X1}) (p_{k+1} \xrightarrow{k=b} \lambda_{X2}) \rangle \langle p_k p_{j+1} \rangle} \\
 &\times \left[ \mathcal{Z}_X(p_{k+1} \xrightarrow{k=b} \lambda_{X2}), \mathcal{Z}_X(p_j \xrightarrow{j=a} \lambda_{X1}), \star, \mathcal{Z}_{(z, \vartheta)}(p_{j+1}), \mathcal{Z}_{(z, \vartheta)}(p_k) \right] \\
 &\times e^{i \sum_{s=a+1}^j (X p_s + \theta p_s \eta_s)} e^{i \sum_{s=j+1}^k (z p_s + \vartheta p_s \eta_s)} e^{i \sum_{s=k+1}^b (X p_s + \theta p_s \eta_s)} \\
 &\times \prod_{X' \neq X} \left. \text{contributions from edge } X' \right\} \Big|_{\theta=0}, \tag{4.13}
 \end{aligned}$$

where we have suppressed the contributions from the other edges. In general, we have to act with the forming operator on the last four lines before doing the integration. In the special case that the forming operator does not contain  $x_i$  derivatives, it does however commute with the integration; we can then do the integral as in the amplitude case. We will focus on this special case for the rest of this subsection and treat a simple example of the general case in the next subsection. The fact that the integral and the  $x_i$  derivatives do in general not commute is exemplified in appendix B.

Let us restrict ourselves to forming operators without  $x_i$  derivatives, i.e. to composite operators built of irreducible fields whose forming operators (2.14) contain only  $\theta$  derivatives. These are the scalars  $\phi$ , the fermions  $\psi$  and the self-dual part of the field strength  $F$ ; they require  $N = n_\theta = 2, 3, 4$   $\theta$  derivatives, respectively. We can then commute the  $(z, \vartheta)$  integral past the forming operator and evaluate this integral in complete analogy to the amplitude case treated in the previous subsection. The full calculation involves the sum over all distributions of the external fields on the different edges, which replace  $a$  and  $b$  in (4.13). We parametrize it via  $1 \leq m_1 \leq m'_1 \leq m''_1 < m_2 \leq m'_2 \leq m''_2 < m_3 \leq \dots < m_L \leq m'_L \leq m''_L < m_1 + n \leq 2n$  and abbreviate this set as  $\{m_i, m'_i, m''_i\}$ . The details of this calculation are quite technical, and we relegate them to appendix A. Here, we just

present the result:

$$\begin{aligned}
 \mathcal{F}_{\mathcal{O}}^{\text{NMHV}}(1, \dots, n; \mathbf{q}) &= \frac{\delta^4(\sum_{r=1}^n \mathbf{p}_r - \mathbf{q})}{\prod_{r=1}^n \langle p_r p_{r+1} \rangle} \sum_{\{m_i, m'_i, m''_i\}} \widetilde{\mathcal{F}}_{\mathcal{O}}(\{m_i, m'_i, m''_i\}) \\
 &\times \sum_{i=1}^L \left\{ \sum_{m''_{i-1} \leq j < k \leq m_{i-1}} \left[ \mathcal{W}_j \xrightarrow{j=m''_{i-1}} \mathcal{Z}_{y_j}(\mathbf{n}), \mathcal{W}_{j+1}, \star, \mathcal{W}_k, \mathcal{W}_{k+1} \xrightarrow{k=m_{i-1}} \mathcal{Z}_{y_k}(\tau_i) \right] \right. \\
 &\quad + \sum_{m_{i-1} \leq j < k \leq m'_i} \left[ \mathcal{W}_j \xrightarrow{j=m_{i-1}} \mathcal{Z}_{y_j}(\tau_i), \mathcal{W}_{j+1}, \star, \mathcal{W}_k, \mathcal{W}_{k+1} \xrightarrow{k=m'_i} \mathcal{Z}_{y_k}(\tau_i) \right] \\
 &\quad \left. + \sum_{m'_i \leq j < k \leq m''_i} \left[ \mathcal{W}_j \xrightarrow{j=m'_i} \mathcal{Z}_{y_j}(\tau_i), \mathcal{W}_{j+1}, \star, \mathcal{W}_k, \mathcal{W}_{k+1} \xrightarrow{k=m''_i} \mathcal{Z}_{y_k}(\mathbf{n}) \right] \right\}, \tag{4.14}
 \end{aligned}$$

where we have defined

$$\begin{aligned}
 \widetilde{\mathcal{F}}_{\mathcal{O}}(\{m_i, m'_i, m''_i\}) &= \prod_{r=1}^L \left\{ \delta_{m''_{r-1}+1, m_r} + (1 - \delta_{m''_{r-1}+1, m_r}) \frac{\langle \mathbf{n} \tau_r \rangle}{\langle \mathbf{n} p_{m''_{r-1}+1} \rangle \langle p_{m_r-1} \tau_r \rangle} \right\} \\
 &\times \langle p_{m_r-1} p_{m_r} \rangle \frac{\left( \sum_{s=m_r}^{m'_r} \langle \tau_r p_s \rangle \{ \xi_r \eta_s \} \right)^{N_r}}{\langle p_{m_r} \tau_r \rangle \langle p_{m'_r} \tau_r \rangle} \langle p_{m'_r} p_{m_r+1} \rangle \\
 &\times \left\{ \delta_{m'_r, m''_r} + (1 - \delta_{m'_r, m''_r}) \frac{\langle \tau_r \mathbf{n} \rangle}{\langle \tau_r p_{m'_r+1} \rangle \langle p_{m''_r} \mathbf{n} \rangle} \right\} \langle p_{m''_r} p_{m_r+1} \rangle. \tag{4.15}
 \end{aligned}$$

Two remarks are in order. First, the result (4.14) still contains the auxiliary spinor  $\mathbf{n}$  from the operator limit (2.16). For the MHV form factor

$$\mathcal{F}_{\mathcal{O}}^{\text{MHV}}(1, \dots, n; \mathbf{q}) = \frac{\delta^4(\sum_{r=1}^n \mathbf{p}_r - \mathbf{q})}{\prod_{r=1}^n \langle p_r p_{r+1} \rangle} \sum_{\{m_i, m'_i, m''_i\}} \widetilde{\mathcal{F}}_{\mathcal{O}}(\{m_i, m'_i, m''_i\}), \tag{4.16}$$

we have explicitly shown in [2] how  $\mathbf{n}$  drops out thanks to a repeated application of the Schouten identity. In the  $n = (L + 1)$ -point case, the spinor  $\mathbf{n}$  trivially drops out of the NMHV form factor due to the summation ranges and the Kronecker deltas in (4.14) and (4.15). We leave a proof of the independence of  $\mathcal{F}_{\mathcal{O}}^{\text{NMHV}}$  on  $\mathbf{n}$  in the general case for future work. Moreover, (4.14) contains the reference twistor  $\mathcal{Z}_\star$ . As in the case of general amplitudes, it is not immediate to see how  $\mathcal{Z}_\star$  drops out.

Second, we have numerically checked in several cases that (4.14) reproduces the known result [26, 37] for the chiral half of the stress-tensor supermultiplet  $T$ :

$$\mathcal{F}_T^{\text{NMHV}}(1, \dots, n; \mathbf{q}) = \mathcal{F}_T^{\text{MHV}}(1, \dots, n; \mathbf{q}) \sum_{j=1}^n \sum_{k=j+2}^{n+j-1} [\mathcal{W}_j, \mathcal{W}_{j+1}, \star, \mathcal{W}_k, \mathcal{W}_{k+1}], \tag{4.17}$$

which factorizes similarly to the amplitude (4.1). It would be desirable to find a factorized form of (4.14) also for general operators.<sup>11</sup>

<sup>11</sup>For more general operators, the immediate generalization of (4.17) is *not* true.

### 4.3 NMHV form factors for general operators

We now turn to form factors of general operators, for which also  $x_i$  derivatives occur in the forming operator (2.14). We show in appendix B that we cannot commute these derivatives out of the integral, which complicates the calculation. To demonstrate that our formalism nevertheless continues to work in this case, we consider a simple example.

We study the next-to-minimal (i.e.  $(L+1)$ -point) NMHV form factor of the operator  $\text{Tr}(\bar{\psi}_2(\phi_{13})^{L-1})$  and consider the external state  $1^{\phi_{12}}2^{\psi_{234}}3^{\phi_{13}} \dots (L+1)^{\phi_{13}}$ . The advantage of this setup is that only one diagram contributes, in which  $\bar{\psi}_2 \rightarrow \phi_{12}\psi_{234}$  and which in particular must be gauge invariant. This diagram is the counterpart of (4.13) with  $a = j = n = L+1$ ,  $j+1 = 1$  and  $k = b = 2$ . Picking up the calculation at this point, we find

$$\begin{aligned}
 & \mathcal{F}_{\text{Tr}(\bar{\psi}_2(\phi_{13})^{L-1})}(1^{\phi_{12}}2^{\psi_{234}}3^{\phi_{13}} \dots (L+1)^{\phi_{13}}; x) \\
 &= i(4\pi)^2 \lim_{\Omega \rightarrow 0} \left\{ \frac{\partial^5}{\partial \eta_1^1 \partial \eta_1^2 \partial \eta_2^2 \partial \eta_2^3 \partial \eta_2^4} \int \frac{d^4 z d^8 \vartheta}{(2\pi)^4} \frac{e^{iz(\mathbf{p}_1 + \mathbf{p}_2) + i\vartheta(\mathbf{p}_1 \eta_1 + \mathbf{p}_2 \eta_2)}}{\langle p_1 p_2 \rangle \langle p_2 p_1 \rangle} \right. \\
 & \quad \times \left( -\frac{\lambda_{X1}^\alpha \lambda_{X2}^\beta}{\langle \lambda_{X1} \lambda_{X2} \rangle} \right) \left( -i\bar{\tau}^{\dot{\alpha}} \frac{\partial}{\partial x^{\alpha\dot{\alpha}}} \right) \left( -i\xi^a \frac{\partial}{\partial \theta^{\beta a}} \right) \\
 & \quad \times \left[ \mathcal{Z}_{(x,\theta)}(\lambda_{X2}), \mathcal{Z}_{(x,\theta)}(\lambda_{X1}), \star, \mathcal{Z}_{(z,\vartheta)}(p_1), \mathcal{Z}_{(z,\vartheta)}(p_2) \right] \Bigg|_{\xi^a = \delta^{a2}, \theta = 0} e^{ix \sum_{i=3}^{L+1} p_i} \\
 &= -i(4\pi)^2 \int \frac{d^4 z}{(2\pi)^4} e^{iz(\mathbf{p}_1 + \mathbf{p}_2)} \left[ \left( -i\tau^\alpha \bar{\tau}^{\dot{\alpha}} \frac{\partial}{\partial x^{\alpha\dot{\alpha}}} \right) \frac{1}{(x-z)^2} \frac{\langle p_2 | x-z | \zeta \rangle}{\langle \tau | x-z | \zeta \rangle} \right] e^{ix \sum_{i=3}^{L+1} p_i},
 \end{aligned} \tag{4.18}$$

where the  $\eta$  derivatives serve to select the  $\phi_{12}\psi_{234}$  component of the super form factor and the only effective contribution of the other edges is the phase  $e^{ix \sum_{i=3}^{L+1} p_i}$ . The polarization vectors  $\tau$  and  $\bar{\tau}$  correspond to the field  $\bar{\psi}_2$  and we have dropped the index as the polarization vectors of all other fields have already dropped out. We remind the reader that in the operator limit  $\lambda_{X1} \rightarrow \tau$  and  $\lambda_{X2} \rightarrow \tau$ . Thanks to the following identity,

$$\langle A|x|C \rangle \langle B|x|D \rangle - \langle A|x|D \rangle \langle B|x|C \rangle = -x^2 \langle AB \rangle [CD], \tag{4.19}$$

we obtain after evaluating the derivative inside of the  $z$  integral

$$\begin{aligned}
 & \mathcal{F}_{\text{Tr}(\bar{\psi}_2(\phi_{13})^{L-1})}(1^{\phi_{12}}2^{\psi_{234}}3^{\phi_{13}} \dots (L+1)^{\phi_{13}}; x) \\
 &= -(4\pi)^2 \int \frac{d^4 z}{(2\pi)^4} \frac{e^{iz(\mathbf{p}_1 + \mathbf{p}_2)}}{(x-z)^4} \frac{(x-z)^2 \langle \tau p_2 \rangle [\zeta \bar{\tau}] - \langle \tau | x-z | \bar{\tau} \rangle \langle p_2 | x-z | \zeta \rangle}{\langle \tau | x-z | \zeta \rangle} e^{ix \sum_{i=3}^{L+1} p_i} \\
 &= -(4\pi)^2 \int \frac{d^4 z}{(2\pi)^4} e^{iz(\mathbf{p}_1 + \mathbf{p}_2)} \frac{\langle p_2 | z-x | \bar{\tau} \rangle}{(z-x)^4} e^{ix \sum_{i=3}^{L+1} p_i} \\
 &= -\frac{e^{ix \sum_{i=1}^{L+1} p_i}}{(\mathbf{p}_1 + \mathbf{p}_2)^2} \langle p_2 | \mathbf{p}_1 + \mathbf{p}_2 | \bar{\tau} \rangle,
 \end{aligned} \tag{4.20}$$

where we have used (B.20). Fourier transforming in  $x$  and using  $\langle p_2 | \mathbf{p}_1 + \mathbf{p}_2 | \bar{\tau} \rangle = \langle p_1 p_2 \rangle [\bar{\tau} \bar{p}_1]$  as well as  $(\mathbf{p}_1 + \mathbf{p}_2)^2 = -\langle p_1 p_2 \rangle [\bar{p}_1 \bar{p}_2]$ , we find

$$\mathcal{F}_{\text{Tr}(\bar{\psi}_2(\phi_{13})^{L-1})}(1^{\phi_{12}}2^{\psi_{234}}3^{\phi_{13}} \dots (L+1)^{\phi_{13}}; \mathbf{q}) = \delta^4 \left( \mathbf{q} - \sum_{i=1}^{L+1} \mathbf{p}_i \right) \frac{[\bar{\tau} \bar{p}_1]}{[\bar{p}_1 \bar{p}_2]}. \tag{4.21}$$

This is indeed the expected gauge-invariant expression; it can be obtained e.g. from the MHV form factor  $\mathcal{F}_{\text{Tr}(\psi_{134}(\phi_{24})^{L-1})(1^{\phi_{34}}2^{\bar{\psi}_1}3^{\phi_{24}} \dots (L+1)^{\phi_{24}}; \mathbf{q})}$  by conjugation.

In the above computation, it was essential to first act with the space-time derivative in the forming operator before integrating out the interaction, so that we may simplify the expression and in particular get rid of the reference twistor through (4.19). In appendix B.3, we show that reversing the order of integration and derivative does not work. It would be desirable to be able to evaluate the integrals occurring in (4.13) for general operators or to find a general form for the commutator of the forming operator and the integrals that allows for a more efficient evaluation. We leave this for future work.

## 5 Generalized form factors and correlation functions

In this section, we will look at quantities that contain more than one composite operator, namely generalized form factors and correlation functions.

Generalized form factors contain  $m$  composite operators and  $n$  external on-shell fields, see [32]. For the case  $n = 0$ , they reduce to correlation functions. Similar to form factors, we can calculate generalized form factors and correlation functions of general local composite operators in twistor space using the operator vertices (2.17). Equally, we can calculate generalized form factors and correlation functions of Wilson loops and extract the corresponding expressions for composite operators by applying the forming operators (2.13) and taking the operator limit  $\diamond \rightarrow \cdot$  for every space-time point — provided we are careful in treating the situations where integrals and forming operators do not commute as discussed in the last section.

Let us consider correlation functions in some generality. At tree level, the correlation function of two local composite operators before taking the operator limit is shown in figure 11. By a trivial counting of Graßmann numbers, one sees that at tree level only the minimal operator vertices contribute.<sup>12</sup> Operators of different lengths have vanishing correlation functions at tree level. For two operators of the same length  $L$ , the tree-level correlation function can be calculated using the inverse soft limit. Using underlined symbols for the second operator, we find

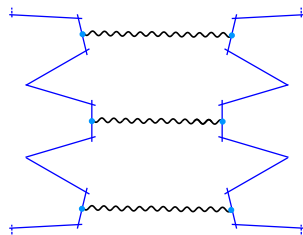
$$\langle \mathcal{O}(x) \underline{\mathcal{O}}(y) \rangle_{\text{tree-level}} = \lim_{\diamond \rightarrow \cdot, \underline{\diamond} \rightarrow \cdot, y} \mathbf{F} \mathcal{O} \mathbf{F} \underline{\mathcal{O}} \sum_{\sigma} \prod_{i=1}^L [Z_i, \underline{Z}'_i, \star, \underline{Z}_{\sigma(i)}, \underline{Z}'_{\sigma(i)}] \Big|_{\theta, \underline{\theta}=0}, \quad (5.1)$$

where the permutation  $\sigma$  is cyclic in the planar limit. We immediately see that this is zero unless complementary fermionic derivatives act on all pairs  $i$  and  $\sigma(i)$ .

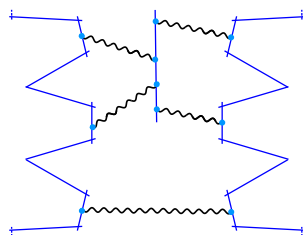
At one-loop order, we have one interaction vertex in addition to the Wilson loops. Counting the Graßmann degree, we find that several diagrams contribute. The first one consists of a four-point interaction vertex connected to minimal operator vertices, an example of which is shown in figure 12. The second possibility consists of a three-point interaction vertex connected to a minimal operator vertex and a next-to-minimal operator vertex, an example of which is shown on the right-hand side of figure 13. The third possibility consists of a two-point interaction vertex connected to two next-to-minimal operator

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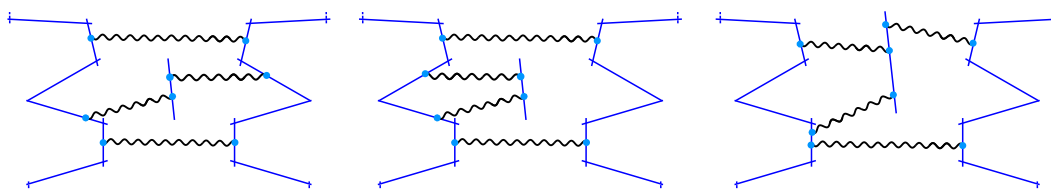
<sup>12</sup>These are given in (4.1) of [2].



**Figure 11.** A diagram for the tree-level two-point correlation function.



**Figure 12.** A diagram with a four-point vertex contributing to the one-loop two-point correlation function.



**Figure 13.** Two diagrams with a two-point vertex and one with a three-point vertex contributing to the one-loop two-point correlation function.

vertices, an example of which is shown in on the left-hand side of figure 13. The fourth possibility is a two-point vertex connected to a next-to-next-to-minimal operator vertex, an example of which is shown in the middle of figure 13. Note that the figures show the correlation functions before taking the operator limit, such that the contraction with the minimal operator vertex receives contributions from the additional field being connected to several different edges of the Wilson loop, of which only one is shown.

In [3], some of the present authors calculated the one-loop dilatation operator in the  $SO(6)$  sector via the UV divergence of the one-loop two-point correlation functions in twistor space. It was stated that the contributions to the dilatation operator stem only from the diagram with the four-point vertex shown in figure 12, see [3] for details of this calculations. In appendix C, we provide the argument promised in [3] that the contributions of the three-vertices and two-vertices shown in figure 13 vanish.

## 6 Conclusion and outlook

In this article, we have shown how to apply the general formalism for composite operators in position twistor space developed in [1, 2] to  $N^k$ MHV form factors as well as to correlation functions.

In *position twistor space*, the form factors of general composite operators can be efficiently calculated using the twistor action in close analogy to the amplitude case treated in [10]. This calculation is based on the inverse soft limit for the position-twistor-space vertices, which we have extended to composite operators in section 2.3. In section 3, we wrote the NMHV form factors in position twistor space explicitly and described how to generalize them to the  $N^k$ MHV case. In order to obtain explicit results for form factors in *momentum (twistor) space*, the implicit Fourier-type integrals over the positions of the interactions have to be calculated. After revisiting the case of NMHV amplitudes, we have shown how this can be efficiently done for NMHV form factors of operators whose forming operators contain no space-time derivatives, i.e. for operators without  $\dot{\alpha}$  indices. In this case, we can commute the forming operator past the aforementioned integrals, and our final result in this case is given in (4.14). For composite operators  $\mathcal{O}$  whose forming operators contain space-time derivatives, our formalism works as well but requires to act with the forming operator before the integration, as we illustrate in section 4.3. Finally, in section 5 we looked at the computation of generalized form factors and correlation functions of composite operators using twistor-space techniques. The last part on one-loop correlation functions in particular closes a gap in an argument in [3] of three of the present authors, as previously promised.

Let us now conclude with an overview of possible directions of future research. We begin by emphasizing that the points raised in the outlook of our last article [2], namely the connection to integrability, the Yangian symmetry of Wilson loops, multi-loop correlation functions in twistor space and the application of twistor-space techniques to theories with less supersymmetry, remain as pertinent as before. Furthermore, the results of our paper allude to several new open questions worth investigating.

First, in section 3.2, we discussed the computation of form factors in position twistor space. The method relies on the computation of amplitudes and there are some points, concerning specifically the boundary-boundary case, in the position twistor space calculation of amplitudes beyond the NMHV level that were left open in [10]. It would be interesting to settle these point w.r.t. the application to form factors.

Second, we have seen in section 4.3 that the Fourier-type integrations that we needed to perform in order to obtain the form factors of operators with  $\dot{\alpha}$  indices in momentum (twistor) space are tricky and in particular do not commute with the application of the forming operator. It would be very interesting to derive a closed expression for the commutator in order to obtain a generating object such that the tree-level form factors in momentum space are given by the application of a derivative operator on that object in analogy to what we have achieved at MHV level and for operators without space-time derivatives at NMHV level. Related is the problem of finding explicit expressions for the general Fourier-type integrals that occur for composite operators whose forming operators contain space-time derivatives.

Third, one should connect the results of our article with some new techniques for the evaluation of form factors, such as the connected prescription of [48, 50, 51] and Graßmannian integrals for form factors [41, 42, 46, 47, 55]. This is also a potential pathway towards connections with integrability.



Fourth, it would be important to relate further to the LHC formalism of [15, 16]. The general MHV form factors we computed [2] were rederived in [17] and it would be interesting to see if the LHC methods offer any improvements over the twistor-space ones for NMHV form factors and if they offer any insights on the non-commutativity of integral and derivative encountered in section 4.3.

Finally, it would be desirable to apply the techniques presented here to higher-point correlation functions, in particular with regard to the recent progress in three-point functions.

## Acknowledgments

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## A Computation of the NMHV form factors

In this appendix, we elaborate on the notationally more involved aspects in the calculation of the NMHV form factors, both in position and in momentum twistor space.

### A.1 Position twistor space

We start with the calculation in position twistor space.

In order to compute the form factor, we use the cogwheel-shaped Wilson loop introduced in [2],<sup>13</sup> which we briefly reviewed in section 2.2. It has three families of edges  $x_i$ ,  $x'_i$  and  $x''_i$ , cf. figure 1. Each of these edges gives rise to a vertex  $\mathbf{W}$ . In order to obtain an  $n$ -point form factor, we have to sum over all ways to distribute the  $n$  external fields on the edges. Let  $m_i$  be the first external field emitted on  $x_i$ ,  $m'_i$  the last external field emitted on  $x_i$  and  $m''_i$  the last external field emitted on  $x''_i$ .<sup>14</sup> We then have to sum over

$$1 \leq m_1 \leq m'_1 \leq m''_1 < m_2 \leq m'_2 \leq m''_2 < m_3 \leq \dots < m_L \leq m'_L \leq m''_L < m_1 + n \leq 2n, \tag{A.1}$$

where  $m_i \leq m'_i$  ensures that there is at least one external field emitted from  $x_i$ ; contributions with no field on  $x_i$  are annihilated by the forming operator (2.14).

<sup>13</sup>See in particular appendix C therein.

<sup>14</sup>External fields that are emitted on an interaction line  $z$  connected to  $x_i$  by a propagator are also considered to be emitted on the edge  $x_i$  for this purpose, and analogously for the other edges and the case of several interaction lines.

For each cog, we divide the computation of the NMHV form factors into three pieces, illustrated in figure 14 and figure 15. In the first diagram shown in figure 14, we need to restrict the range of the parameters as  $m_i - 1 \leq j < k \leq m'_i$ , where  $j = m_i - 1$  implies that the propagator is attached to the leftmost twistor on the line  $x_i$ , while  $k = m'_i$  means that the propagator is attached to the rightmost one. In addition,  $k = j + 1$  means that we are dealing with a two-point interaction vertex; as in the case of amplitudes, this can be dropped. Similarly, we need to have  $m''_{i-1} \leq j < k \leq m_i - 1$  for the left diagrams in figure 15, while for the right one we need  $m'_i \leq j < k \leq m''_i$ .

In (3.6), we have given an expression for the interaction vertex connected to a general edge. Specifying (3.6) to the three cases above and suppressing the vertices of the other cogs, the diagrams of figure 14 contribute as

$$\begin{aligned}
 \mathbf{A}_i &= i(4\pi)^2 \sum_{m_i-1 \leq j < k \leq m'_i} \mathbf{W}_{x'_i}(\mathcal{Z}''_{i-1}, \mathcal{Z}_i; \tilde{\mathcal{Z}}_{m''_{i-1}+1}, \dots, \tilde{\mathcal{Z}}_{m_i-1}) \\
 &\quad \times \mathbf{W}_{x_i}(\mathcal{Z}_i, \mathcal{Z}'_i; \tilde{\mathcal{Z}}_{m_i}, \dots, \tilde{\mathcal{Z}}_j, \tilde{\mathcal{Z}}_{k+1}, \dots, \tilde{\mathcal{Z}}_{m'_i}) \mathbf{W}_{x''_i}(\mathcal{Z}'_i, \mathcal{Z}''_i; \tilde{\mathcal{Z}}_{m'_i+1}, \dots, \tilde{\mathcal{Z}}_{m''_i}) \\
 &\quad \times \mathbf{V}(\tilde{\mathcal{Z}}_{j+1}, \dots, \tilde{\mathcal{Z}}_k) \left[ (\tilde{\mathcal{Z}}_{k+1} \xrightarrow{k=m'_i} \mathcal{Z}'_i), (\tilde{\mathcal{Z}}_j \xrightarrow{j=m_i-1} \mathcal{Z}_i), \star, \tilde{\mathcal{Z}}_{j+1}, \tilde{\mathcal{Z}}_k \right],
 \end{aligned} \tag{A.2}$$

while those on the left- and right-hand side of figure 15, called  $\mathbf{B}_i$  and  $\mathbf{C}_i$  respectively, contribute

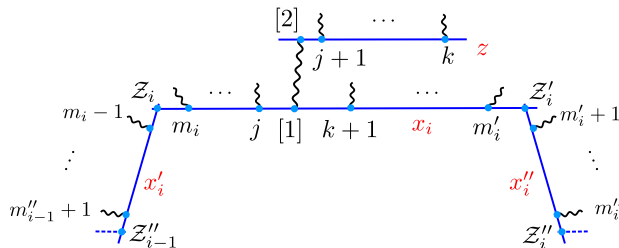
$$\begin{aligned}
 \mathbf{B}_i &= i(4\pi)^2 \sum_{m''_{i-1} \leq j < k \leq m_i-1} \mathbf{W}_{x'_i}(\mathcal{Z}''_{i-1}, \mathcal{Z}_i; \tilde{\mathcal{Z}}_{m''_{i-1}+1}, \dots, \tilde{\mathcal{Z}}_j, \tilde{\mathcal{Z}}_{k+1}, \dots, \tilde{\mathcal{Z}}_{m_i-1}) \\
 &\quad \times \mathbf{W}_{x_i}(\mathcal{Z}_i, \mathcal{Z}'_i; \tilde{\mathcal{Z}}_{m_i}, \dots, \tilde{\mathcal{Z}}_{m'_i}) \mathbf{W}_{x''_i}(\mathcal{Z}'_i, \mathcal{Z}''_i; \tilde{\mathcal{Z}}_{m'_i+1}, \dots, \tilde{\mathcal{Z}}_{m''_i}) \\
 &\quad \times \mathbf{V}(\tilde{\mathcal{Z}}_{j+1}, \dots, \tilde{\mathcal{Z}}_k) \left[ (\tilde{\mathcal{Z}}_{k+1} \xrightarrow{k=m_i-1} \mathcal{Z}_i), (\tilde{\mathcal{Z}}_j \xrightarrow{j=m''_{i-1}} \mathcal{Z}''_{i-1}), \star, \tilde{\mathcal{Z}}_{j+1}, \mathcal{Z}_k \right], \\
 \mathbf{C}_i &= i(4\pi)^2 \sum_{m'_i \leq j < k \leq m''_i} \mathbf{W}_{x'_i}(\mathcal{Z}''_{i-1}, \mathcal{Z}_i; \tilde{\mathcal{Z}}_{m''_{i-1}+1}, \dots, \tilde{\mathcal{Z}}_{m_i-1}) \\
 &\quad \times \mathbf{W}_{x_i}(\mathcal{Z}_i, \mathcal{Z}'_i; \tilde{\mathcal{Z}}_{m_i}, \dots, \tilde{\mathcal{Z}}_{m'_i}) \mathbf{W}_{x''_i}(\mathcal{Z}'_i, \mathcal{Z}''_i; \tilde{\mathcal{Z}}_{m'_i+1}, \dots, \tilde{\mathcal{Z}}_j, \tilde{\mathcal{Z}}_{k+1}, \dots, \tilde{\mathcal{Z}}_{m''_i}) \\
 &\quad \times \mathbf{V}(\tilde{\mathcal{Z}}_{j+1}, \dots, \tilde{\mathcal{Z}}_k) \left[ (\tilde{\mathcal{Z}}_{k+1} \xrightarrow{k=m''_i} \mathcal{Z}_i), (\mathcal{Z}_j \xrightarrow{j=m'_i} \mathcal{Z}''_{i-1}), \star, \tilde{\mathcal{Z}}_{j+1}, \tilde{\mathcal{Z}}_k \right].
 \end{aligned} \tag{A.3}$$

Finally, the complete tree-level NMHV form factor of the Wilson loop in position twistor space is obtained by adding (A.2) to the two terms in (A.3), multiplying with the contributions of the cogs of the Wilson loop that are not connected to the vertex  $\mathbf{V}$  and then summing over the cogs  $i$  that are as well as over the possible distributions of the integers  $m_i, m'_i$  and  $m''_i$  subject to (A.1):

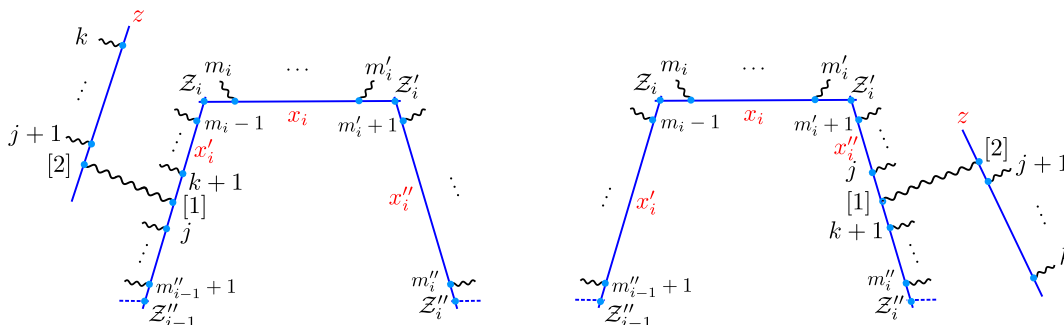
$$\mathbb{F}_{\mathcal{W}}^{\text{NMHV}} = \sum_{\{m_s, m'_s, m''_s\}} \sum_{i=1}^L \left( \prod_{r=1}^{i-1} \mathbf{D}_r \right) (\mathbf{A}_i + \mathbf{B}_i + \mathbf{C}_i) \left( \prod_{r=i+1}^L \mathbf{D}_r \right), \tag{A.4}$$

where the contribution of a cog that is not connected to the interaction vertex is

$$\begin{aligned}
 \mathbf{D}_i &= \mathbf{W}_{x'_i}(\mathcal{Z}''_{i-1}, \mathcal{Z}_i; \tilde{\mathcal{Z}}_{m''_{i-1}+1}, \dots, \tilde{\mathcal{Z}}_{m_i-1}) \mathbf{W}_{x_i}(\mathcal{Z}_i, \mathcal{Z}'_i; \tilde{\mathcal{Z}}_{m_i}, \dots, \tilde{\mathcal{Z}}_{m'_i}) \\
 &\quad \times \mathbf{W}_{x''_i}(\mathcal{Z}'_i, \mathcal{Z}''_i; \tilde{\mathcal{Z}}_{m'_i+1}, \dots, \tilde{\mathcal{Z}}_{m''_i}).
 \end{aligned} \tag{A.5}$$



**Figure 14.** An NMHV diagram of type **A** — the interaction vertex is connected to the operator-bearing edge  $x_i$  of the Wilson loop.



**Figure 15.** The NMHV diagrams of type **B** and **C** — the interaction vertex is connected to the edges  $x_i'$  and  $x_i''$  of the Wilson loop, which do not carry the operator.

To obtain the tree-level NMHV form factor of a local operator  $\mathcal{O}$  in position twistor space, we have to act with the forming operator  $\mathbf{F}_{\mathcal{O}}$  defined in (2.13) on the *integrand* of (A.4), i.e. *before* doing the integrations over the vertex positions; see the discussion in the main text.

## A.2 Momentum twistor space

Let us now describe the transition to momentum (twistor) space.

First, we need to recall some of the notation introduced in [2]. For the operator  $\mathcal{O}$  of (2.9), we denote by  $N_i$  the total number of derivatives (space-time and fermionic) required to generate the field  $D^{k_i} A_i$  in (2.14). Specifically, for  $A_i = \bar{F}, \bar{\psi}$  or  $\phi$ , we have  $N_i = k_i + 2$ . Otherwise,  $N_i = k_i + 3$  for  $A_i = \psi$  and  $N_i = k_i + 4$  for  $A_i = F$ . Moreover, we denote by  $n_{\theta_i}$  the number of  $\frac{\partial}{\partial \theta_i}$  derivatives required to generate the field  $D^{k_i} A_i$  in (2.14). Concretely,  $n_{\theta_i} = 0, 1, 2, 3, 4$  for  $A_i = \bar{F}, \bar{\psi}, \phi, \psi, F$ , respectively. For the operators without  $\dot{\alpha}$  indices, we have  $k_i = 0$  and also  $N_i = n_{\theta_i} = 2, 3, 4$ .

The computation of the NMHV form factor parallels the calculations done for amplitudes in subsection 4.1. Starting from the position twistor expression (A.4), we insert on-shell momentum eigenstates. Inserting on-shell momentum eigenstates in the **W** vertex gives

$$\begin{aligned} \mathcal{U}_x(\mathcal{Z}_1, \mathcal{Z}_2; \mathbf{P}_1, \dots, \mathbf{P}_m) &\equiv \mathbf{W}_x(\mathcal{Z}_1, \mathcal{Z}_2; \mathcal{A}_{\mathbf{P}_1}, \dots, \mathcal{A}_{\mathbf{P}_m}) \\ &= \frac{\langle \lambda_1 \lambda_2 \rangle}{\langle \lambda_1 p_1 \rangle \left( \prod_{k=1}^{m-1} \langle p_k p_{k+1} \rangle \right) \langle p_m \lambda_2 \rangle} e^{i \sum_{k=1}^m (x p_k + \theta p_k \eta_k)}, \end{aligned} \quad (\text{A.6})$$

for  $m \geq 1$  and  $\mathcal{U}_x(\mathcal{Z}_1, \mathcal{Z}_2; ) = 1$  for  $m = 0$ ; cf. (C.1) in [2]. This leads to the general formula (4.13). Analogously, we obtain for the contribution  $\mathbf{A}_i$  (A.2) in which the vertex is connected to the edge  $x_i$  of the cog:

$$\begin{aligned}
 \hat{\mathbf{A}}_i &= i(4\pi)^2 \sum_{m_i-1 \leq j < k \leq m'_i} \mathcal{U}_{x'_i}(\mathcal{Z}''_{i-1}, \mathcal{Z}_i; \mathbf{P}_{m''_{i-1}+1}, \dots, \mathbf{P}_{m_i-1}) \mathcal{U}_{x''_i}(\mathcal{Z}'_i, \mathcal{Z}''_i; \mathbf{P}_{m'_i+1}, \dots, \mathbf{P}_{m''_i}) \\
 &\times \int \frac{d^4 z d^8 \vartheta}{(2\pi)^4} \frac{e^{i \sum_{s=j+1}^k (z p_s + \vartheta p_s \eta_s)}}{(\prod_{s=j+1}^{k-1} \langle p_s p_{s+1} \rangle) \langle p_k p_{j+1} \rangle} \\
 &\times \mathbf{F}_{D^{k_i} A_i} \mathcal{U}_{x_i}(\mathcal{Z}_i, \mathcal{Z}'_i; \mathbf{P}_{m_i}, \dots, \mathbf{P}_j, \mathbf{P}_{k+1}, \dots, \mathbf{P}_{m'_i}) \\
 &\times \left[ \mathcal{Z}_{(x_i, \theta_i)}(p_{k+1} \xrightarrow{k=m'_i} \lambda'_i), \mathcal{Z}_{(x_i, \theta_i)}(p_j \xrightarrow{j=m_i-1} \lambda_i), \star, \mathcal{Z}_{(z, \vartheta)}(p_{j+1}), \mathcal{Z}_{(z, \vartheta)}(p_k) \right], \tag{A.7}
 \end{aligned}$$

where we have placed a  $\hat{\phantom{A}}$  on the contribution  $\mathbf{A}_i$  from (A.2) to indicate the fact that it uses the momentum on-shell external states (2.7) and that we have included the forming operator (2.14). Expressions similar to (A.7) can also be obtained for the contributions  $\hat{\mathbf{B}}_i$  and  $\hat{\mathbf{C}}_i$  coming from the other edges of the cog, see (A.3).

In order to proceed as for amplitudes at this point, it is necessary to be able to change the order of the  $z$  integration from the vertex  $\mathbf{V}$  and the derivatives in the forming operator  $\mathbf{F}_{D^{k_i} A_i}$ . For operators without space-time derivatives in the forming operator, i.e.  $N_i = n_{\theta_i}$ , this is possible. As with amplitudes, we then shift  $z \rightarrow z + x_i$  and  $\vartheta \rightarrow \vartheta + \theta_i$ . After doing the integration, we obtain

$$\begin{aligned}
 \hat{\mathbf{A}}_i &= \sum_{m_i-1 \leq j < k \leq m'_i} \mathcal{U}_{x'_i}(\mathcal{Z}''_{i-1}, \mathcal{Z}_i; \mathbf{P}_{m''_{i-1}+1}, \dots, \mathbf{P}_{m_i-1}) \mathcal{U}_{x''_i}(\mathcal{Z}'_i, \mathcal{Z}''_i; \mathbf{P}_{m'_i+1}, \dots, \mathbf{P}_{m''_i}) \\
 &\times \left( \mathbf{F}_{D^{k_i} A_i}^{N_i=n_{\theta_i}} \mathcal{U}_{x_i}(\mathcal{Z}_i, \mathcal{Z}'_i; \mathbf{P}_{m_i}, \dots, \mathbf{P}_{m'_i}) \right) \frac{\langle (p_j \xrightarrow{j=m_i-1} \lambda_i) p_{j+1} \rangle \langle p_k (p_{k+1} \xrightarrow{k=m'_i} \lambda'_i) \rangle}{\langle (p_j \xrightarrow{j=m_i-1} \lambda_i) (p_{k+1} \xrightarrow{k=m'_i} \lambda'_i) \rangle \langle p_k p_{j+1} \rangle} \\
 &\times \left[ \mathcal{Z}_0(p_{k+1} \xrightarrow{k=m'_i} \lambda'_i), \mathcal{Z}_0(p_j \xrightarrow{j=m_i-1} \lambda_i), \star, \mathcal{Z}_{\sum_{s=j+1}^k \mathbf{P}_s}(p_{j+1}), \mathcal{Z}_{\sum_{s=j+1}^k \mathbf{P}_s}(p_k) \right], \tag{A.8}
 \end{aligned}$$

where we have used (4.7) and rearranged the second  $\mathcal{U}$ . As in the case of amplitudes, the five-bracket in (A.8) is independent of  $x_i$  and  $\theta_i$ . This has the important consequence that the derivatives specifying the operator only act on the exponential functions contained in  $\mathcal{U}_{x_i}$ . We can now act with the forming operator (2.13), which affects only  $\mathcal{U}_{x_i}$  and was computed in appendix C of [2]. We then take the operator limit, which sends all  $x_i$  to  $x$ ,  $\lambda_i, \lambda'_i$  to  $\tau_i$  and  $\lambda''_i$  to the reference spinor  $\mathbf{n}$ ; see figure 2 and also appendix A of [2]. The computation is very similar to the one presented in [2] for the MHV form factor, up to the five-bracket that we need to treat carefully. We find

$$\begin{aligned}
 \hat{\mathbf{A}}_i &= e^{i \sum_{s=m''_{i-1}+1}^{m'_i} x p_s} \sum_{m_i-1 \leq j < k \leq m'_i} \tilde{\mathcal{L}}'_i(m''_{i-1}, m_i) \tilde{\mathcal{L}}_i(m_i, m'_i) \tilde{\mathcal{L}}''_i(m'_i, m''_i) \\
 &\times \left[ \mathcal{Z}_0(p_j \xrightarrow{j=m_i-1} \tau_i), \mathcal{Z}_0(p_{j+1}), \star, \mathcal{Z}_{y_k - y_j}(p_k), \mathcal{Z}_{y_k - y_j}(p_{k+1} \xrightarrow{k=m'_i} \tau_i) \right], \tag{A.9}
 \end{aligned}$$

where we have simplified the five-bracket by absorbing some angular brackets as in (4.9). The factors  $\tilde{\mathcal{L}}_i$ ,  $\tilde{\mathcal{L}}'_i$  and  $\tilde{\mathcal{L}}''_i$  were defined in appendix C of [2] and we reproduce them here for the reader's convenience:

$$\begin{aligned}\tilde{\mathcal{L}}_j(m_j, m'_j) &= -\frac{\left(\sum_{k=m_j}^{m'_j} \langle \tau_j p_k \rangle [\bar{p}_k \bar{\tau}_j]\right)^{N_j - n_{\theta_j}} \left(\sum_{k=m_j}^{m'_j} \langle \tau_j p_k \rangle \{\xi_j \eta_k\}\right)^{n_{\theta_j}}}{\langle \tau_j p_{m_j} \rangle \prod_{k=m_j}^{m'_j-1} \langle p_k p_{k+1} \rangle \langle p_{m'_j} \tau_j \rangle}, \\ \tilde{\mathcal{L}}'_j(m''_{j-1}, m_j) &= \frac{\langle \mathbf{n} \tau_j \rangle}{\langle \mathbf{n} p_{m''_{j-1}+1} \rangle \prod_{k=m''_{j-1}+1}^{m_j-2} \langle p_k p_{k+1} \rangle \langle p_{m_j-1} \tau_j \rangle}, \\ \tilde{\mathcal{L}}''_j(m'_j, m''_j) &= \frac{\langle \tau_j \mathbf{n} \rangle}{\langle \tau_j p_{m'_j+1} \rangle \prod_{k=m'_j+1}^{m''_j-1} \langle p_k p_{k+1} \rangle \langle p_{m''_j} \mathbf{n} \rangle},\end{aligned}\tag{A.10}$$

where we recall that we are currently treating only the case  $N_i = n_{\theta_i}$ . These objects are just the operator limit of (A.6) for each of the three edges of the cog with label  $i$ , stripped of the exponential factors. We need to keep in mind the condition (A.1) on the integers  $\{m_i, m'_i, m''_i\}$ . Furthermore, when no particle is emitted from a specific edge, the corresponding contribution equals one, i.e.

$$\tilde{\mathcal{L}}'_j(m''_{j-1}, m_j = m''_{j-1}) = 1, \quad \tilde{\mathcal{L}}''_j(m'_j, m''_j = m'_j) = 1.\tag{A.11}$$

In particular, the dependence on the operator  $\mathcal{O}$  is completely contained in the terms  $\tilde{\mathcal{L}}_i(m_i, m'_i)$ . Using the momentum supertwistors for the form factors and the same technique as in (4.10) for the five-brackets, we write

$$\begin{aligned}\hat{\mathbf{A}}_i &= e^{i \sum_{s=m''_{i-1}+1}^{m''_i} x_{\mathbf{p}_s}} \tilde{\mathcal{L}}'_i(m''_{i-1}, m_i) \tilde{\mathcal{L}}_i(m_i, m'_i) \tilde{\mathcal{L}}''_i(m'_i, m''_i) \\ &\times \sum_{m_i-1 \leq j < k \leq m'_i} \left[ \mathcal{W}_j \xrightarrow{j=m_i-1} \mathcal{Z}_{y_j}(\tau_i), \mathcal{W}_{j+1}, \star, \mathcal{W}_k, \mathcal{W}_{k+1} \xrightarrow{k=m'_i} \mathcal{Z}_{y_k}(\tau_i) \right].\end{aligned}\tag{A.12}$$

Similarly, we obtain for  $\hat{\mathbf{B}}_i$  and  $\hat{\mathbf{C}}_i$  the formulae

$$\begin{aligned}\hat{\mathbf{B}}_i &= e^{i \sum_{s=m''_{i-1}+1}^{m''_i} x_{\mathbf{p}_s}} \tilde{\mathcal{L}}'_i(m''_{i-1}, m_i) \tilde{\mathcal{L}}_i(m_i, m'_i) \tilde{\mathcal{L}}''_i(m'_i, m''_i) \\ &\times \sum_{m''_{i-1} \leq j < k \leq m_i-1} \left[ \mathcal{W}_j \xrightarrow{j=m''_{i-1}} \mathcal{Z}_{y_j}(\mathbf{n}), \mathcal{W}_{j+1}, \star, \mathcal{W}_k, \mathcal{W}_{k+1} \xrightarrow{k=m_i-1} \mathcal{Z}_{y_k}(\tau_i) \right], \\ \hat{\mathbf{C}}_i &= e^{i \sum_{s=m''_{i-1}+1}^{m''_i} x_{\mathbf{p}_s}} \tilde{\mathcal{L}}'_i(m''_{i-1}, m_i) \tilde{\mathcal{L}}_i(m_i, m'_i) \tilde{\mathcal{L}}''_i(m'_i, m''_i) \\ &\times \sum_{m'_i \leq j < k \leq m''_i} \left[ \mathcal{W}_j \xrightarrow{j=m'_i} \mathcal{Z}_{y_j}(\tau_i), \mathcal{W}_{j+1}, \star, \mathcal{W}_k, \mathcal{W}_{k+1} \xrightarrow{k=m''_i} \mathcal{Z}_{y_k}(\mathbf{n}) \right].\end{aligned}\tag{A.13}$$

Finally, we also need the contributions from the cogs that are not attached to a vertex

$$\hat{\mathbf{D}}_i = e^{i \sum_{s=m''_{i-1}+1}^{m''_i} x_{\mathbf{p}_s}} \tilde{\mathcal{L}}'_i(m''_{i-1}, m_i) \tilde{\mathcal{L}}_i(m_i, m'_i) \tilde{\mathcal{L}}''_i(m'_i, m''_i).\tag{A.14}$$

Putting everything together, the NMHV form factor is obtained after summing over all the cogs of the Wilson loop as well as over all the distributions of the integers  $\{m_s, m'_s, m''_s\}$  ordered as in (A.1) and applying the Fourier transformation  $\int \frac{d^4x}{(2\pi)^4} e^{-iqx}$ :

$$\mathcal{F}_O^{\text{NMHV}}(1, \dots, n; \mathbf{q}) = \int \frac{d^4x}{(2\pi)^4} e^{-iqx} \sum_{\{m_s, m'_s, m''_s\}} \sum_{i=1}^L \left( \prod_{r=1}^{i-1} \hat{\mathbf{D}}_r \right) (\hat{\mathbf{A}}_i + \hat{\mathbf{B}}_i + \hat{\mathbf{C}}_i) \left( \prod_{r=i+1}^L \hat{\mathbf{D}}_r \right), \tag{A.15}$$

where the quantities  $\hat{\mathbf{A}}_r, \dots, \hat{\mathbf{D}}_r$  are defined in (A.12), (A.13) and (A.14). Pulling out the Parke-Taylor denominator and the momentum-conserving delta function, we can bring (A.15) to the form (4.14).

## B Fourier-type integrals

In this appendix, we derive several Fourier-type integral identities that are used in section 4. Moreover, we demonstrate the non-commutativity of the integral with space-time derivatives coming from the forming operator in a given example.

### B.1 Fourier-type integrals with ratios of scalar products

The main bosonic Fourier-type integral identity that we want to show reads

$$\int \frac{d^4x}{(2\pi)^4} \frac{1}{x^2} \frac{\langle s_1|x|\zeta \rangle \cdots \langle s_k|x|\zeta \rangle}{\langle t_1|x|\zeta \rangle \cdots \langle t_k|x|\zeta \rangle} e^{iqx} = \frac{1}{i(4\pi)^2} \frac{1}{\mathbf{q}^2} \frac{\langle s_1|\mathbf{q}|\zeta \rangle \cdots \langle s_k|\mathbf{q}|\zeta \rangle}{\langle t_1|\mathbf{q}|\zeta \rangle \cdots \langle t_k|\mathbf{q}|\zeta \rangle}. \tag{B.1}$$

In contrast to usual Fourier integrals, this integral, which stems from the bosonic part of the twistor-space propagator, is not only badly behaved for  $x^2 = 0$  but also for  $\langle t_i|x|\zeta \rangle = 0$ . In addition to calculating the integral, we thus also require a prescription that makes it well-defined. We will first address the latter issue with an explicit calculation of the case  $k = 1$ . Assuming that a similar prescription also exists for  $k \geq 2$ , we then argue that the integral must be given by (B.1).

**Case  $k = 1$ .** Regularizing the integral (B.1) is not trivial, due to the fact that the term  $\langle t_1|x|\zeta \rangle$  introduces an extra pole in the  $x_0$  integration. The guiding principle behind the regularization prescription that we choose is based on the following a-posteriori thinking: the expression (B.1) correctly translates the amplitudes from position twistor space to the known amplitudes in momentum space, i.e. the additional pole cannot contribute. A way to get rid of the extra pole starts by taking the vector  $\mathbf{t}_1 \equiv t_1^\alpha \zeta^{\dot{\alpha}}$  and giving it a small mass. Then, we can boost it to be parallel to the  $x_0$  direction and replace the  $\mathbf{t}_1 x$  in the denominator by  $\mathbf{t}_1 x + i\varepsilon \mathbf{q} \mathbf{t}_1 = 2\mathbf{t}_{1,0}(x_0 + i\varepsilon \mathbf{q}_0)$ . Since the sign of  $\mathbf{q}_0$  determines how to close the contour due to the exponent  $e^{iqx}$ , i.e. whether we close the contour in the upper or lower  $x_0$  plane, the extra pole is always avoided. One issue with this approach is that the intermediate steps of the computation break Lorentz invariance and it is quite non-trivial to see how it is restored in the end. Thus, we prefer to do the computation in Euclidean space and to Wick rotate in the end.

Let us furthermore replace  $s_1^\alpha \zeta^{\dot{\alpha}}$  and  $t_1^\alpha \zeta^{\dot{\alpha}}$  by arbitrary vectors  $s^{\alpha\dot{\alpha}}$  and  $t^{\alpha\dot{\alpha}}$ . Thus, we consider the integral

$$\tilde{I} = \int \frac{d^4x}{(2\pi)^4} \frac{1}{x^2} \frac{x \cdot s}{x \cdot t} e^{iq \cdot x} = \int \frac{d^4x}{(2\pi)^4} \frac{1}{x^2} \frac{2x \cdot s}{2x \cdot t} e^{2iq \cdot x} = \frac{1}{4} \int \frac{d^4x}{(2\pi)^4} \frac{1}{x^2} \frac{x \cdot s}{x \cdot t} e^{iq \cdot x}, \quad (\text{B.2})$$

where we have rescaled  $x$  to absorb the factor of 2 in the scalar product; recall that  $xy \equiv x^\alpha y_{\dot{\alpha}} = 2x_\mu y^\mu \equiv 2x \cdot y$ . We will be interested in the limit in which the vectors  $s$  and  $t$  become complex and obey  $s^2 = t^2 = s \cdot t = 0$ , but for now we take them to be real. The above integral (B.2) is ill-defined due to the  $x \cdot t$  in the denominator and we propose to define it as<sup>15</sup>

$$\tilde{I} = \frac{1}{4} \lim_{\varepsilon \rightarrow 0} \int \frac{d^4x}{(2\pi)^4} \frac{1}{x^2} \frac{x \cdot s}{x \cdot t + i\varepsilon(\mathbf{q} \cdot \hat{\mathbf{t}})} e^{iq \cdot x}. \quad (\text{B.3})$$

We decompose  $x$  as  $x = x_t \hat{\mathbf{t}} + x_\perp$  with  $\hat{\mathbf{t}} = \mathbf{t}/|\mathbf{t}|$  and  $\mathbf{t} \cdot x_\perp = 0$ . We see that we can compute the integration over  $x_t$  in (B.3) by doing a contour integral that we close in the upper (lower) half plane for  $\mathbf{q} \cdot \hat{\mathbf{t}} > 0$  ( $\mathbf{q} \cdot \hat{\mathbf{t}} < 0$ ). The poles are at  $x_t = \pm i|x_\perp|$  and  $x_t = -i\varepsilon \mathbf{q} \cdot \hat{\mathbf{t}}$ . Due to the way that we close the contour, the last pole never contributes, regardless of the value of  $\mathbf{q} \cdot \hat{\mathbf{t}}$ . Hence, we obtain after computing the residues and taking the limit  $\varepsilon \rightarrow 0$

$$\begin{aligned} \tilde{I} &= \frac{1}{4} \frac{2\pi i}{(2\pi)^4} \int d^3x_\perp \left[ \Theta(\mathbf{q} \cdot \hat{\mathbf{t}}) \frac{(x_\perp + i|x_\perp|\hat{\mathbf{t}}) \cdot \mathbf{s}}{(i|x_\perp||\mathbf{t}|)} e^{-|x_\perp|\mathbf{q} \cdot \hat{\mathbf{t}}} \right. \\ &\quad \left. - \Theta(-\mathbf{q} \cdot \hat{\mathbf{t}}) \frac{(x_\perp - i|x_\perp|\hat{\mathbf{t}}) \cdot \mathbf{s}}{(i|x_\perp||\mathbf{t}|)} e^{|x_\perp|\mathbf{q} \cdot \hat{\mathbf{t}}} \right] \frac{e^{i\mathbf{q} \cdot x_\perp}}{2i|x_\perp|} \\ &= \frac{1}{4} \frac{2\pi i}{(2\pi)^4} \int d^3x_\perp \left[ \frac{\mathbf{s} \cdot \mathbf{t}}{t^2} + \text{sgn}(\mathbf{q} \cdot \hat{\mathbf{t}}) \frac{\mathbf{s} \cdot x_\perp}{i|\mathbf{t}||x_\perp|} \right] \frac{e^{i\mathbf{q} \cdot x_\perp - |x_\perp|\mathbf{q} \cdot \hat{\mathbf{t}}}}{2i|x_\perp|}, \end{aligned} \quad (\text{B.4})$$

where  $\Theta$  denotes the Heaviside step function and  $\text{sgn}(\mathbf{q} \cdot \hat{\mathbf{t}})$  the sign of  $\mathbf{q} \cdot \hat{\mathbf{t}}$ . Going to spherical coordinates in the space perpendicular to  $\mathbf{t}$  via  $x_\perp = r\hat{x}_\perp$ , we find

$$\begin{aligned} \tilde{I} &= \frac{1}{4} \frac{2\pi i}{(2\pi)^4} \int_0^\infty r^2 dr \int_{S^2} d\hat{x}_\perp^2 \frac{1}{2ir} e^{-r(|\mathbf{q} \cdot \hat{\mathbf{t}}| - i\mathbf{q} \cdot \hat{x}_\perp)} \left( \frac{\mathbf{s} \cdot \mathbf{t}}{t^2} + \text{sgn}(\mathbf{q} \cdot \hat{\mathbf{t}}) \frac{\mathbf{s} \cdot \hat{x}_\perp}{i|\mathbf{t}|} \right) \\ &= \frac{1}{4} \frac{\pi}{(2\pi)^4} \int_{S^2} d\hat{x}_\perp^2 \frac{1}{(|\mathbf{q} \cdot \hat{\mathbf{t}}| - i\mathbf{q} \cdot \hat{x}_\perp)^2} \left( \frac{\mathbf{s} \cdot \mathbf{t}}{t^2} + \text{sgn}(\mathbf{q} \cdot \hat{\mathbf{t}}) \frac{\mathbf{s} \cdot \hat{x}_\perp}{i|\mathbf{t}|} \right). \end{aligned} \quad (\text{B.5})$$

We now define  $\mathbf{q}_\perp = \mathbf{q} - \mathbf{t} \frac{\mathbf{q} \cdot \mathbf{t}}{t^2}$  and similarly for  $\mathbf{s}_\perp$ . Setting  $\hat{\mathbf{q}}_\perp = \mathbf{q}_\perp/|\mathbf{q}_\perp|$ , we decompose  $\hat{x}_\perp$  as

$$\hat{x}_\perp = \cos(\theta)\hat{\mathbf{q}}_\perp + \sin(\theta)(\cos(\varphi)\hat{z} + \sin(\varphi)\hat{w}), \quad (\text{B.6})$$

where  $\hat{z}$  and  $\hat{w}$  are two orthonormal vectors spanning the plane orthogonal to  $\mathbf{t}$  and  $\hat{\mathbf{q}}_\perp$ . Then,  $d\hat{x}_\perp^2 = \sin(\theta)d\theta d\varphi$  and the  $\varphi$  integral is easily done. It yields

$$\tilde{I} = \frac{1}{4} \frac{2\pi^2}{(2\pi)^4} \int_0^\pi \frac{\sin(\theta)d\theta}{(|\mathbf{q} \cdot \hat{\mathbf{t}}| + i|\mathbf{q}_\perp| \cos(\theta))^2} \left( \frac{\mathbf{s} \cdot \mathbf{t}}{t^2} + \text{sgn}(\mathbf{q} \cdot \hat{\mathbf{t}}) \cos(\theta) \frac{\mathbf{s}_\perp \cdot \hat{\mathbf{q}}_\perp}{i|\mathbf{t}|} \right). \quad (\text{B.7})$$

Integrating by parts and using  $\log \frac{a+bi}{a-bi} = 2i \arctan \frac{b}{a}$ , we find

$$\tilde{I} = \frac{1}{(4\pi)^2 \mathbf{q}^2} \left[ \frac{\mathbf{s} \cdot \mathbf{t}}{t^2} - \frac{(\mathbf{q} \cdot \hat{\mathbf{t}})(\mathbf{q}_\perp \cdot \mathbf{s}_\perp)}{|\mathbf{t}|\mathbf{q}_\perp^2} \right] + \frac{1}{(4\pi)^2} \text{sgn}(\mathbf{q} \cdot \hat{\mathbf{t}}) \frac{\mathbf{q}_\perp \cdot \mathbf{s}_\perp}{|\mathbf{t}||\mathbf{q}_\perp|^3} \arctan \frac{|\mathbf{q}_\perp|}{|\mathbf{q} \cdot \hat{\mathbf{t}}|}. \quad (\text{B.8})$$

<sup>15</sup>We thank Erik Panzer for a very helpful discussion on this point.

We now insert  $|\mathbf{q}_\perp|^2 = \mathbf{q}^2 - (\mathbf{q} \cdot \mathbf{t})^2/t^2$  and  $\mathbf{s}_\perp \cdot \mathbf{q}_\perp = \mathbf{s} \cdot \mathbf{q} - (\mathbf{q} \cdot \mathbf{t})(\mathbf{s} \cdot \mathbf{t})/t^2$  so that after a couple of trivial manipulations

$$\tilde{I} = \frac{1}{(4\pi)^2} \left[ \frac{1}{\mathbf{q}^2} \frac{(\mathbf{s} \cdot \mathbf{q})(\mathbf{t} \cdot \mathbf{q}) - (\mathbf{s} \cdot \mathbf{t})\mathbf{q}^2}{(\mathbf{q} \cdot \mathbf{t})^2 - t^2\mathbf{q}^2} + \frac{(\mathbf{s} \cdot \mathbf{t})(\mathbf{q} \cdot \mathbf{t}) - t^2(\mathbf{s} \cdot \mathbf{q})}{|(\mathbf{q} \cdot \mathbf{t})^2 - t^2\mathbf{q}^2|^{\frac{3}{2}}} \arctan \frac{\sqrt{|t^2\mathbf{q}^2 - (\mathbf{q} \cdot \mathbf{t})^2|}}{|\mathbf{q} \cdot \hat{\mathbf{t}}|} \right]. \quad (\text{B.9})$$

It is now obvious how to take the limit  $\mathbf{s}^2 = \mathbf{t}^2 = \mathbf{s} \cdot \mathbf{t} = 0$ . In addition, if we now Wick rotate the integral, we pick up a factor of  $-i$  from the measure, so that

$$\int \frac{d^4x}{(2\pi)^4} \frac{1}{x^2} \frac{\langle \mathbf{s}_1 | x | \zeta \rangle}{\langle \mathbf{t}_1 | x | \zeta \rangle} e^{i\mathbf{q}x} = \frac{1}{i(4\pi)^2} \frac{1}{\mathbf{q}^2} \frac{\langle \mathbf{s}_1 | \mathbf{q} | \zeta \rangle}{\langle \mathbf{t}_1 | \mathbf{q} | \zeta \rangle}. \quad (\text{B.10})$$

**Case  $k \geq 2$ .** Let us now define  $(\mathbf{s}_i)_{\alpha\dot{\alpha}} = s_{i\alpha}\zeta_{\dot{\alpha}}$  and  $(\mathbf{t}_i)_{\alpha\dot{\alpha}} = t_{i\alpha}\zeta_{\dot{\alpha}}$  so as to rewrite (B.1) as

$$\int \frac{d^4x}{(2\pi)^4} \frac{1}{x^2} \frac{(\mathbf{s}_1 x) \cdots (\mathbf{s}_k x)}{(\mathbf{t}_1 x) \cdots (\mathbf{t}_k x)} e^{i\mathbf{q}x} = \frac{1}{i(4\pi)^2} \frac{1}{\mathbf{q}^2} \frac{(\mathbf{s}_1 \mathbf{q}) \cdots (\mathbf{s}_k \mathbf{q})}{(\mathbf{t}_1 \mathbf{q}) \cdots (\mathbf{t}_k \mathbf{q})}, \quad (\text{B.11})$$

for  $\mathbf{s}_i \mathbf{s}_j = \mathbf{s}_i \mathbf{t}_j = \mathbf{t}_i \mathbf{t}_j = 0$ . We shall assume that a prescription similar to the case  $k = 1$  exists that makes the integral well-defined. We observe first that the left-hand side of (B.11) is homogeneous of degree 1 independently in each  $\mathbf{s}_i$  and of degree  $-1$  in each  $\mathbf{t}_i$ . Furthermore, a simple change of variables shows that it is homogeneous of degree  $-2$  in  $\mathbf{q}$ . Due to Lorentz invariance and to the conditions that we wish to impose on the vectors  $\mathbf{s}_i$  and  $\mathbf{t}_i$ , we can only use the scalar products  $\mathbf{q}^2$ ,  $\mathbf{q}\mathbf{s}_i$  and  $\mathbf{q}\mathbf{t}_i$  to build the answer. Using these ingredients, we cannot build a cross-ratio that is invariant under all independent rescalings of the variables. Hence, up to a constant, the answer of the integral can only be equal to the right-hand side of (B.11). The constant, however, is fixed by considering the limit where  $\mathbf{s}_i \rightarrow \mathbf{t}_i$  for  $i = 2, \dots, k$ , thus seeing that it is independent of  $k$  and consequently given by the one found in the case  $k = 1$ . This concludes the derivation of (B.1).

**Fourier transform of the R-invariants.** Armed with the identity (B.1), we can prove an important result for the R-invariants, namely the following super Fourier-type identity:

$$\begin{aligned} I_s &\equiv \int \frac{d^4z d^8\vartheta}{(2\pi)^4} e^{iz\mathbf{q} + i\vartheta\Gamma} [\mathcal{Z}_{(x,\theta)}(\lambda_1), \mathcal{Z}_{(x,\theta)}(\lambda_2), \star, \mathcal{Z}_{(z,\vartheta)}(\lambda_3), \mathcal{Z}_{(z,\vartheta)}(\lambda_4)] \\ &= \frac{1}{i(4\pi)^2} e^{ix\mathbf{q} + i\theta\Gamma} [\mathcal{Z}_{(0,0)}(\lambda_1), \mathcal{Z}_{(0,0)}(\lambda_2), \star, \mathcal{Z}_{(\mathbf{q},\Gamma)}(\lambda_3), \mathcal{Z}_{(\mathbf{q},\Gamma)}(\lambda_4)], \end{aligned} \quad (\text{B.12})$$

with  $\Gamma \equiv \Gamma_{\alpha\dot{\alpha}}$  being 8 fermionic variables.

*Proof.* We start by using the identity

$$[\mathcal{Z}_{(x,\theta)}(\lambda_1), \mathcal{Z}_{(x,\theta)}(\lambda_2), \star, \mathcal{Z}_{(z,\vartheta)}(\lambda_3), \mathcal{Z}_{(z,\vartheta)}(\lambda_4)] = - \frac{\langle \lambda_1 \lambda_2 \rangle \langle \lambda_3 \lambda_4 \rangle \prod_{a=1}^4 \langle (\theta - \vartheta)^a | x - z | \zeta \rangle}{(x - z)^2 \prod_{j=1}^4 \langle \lambda_j | x - z | \zeta \rangle} \quad (\text{B.13})$$

for the R-invariant and by shifting  $z \rightarrow z + x$  and  $\vartheta \rightarrow \vartheta + \theta$ . After expressing  $\vartheta$  through  $\chi_3$  and  $\chi_4$  as  $\vartheta^{\alpha\dot{\alpha}} = \frac{-i}{\langle \lambda_3 \lambda_4 \rangle} (\lambda_3^\alpha \chi_4^{\dot{\alpha}} - \lambda_4^\alpha \chi_3^{\dot{\alpha}})$ , we obtain

$$I_s = e^{ix\mathbf{q} + i\theta\Gamma} \int \frac{d^4z d^8\vartheta}{(2\pi)^4} \frac{(-1) \langle \lambda_1 \lambda_2 \rangle}{\langle \lambda_3 \lambda_4 \rangle^3} \frac{1}{z^2} \frac{\prod_{a=1}^4 (\langle \lambda_3 | z | \zeta \rangle \chi_4^a - \langle \lambda_4 | z | \zeta \rangle \chi_3^a)}{\prod_{j=1}^4 \langle \lambda_j | z | \zeta \rangle}. \quad (\text{B.14})$$



Using (B.1) to perform the integral over  $z$  and expressing the result via  $\vartheta$ , we find

$$\begin{aligned}
 I_s &= \frac{e^{ix\mathbf{q}+i\theta\Gamma}}{i(4\pi)^2} \int d^8\vartheta e^{i\vartheta\Gamma} \frac{(-1)\langle\lambda_1\lambda_2\rangle\langle\lambda_3\lambda_4\rangle \prod_{a=1}^4\langle\vartheta^a|\mathbf{q}|\zeta\rangle}{\mathbf{q}^2 \prod_{j=1}^4\langle\lambda_j|\mathbf{q}|\zeta\rangle} \\
 &= \frac{e^{ix\mathbf{q}+i\theta\Gamma}}{i(4\pi)^2} \int d^8\vartheta \frac{(\vartheta\Gamma)^4}{4!} \frac{(-1)\langle\lambda_1\lambda_2\rangle\langle\lambda_3\lambda_4\rangle \prod_{a=1}^4\langle\vartheta^a|\mathbf{q}|\zeta\rangle}{\mathbf{q}^2 \prod_{j=1}^4\langle\lambda_j|\mathbf{q}|\zeta\rangle} \\
 &= \frac{e^{ix\mathbf{q}+i\theta\Gamma}}{i(4\pi)^2} \frac{(-1)\langle\lambda_1\lambda_2\rangle\langle\lambda_3\lambda_4\rangle \prod_{a=1}^4\langle\Gamma_a|\mathbf{q}|\zeta\rangle}{\mathbf{q}^2 \prod_{j=1}^4\langle\lambda_j|\mathbf{q}|\zeta\rangle},
 \end{aligned} \tag{B.15}$$

where we used that only the fourth order in the expansion of  $e^{i\vartheta\Gamma}$  contributes as there are exactly four powers of  $\vartheta$  in  $\prod_{a=1}^4\langle\vartheta^a|\mathbf{q}|\zeta\rangle$ . Using an identity analogous to (B.13), we see that (B.15) is exactly the desired result (B.12).

## B.2 An integral for a form factor with $\dot{\alpha}$ index

In section 4.3, we need to compute the following integral

$$\int \frac{d^4z}{(2\pi)^4} e^{iz\mathbf{q}} \frac{\mathbf{s}(z-x)}{((z-x)^2)^2} = e^{ix\mathbf{q}} \int \frac{d^4z}{(2\pi)^4} e^{iz\mathbf{q}} \frac{\mathbf{s}z}{(z^2)^2}. \tag{B.16}$$

We solve it in Euclidean space, i.e. we calculate

$$I_x = \int \frac{d^4z}{(2\pi)^4} e^{2iz\cdot\mathbf{q}} \frac{2\mathbf{s}\cdot z}{(z^2)^2}, \tag{B.17}$$

where we have also dropped the phase. We can rotate  $z$  so that  $\mathbf{s}$  is parallel to the  $z_0$  direction with component  $\mathbf{s}_0$ . Then, we have  $\mathbf{s}\cdot z \equiv \mathbf{s}_\mu z^\mu = \mathbf{s}_0 z_0$ . We write  $\vec{\mathbf{q}}$  for the component of  $\mathbf{q}$  that is perpendicular to the 0-direction. Doing a contour integral for  $z_0$ , we find

$$I_x = \frac{2\mathbf{s}_0}{(2\pi)^4} \int d^3\vec{z} e^{2i\vec{z}\cdot\vec{\mathbf{q}}} \int_{-\infty}^{\infty} dz_0 \frac{z_0 e^{2iz_0\mathbf{q}_0}}{(z_0^2 + |\vec{z}|^2)^2} = \frac{2\mathbf{s}_0}{(2\pi)^4} \int d^3\vec{z} e^{2i\vec{z}\cdot\vec{\mathbf{q}}} \frac{\pi i \mathbf{q}_0 e^{-2|\vec{z}||\mathbf{q}_0|}}{|\vec{z}|}. \tag{B.18}$$

Recognizing  $\mathbf{s}_0\mathbf{q}_0$  as  $\mathbf{s}\cdot\mathbf{q}$  and introducing polar coordinates for the remaining  $\vec{z}$  integration, we find

$$I_x = \frac{\pi i 2\mathbf{s}\cdot\mathbf{q}}{(2\pi)^4} 2\pi \int_0^\infty r^2 dr \int_{-1}^1 du e^{2iru|\vec{\mathbf{q}}|} \frac{e^{-2r|\mathbf{q}_0|}}{r} = \frac{\pi i}{(2\pi)^3} \frac{2\mathbf{s}\cdot\mathbf{q}}{2\mathbf{q}^2} = \frac{i}{(4\pi)^2} \frac{2\mathbf{s}\cdot\mathbf{q}}{\mathbf{q}^2}. \tag{B.19}$$

By Wick rotating to Minkowski space as before, which gives an extra  $-i$  factor, we obtain the final result:

$$\int \frac{d^4z}{(2\pi)^4} e^{iz\mathbf{q}} \frac{\mathbf{s}(z-x)}{((z-x)^2)^2} = \frac{\mathbf{s}\mathbf{q}}{(4\pi)^2} \frac{e^{ix\mathbf{q}}}{\mathbf{q}^2}. \tag{B.20}$$

We remark that, unlike in the case that we shall present in the next appendix B.3, we can compute (B.20) by exchanging integral and derivative.<sup>16</sup> Namely, we can write after

<sup>16</sup>The reason why we can exchange the derivative and integral here but not in the next section is related to the additional factor  $tz$  in the denominator.

taking the derivative out of the integral and shifting  $z$  by  $x$ :

$$\begin{aligned} \int \frac{d^4 z}{(2\pi)^4} e^{izq} \frac{s(z-x)}{((z-x)^2)^2} &= s_{\alpha\dot{\alpha}} \frac{\partial}{\partial x_{\alpha\dot{\alpha}}} \int \frac{d^4 z}{(2\pi)^4} e^{izq} \frac{1}{(z-x)^2} \\ &= s_{\alpha\dot{\alpha}} \frac{\partial}{\partial x_{\alpha\dot{\alpha}}} e^{ixq} \int \frac{d^4 z}{(2\pi)^4} \frac{e^{izq}}{z^2} = \frac{s q}{(4\pi)^2} \frac{e^{ixq}}{q^2}, \end{aligned} \quad (\text{B.21})$$

where we have used (B.1) for  $k=0$  and the definition  $x^2 \equiv \frac{x^{\alpha\dot{\alpha}} x_{\alpha\dot{\alpha}}}{2}$  below (2.2).

### B.3 Commutators of integrals and derivatives

In this subsection, we demonstrate that the derivative and integration in (4.18) do not commute. Therefore, it is not possible to first perform the Fourier transformations and take the space-time derivative only in the very end. We take as our example the integral in the last line of (4.18), defining

$$I_{\text{nc}} \equiv \int \frac{d^4 z}{(2\pi)^4} e^{iz(\mathbf{p}_1+\mathbf{p}_2)} \left[ \tau^{\alpha\bar{\tau}\dot{\alpha}} \frac{\partial}{\partial x^{\alpha\dot{\alpha}}} \frac{1}{(x-z)^2} \frac{\langle p_2 | x-z | \zeta \rangle}{\langle \tau | x-z | \zeta \rangle} \right]. \quad (\text{B.22})$$

If we first evaluate the  $x$  derivative and then evaluate the integral with the help of appendix B.2, we obtain the result leading to the last line of (4.20):

$$I_{\text{nc}} = \frac{1}{i(4\pi)^2} \frac{e^{ix(\mathbf{p}_1+\mathbf{p}_2)}}{(\mathbf{p}_1+\mathbf{p}_2)^2} \langle p_2 | \mathbf{p}_1 + \mathbf{p}_2 | \bar{\tau} \rangle. \quad (\text{B.23})$$

Alternatively, if we pull out the derivative in front of the integral and then shift  $z$  by  $x$ , we find

$$\begin{aligned} \tilde{I}_{\text{nc}} &\equiv \tau^{\alpha\bar{\tau}\dot{\alpha}} \frac{\partial}{\partial x^{\alpha\dot{\alpha}}} \int \frac{d^4 z}{(2\pi)^4} e^{iz(\mathbf{p}_1+\mathbf{p}_2)} \frac{1}{(x-z)^2} \frac{\langle p_2 | x-z | \zeta \rangle}{\langle \tau | x-z | \zeta \rangle} \\ &= \tau^{\alpha\bar{\tau}\dot{\alpha}} \frac{\partial}{\partial x^{\alpha\dot{\alpha}}} \int \frac{d^4 z}{(2\pi)^4} e^{i(z+x)(\mathbf{p}_1+\mathbf{p}_2)} \frac{1}{z^2} \frac{\langle p_2 | z | \zeta \rangle}{\langle \tau | z | \zeta \rangle} \\ &= \tau^{\alpha\bar{\tau}\dot{\alpha}} \frac{\partial}{\partial x^{\alpha\dot{\alpha}}} \left\{ e^{i(\mathbf{p}_1+\mathbf{p}_2)x} \frac{1}{i(4\pi)^2} \frac{1}{(\mathbf{p}_1+\mathbf{p}_2)^2} \frac{\langle p_2 | \mathbf{p}_1 + \mathbf{p}_2 | \zeta \rangle}{\langle \tau | \mathbf{p}_1 + \mathbf{p}_2 | \zeta \rangle} \right\}, \end{aligned} \quad (\text{B.24})$$

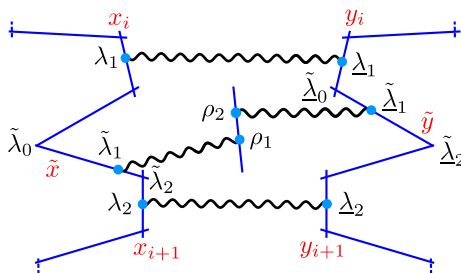
where we have used the integral formula (B.10). Evaluating the derivative and comparing the result with (B.23) yields

$$\begin{aligned} \tilde{I}_{\text{nc}} &= \frac{1}{i(4\pi)^2} \frac{e^{ix(\mathbf{p}_1+\mathbf{p}_2)}}{(\mathbf{p}_1+\mathbf{p}_2)^2} \frac{\langle \tau | \mathbf{p}_1 + \mathbf{p}_2 | \bar{\tau} \rangle \langle p_2 | \mathbf{p}_1 + \mathbf{p}_2 | \zeta \rangle}{\langle \tau | \mathbf{p}_1 + \mathbf{p}_2 | \zeta \rangle} \\ &= I_{\text{nc}} - \frac{1}{i(4\pi)^2} e^{ix(\mathbf{p}_1+\mathbf{p}_2)} \frac{\langle \tau p_2 \rangle [\zeta \bar{\tau}]}{\langle \tau | \mathbf{p}_1 + \mathbf{p}_2 | \zeta \rangle}, \end{aligned} \quad (\text{B.25})$$

where we have used (4.19) in the last line. Hence, we see that we are in general not allowed to pull the  $x$  derivative in front of the integral.

## C On the one-loop two-point correlation function in the SO(6) sector

In [3], some of the present authors calculated the one-loop dilatation operator in the SO(6) sector via the UV-divergent contributions to the one-loop two-point correlation function



**Figure 16.** A one-loop correlation function diagram with the two-point vertex attached to both operators.

in twistor space. It was stated that the contributions to the dilatation operator stem only from the diagram with the four-point vertex in figure 12, see [3] for details of this calculations. Looking at section 5, we see that we have in addition a diagram with a three-point vertex and two diagrams with a two-point vertex. In this appendix, we will discuss their contributions to the dilatation operator and show that they vanish; this complements the computation of [3].

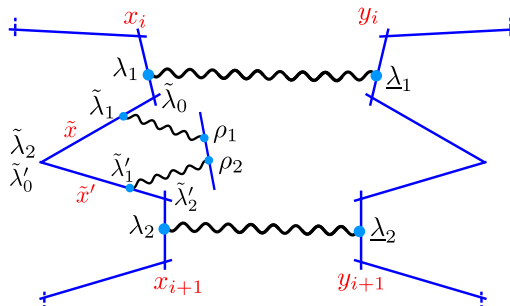
Note that, in this appendix, we refer to the spinors  $\lambda_i$  in the interaction vertex (2.4) by  $\rho_i$ . Furthermore, since we are only interested in showing whether a contribution is divergent or not, we shall often drop irrelevant overall factors.

**The two-point vertex connected to both sides of the diagram.** We are considering diagrams of the type depicted on the left-hand side of figure 13, which we reproduce in figure 16 including the new notation. The two-point vertex is attached to two gluons  $g^+$  emitted from the positions  $\tilde{\lambda}_1$  and  $\tilde{\lambda}_1$ . We have to sum over the planar ways of attaching the vertex, so that  $\tilde{x}$  has to be summed over  $x_i, x_i'', x_{i+1}'$  and  $x_{i+1}$  and similarly for  $\tilde{y}$ . Accordingly, the spinors  $\tilde{\lambda}_0, \tilde{\lambda}_2, \tilde{\lambda}_0$  and  $\tilde{\lambda}_2$  take the appropriate values of the vertices of the cogwheel Wilson loops, see figure 1. The result of the particular diagram shown in figure 16 is

$$\begin{aligned}
 V_2 = & \int \frac{d^4 z d^8 \vartheta}{(2\pi)^4} \int D\lambda_1 D\lambda_2 D\lambda_1 D\lambda_2 D\tilde{\lambda}_1 D\tilde{\lambda}_1 \frac{D\rho_1 D\rho_2}{\langle \rho_1 \rho_2 \rangle \langle \rho_2 \rho_1 \rangle} \\
 & \times \bar{\delta}^2(\mathcal{Z}_{x_i}(\lambda_1), \star, \mathcal{Z}_{y_i}(\lambda_1)) \bar{\delta}^2(\mathcal{Z}_{x_{i+1}}(\lambda_2), \star, \mathcal{Z}_{y_{i+1}}(\lambda_2)) F(\lambda_1, \lambda_2, \lambda_1, \lambda_2) \\
 & \times \frac{\langle \tilde{\lambda}_0 \tilde{\lambda}_2 \rangle}{\langle \tilde{\lambda}_0 \tilde{\lambda}_1 \rangle \langle \tilde{\lambda}_1 \tilde{\lambda}_2 \rangle} \frac{\langle \tilde{\lambda}_0 \tilde{\lambda}_2 \rangle}{\langle \tilde{\lambda}_0 \tilde{\lambda}_1 \rangle \langle \tilde{\lambda}_1 \tilde{\lambda}_2 \rangle} \bar{\delta}^{2|4}(\mathcal{Z}_{\tilde{x}}(\tilde{\lambda}_1), \star, \mathcal{Z}_z(\rho_1)) \bar{\delta}^{2|4}(\mathcal{Z}_{\tilde{y}}(\tilde{\lambda}_1), \star, \mathcal{Z}_z(\rho_2)),
 \end{aligned} \tag{C.1}$$

where  $F$  is a function of the appropriate homogeneity that takes into account which operators are inserted at the operator-bearing edges; its precise form does not matter in the present discussion. The fraction of angular brackets comes from the expansion of the two frames  $U$ . We recall that according to our prescription we are to take the integral over  $z$  in the interaction vertex last. The fermionic integration over  $\vartheta$  can however be taken immediately, which leads to an additional factor of  $\langle \rho_1 \rho_2 \rangle^4$  and removes the fermionic pieces from the  $\bar{\delta}^{2|4}$  functions. We can replace

$$\bar{\delta}^2(Z_1, Z_2, Z_3) = \int \frac{ds dt}{st} \bar{\delta}^4(Z_1 + sZ_2 + tZ_3), \tag{C.2}$$



**Figure 17.** A two-point vertex 1-loop correlation function diagram with the two-point vertex attached uniquely to one of the two cogwheel Wilson loops.

where one has to be careful that the above is homogeneous of degree 0 in  $Z_2$  and  $Z_3$  but of degree  $-4$  in  $Z_1$ . The appropriate form of the delta function has to be taken depending on the homogeneity of the integrand in each of the  $Z_i$  so that the integral is homogeneous of degree 0. Thus, we obtain

$$V_2 \propto \frac{F(\lambda_1, \lambda_2, \underline{\lambda}_1, \underline{\lambda}_2) \langle \tilde{\lambda}_0 \tilde{\lambda}_2 \rangle \langle \tilde{\lambda}_0 \tilde{\lambda}_2 \rangle}{|x_i - y_i|^2 |x_{i+1} - y_{i+1}|^2} \int \frac{d^4 z}{|\tilde{x} - z|^2 |\tilde{y} - z|^2} \frac{\langle \rho_1 \rho_2 \rangle^2}{\langle \tilde{\lambda}_0 \tilde{\lambda}_1 \rangle \langle \tilde{\lambda}_1 \tilde{\lambda}_2 \rangle \langle \tilde{\lambda}_0 \tilde{\lambda}_1 \rangle \langle \tilde{\lambda}_1 \tilde{\lambda}_2 \rangle}, \quad (\text{C.3})$$

where the spinors that we integrate over in (C.1) have been *fixed* by the delta function to the values

$$\begin{aligned} \lambda_{1\alpha} &= -\underline{\lambda}_{1\alpha} = \frac{i(x_i - y_i)_{\alpha\dot{\alpha}} \zeta^{\dot{\alpha}}}{(x_i - y_i)^2}, & \lambda_{2\alpha} &= -\underline{\lambda}_{2\alpha} = \frac{i(x_{i+1} - y_{i+1})_{\alpha\dot{\alpha}} \zeta^{\dot{\alpha}}}{(x_{i+1} - y_{i+1})^2}, \\ \tilde{\lambda}_{1\alpha} &= -\rho_{1\alpha} = \frac{i(\tilde{x} - z)_{\alpha\dot{\alpha}} \zeta^{\dot{\alpha}}}{(\tilde{x} - z)^2}, & \tilde{\lambda}_{1\alpha} &= -\rho_{2\alpha} = \frac{i(\tilde{y} - z)_{\alpha\dot{\alpha}} \zeta^{\dot{\alpha}}}{(\tilde{y} - z)^2}, \end{aligned} \quad (\text{C.4})$$

and the factors of  $|\tilde{x} - z|^2 |\tilde{y} - z|^2$  and  $|x_i - y_i|^2 |x_{i+1} - y_{i+1}|^2$  in the denominator arose due to the Jacobians of the integration. Here, as elsewhere,  $\zeta^{\dot{\alpha}}$  is the lower component of the (bosonic) reference twistor  $Z_*$ . We now have to take the sum over the insertion points and the operator limit, *before* we perform the integration over  $z$ . The contributions from the other insertion points are similar and the sum can be done using the Schouten identity in the same way as described in figure 8 of [2]. It effectively leads to replacing in (C.3)  $\tilde{\lambda}_0$  by  $\lambda_1$ ,  $\tilde{\lambda}_2$  by  $\lambda_2$ ,  $\tilde{\lambda}_0$  by  $\underline{\lambda}_1$  and  $\tilde{\lambda}_2$  by  $\underline{\lambda}_2$ . Finally, in the operator limit we have  $x_i, x_{i+1} \rightarrow x$  and  $y_j, y_{j+1} \rightarrow y$ , so that the prefactor  $\langle \lambda_1 \lambda_2 \rangle \langle \underline{\lambda}_1 \underline{\lambda}_2 \rangle$  in front of the integral vanishes and hence the contribution is zero.

**The two-point vertex connected to one side of the diagram.** For the other diagram containing a two-point vertex, shown in the middle of figure 13 and in more detail in figure 17, we have to connect the two propagators coming from the vertex to any combination of the edges between  $\lambda_1$  and  $\lambda_2$ . Below, we compute just the combination shown in figure 17. We find that, after the  $\vartheta$  integration and the integration over  $\lambda_1, \lambda_2, \underline{\lambda}_1$  and

$\underline{\lambda}_2$  which localizes these spinors in a similar way as (C.4), it contributes

$$V_2' \propto \frac{F(\lambda_1, \lambda_2, \underline{\lambda}_1, \underline{\lambda}_2)}{|x_i - y_i|^2 |x_{i+1} - y_{i+1}|^2} \int \frac{d^4 z}{(2\pi)^4} D\rho_1 D\rho_2 D\tilde{\lambda}_1 D\tilde{\lambda}'_1 \left[ \frac{\langle \rho_1 \rho_2 \rangle^4}{\langle \rho_1 \rho_2 \rangle^2} \frac{\langle \tilde{\lambda}_0 \tilde{\lambda}_2 \rangle}{\langle \tilde{\lambda}_0 \tilde{\lambda}_1 \rangle \langle \tilde{\lambda}_1 \tilde{\lambda}_2 \rangle} \frac{\langle \tilde{\lambda}'_0 \tilde{\lambda}'_2 \rangle}{\langle \tilde{\lambda}'_0 \tilde{\lambda}'_1 \rangle \langle \tilde{\lambda}'_1 \tilde{\lambda}'_2 \rangle} \right. \\ \left. \times \bar{\delta}^2(Z_{\tilde{x}}(\tilde{\lambda}_1), \star, Z_z(\rho_1)) \bar{\delta}^2(Z_{\tilde{x}'}(\tilde{\lambda}'_1), \star, Z_z(\rho_2)) \right]. \quad (\text{C.5})$$

The term  $\langle \rho_1 \rho_2 \rangle^2$  in the denominator comes directly from the interaction vertex in the action, while the factor  $\langle \rho_1 \rho_2 \rangle^4$  in the numerator is the result of the  $d^8 \vartheta$  integration as for the previous two-point vertex. Doing the remaining integrations and summing over the possible ways to connect the diagram leads in the operator limit (implying  $\tilde{x}, \tilde{x}' \rightarrow x$ ) to

$$\sum_{\text{connections}} V_2' \propto \frac{F(\lambda_1, \lambda_2, \underline{\lambda}_1, \underline{\lambda}_2)}{|x - y|^4} \int \frac{d^4 z}{|x - z|^4} \frac{\langle \lambda_1 \lambda_2 \rangle \langle \rho_1 \rho_2 \rangle^2}{\langle \lambda_1 \tilde{\lambda}_1 \rangle \langle \tilde{\lambda}_1 \tilde{\lambda}'_1 \rangle \langle \tilde{\lambda}'_1 \tilde{\lambda}_2 \rangle}, \quad (\text{C.6})$$

where the cancellation happens because we have *localized* the spinors

$$\lambda_{1\alpha} = \underline{\lambda}_{1\alpha} \propto (x - y)_{\alpha\dot{\alpha}} \zeta^{\dot{\alpha}}, \quad \lambda_{2\alpha} = \underline{\lambda}_{2\alpha} \propto (x - y)_{\alpha\dot{\alpha}} \zeta^{\dot{\alpha}}, \quad (\text{C.7}) \\ \tilde{\lambda}_{1\alpha} = -\rho_{1\alpha} \propto (x - z)_{\alpha\dot{\alpha}} \zeta^{\dot{\alpha}}, \quad \tilde{\lambda}'_{1\alpha} = -\rho_{2\alpha} \propto (x - z)_{\alpha\dot{\alpha}} \zeta^{\dot{\alpha}},$$

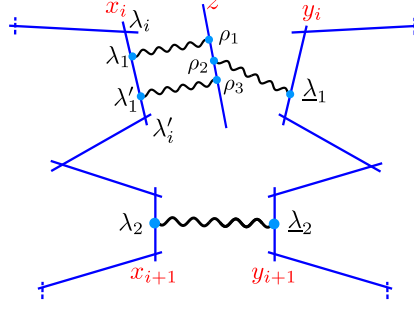
via the delta functions of the propagators. Hence, we see that the contribution becomes zero due to the vanishing of the product of brackets  $\langle \lambda_1 \lambda_2 \rangle \langle \rho_1 \rho_2 \rangle$  in the operator limit, which we take before performing the integration over  $z$ .

**Diagrams with a three-point vertex.** Finally, we consider the diagrams with a three-point vertex, where we take the operator vertex up to the second term in the expansion of the parallel propagator  $U_{(x_i, \theta)}(\mathcal{Z}_i, \mathcal{Z}_{i+1})$ , see figure 18. The particles emitted from the edge  $x_i$  of the loop can be either a scalar  $\phi$  and a positive-helicity gluon  $g^+$  or two anti-fermions  $\bar{\psi}$ . Therefore, we cannot factor out the forming operator as in the cases with the two-point vertex. In the case of two anti-fermions  $\bar{\psi}$ , both are attached to the operating-bearing edge  $x_i$  of the Wilson loop. However, when there is a scalar  $\phi$  and a positive-helicity gluon  $g^+$ , the gluon can be attached to many different edges and we have to sum over all of them. Here, we will just treat the attachment shown in figure 18 in detail and then argue that the other terms are similar and can be combined using the Schouten identity after the operator limit is taken. We obtain for the vertex on the cog  $x_i$  (see appendix B of [2] for more details)

$$\mathbf{W}_\phi = \int D\lambda_1 D\lambda'_1 \left[ \frac{\partial^2 \mathcal{A}(\mathcal{Z}_{(x, \theta)}(\lambda_1))}{\partial \chi_1^a \partial \chi_1^b} \frac{\langle \lambda_1 \lambda'_i \rangle}{\langle \lambda_1 \lambda'_1 \rangle \langle \lambda'_1 \lambda'_i \rangle} \mathcal{A}(\mathcal{Z}_{(x, \theta)}(\lambda'_1)) \right. \\ \left. + \frac{\partial \mathcal{A}(\mathcal{Z}_{(x, \theta)}(\lambda_1))}{\partial \chi_1^a} \frac{1}{\langle \lambda_1 \lambda'_1 \rangle} \frac{\partial \mathcal{A}(\mathcal{Z}_{(x, \theta)}(\lambda'_1))}{\partial \chi_1^b} \right] \Big|_{\theta=0}. \quad (\text{C.8})$$

The first term corresponds to a scalar  $\phi$  and a positive-helicity gluon  $g^+$  being emitted from  $x_i$  and the second to two anti-fermions  $\bar{\psi}$  being emitted. We treat them separately.<sup>17</sup>

<sup>17</sup>Of course, there is also a term in which the indices  $a$  and  $b$  in the second line are interchanged, but there is also a term in which the  $\mathcal{A}(\mathcal{Z}_{(x, \theta)}(\lambda_1))$  in the first term is in front of the term with two derivatives acting on  $\mathcal{A}(\mathcal{Z}_{(x, \theta)}(\lambda'_1))$ .



**Figure 18.** Three-point vertex contributions to the one-loop two-point correlation function.

On the opposite side of the correlation function, we have the operator vertex

$$\mathbf{W}_{\underline{\phi}} = \int D\lambda_1 \frac{\partial^2 \mathcal{A}(\mathcal{Z}_{(y,\theta)}(\lambda_1))}{\partial \chi_1^a \partial \chi_1^b} \Big|_{\theta=0}. \quad (\text{C.9})$$

The three  $\mathcal{A}$  fields, two at line  $x$  and one at line  $y$ , are connected to the three-point vertex

$$\mathbf{V}_3 = \int \frac{d^4 z d^8 \vartheta}{(2\pi)^4} \int \frac{D\rho_1 D\rho_2 D\rho_3}{\langle \rho_1 \rho_2 \rangle \langle \rho_2 \rho_3 \rangle \langle \rho_3 \rho_1 \rangle} \text{Tr} (\mathcal{A}(\mathcal{Z}_{(z,\vartheta)}(\rho_1)) \mathcal{A}(\mathcal{Z}_{(z,\vartheta)}(\rho_2)) \mathcal{A}(\mathcal{Z}_{(z,\vartheta)}(\rho_3))). \quad (\text{C.10})$$

The expression for the diagram corresponding to the first term in (C.8) therefore yields

$$\frac{F(\lambda_2, \underline{\lambda}_2)}{|x_{i+1} - y_{i+1}|} \int D\lambda_1 D\lambda'_1 \frac{d^4 z d^8 \vartheta}{(2\pi)^4} \frac{D\rho_1 D\rho_2 D\rho_3}{\langle \rho_1 \rho_2 \rangle \langle \rho_2 \rho_3 \rangle \langle \rho_3 \rho_1 \rangle} \frac{\partial^2}{\partial \chi_1^a \partial \chi_1^b} \delta^{2|4}(\mathcal{Z}_{(x_i,\theta)}(\lambda_1), \star, \mathcal{Z}_{(z,\vartheta)}(\rho_1)) \frac{\langle \lambda_1 \lambda'_i \rangle}{\langle \lambda_1 \lambda'_1 \rangle \langle \lambda'_1 \lambda'_i \rangle} \delta^{2|4}(\mathcal{Z}_{(x_i,\theta)}(\lambda'_1), \star, \mathcal{Z}_{(z,\vartheta)}(\rho_3)) \frac{\partial^2}{\partial \chi_1^a \partial \chi_1^b} \delta^{2|4}(\mathcal{Z}_{(z,\vartheta)}(\rho_2), \star, \mathcal{Z}_{(y_i,\theta)}(\lambda_1)), \quad (\text{C.11})$$

where we have localized two spinors as  $\lambda_{2\alpha} \propto \underline{\lambda}_{2\alpha} \propto (x_{i+1} - y_{i+1})_{\alpha\dot{\alpha}} \zeta^{\dot{\alpha}}$  and where  $F(\lambda_2, \underline{\lambda}_2)$  is a function that takes into account what operator are inserted at  $\lambda_2$  and  $\underline{\lambda}_2$ . Integrating over  $d^8 \vartheta$  gives a factor of  $\langle \rho_3 \rho_1 \rangle^2 \langle \rho_3 \rho_2 \rangle^2$ . Evaluating the remaining integrations and summing over the possible ways to connect the diagram leads in the operator limit to

$$\frac{F(\lambda_2, \underline{\lambda}_2)}{|x - y|^2} \int \frac{d^4 z}{(2\pi)^4} \frac{1}{|x - z|^4 |y - z|^2} \left[ \frac{\langle \rho_2 \lambda'_i \rangle}{\langle \rho_2 \rho_1 \rangle \langle \rho_1 \lambda'_i \rangle} \frac{\langle \rho_2 \rho_1 \rangle^2 \langle \rho_3 \rho_1 \rangle^2}{\langle \rho_1 \rho_2 \rangle \langle \rho_2 \rho_3 \rangle \langle \rho_3 \rho_1 \rangle} \right], \quad (\text{C.12})$$

$$= \frac{\langle \rho_2 \lambda'_i \rangle \langle \rho_1 \rho_3 \rangle}{\langle \rho_1 \lambda'_i \rangle \langle \rho_2 \rho_3 \rangle} \stackrel{\square \rightarrow \cdot}{=} 1$$

due to the fact that the spinorial integration leads to

$$\lambda_1 = \rho_2, \quad \lambda'_1 = \rho_1, \quad \underline{\lambda}_1 = \rho_3. \quad (\text{C.13})$$

Similarly, the second term in (C.8) yields

$$F(\lambda_2, \underline{\lambda}_2) \int D\lambda_1 D\lambda'_1 \frac{d^4 z d^8 \vartheta}{(2\pi)^4} \frac{D\rho_1 D\rho_2 D\rho_3}{\langle \rho_1 \rho_2 \rangle \langle \rho_2 \rho_3 \rangle \langle \rho_3 \rho_1 \rangle} \frac{\partial}{\partial \chi_1^a} \delta^{2|4}(\mathcal{Z}_{(x_i,\theta)}(\lambda_1), \star, \mathcal{Z}_{(z,\vartheta)}(\rho_1)) \frac{1}{\langle \lambda_1 \lambda'_1 \rangle} \frac{\partial}{\partial \chi_1^b} \delta^{2|4}(\mathcal{Z}_{(x_i,\theta)}(\lambda'_1), \star, \mathcal{Z}_{(z,\vartheta)}(\rho_3)) \frac{\partial^2}{\partial \chi_1^a \partial \chi_1^b} \delta^{2|4}(\mathcal{Z}_{(z,\vartheta)}(\rho_2), \star, \mathcal{Z}_{(y_i,\theta)}(\lambda_1)). \quad (\text{C.14})$$

Integrating over  $d^8\vartheta$  yields a factor of  $-\langle\rho_3\rho_2\rangle\langle\rho_3\rho_1\rangle\langle\rho_2\rho_1\rangle^2$  so that in the operator limit we find

$$\frac{F(\lambda_2, \lambda_2)}{|x-y|^2} \int \frac{d^4z}{(2\pi)^4} \frac{1}{|x-z|^4|y-z|^2} \left[ - \underbrace{\frac{1}{\langle\rho_2\rho_1\rangle} \frac{\langle\rho_3\rho_2\rangle\langle\rho_3\rho_1\rangle\langle\rho_1\rho_2\rangle^2}{\langle\rho_1\rho_2\rangle\langle\rho_2\rho_3\rangle\langle\rho_3\rho_1\rangle}}_{=1} \right], \quad (\text{C.15})$$

where also here the spinors are localized according to (C.13). We observe that the contributions (C.12) and (C.15) precisely cancel. Hence, the three-point vertex does not contribute to the one-loop UV divergent part of the correlation function. This completes our treatment of the dilatation operator of [3].

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