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Directed Steiner tree packing and directed tree connectivity

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Abstract

For a digraph $D = (V(D), A(D))$, and a set $S \subseteq V(D)$ with $r \in S$ and $|S| \geq 2$, an (S, r)-tree is an out-tree T rooted at r with $S \subseteq V(T)$. Two (S, r)-trees T_1 and T_2 are said to be arc-disjoint if $A(T_1) \cap A(T_2) = \emptyset$. Two arc-disjoint (S, r)-trees T_1 and T_2 are said to be internally disjoint if $V(T_1) \cap V(T_2) = S$. Let $\kappa_{S,r}(D)$ and $\lambda_{S,r}(D)$ be the maximum number of internally disjoint and arc-disjoint (S, r)-trees in D, respectively. The generalized k-vertex-strong connectivity of D is defined as

$$\kappa_k(D) = \min\{\kappa_{S,r}(D) \mid S \subseteq V(D), |S| = k, r \in S\}.$$

Similarly, the generalized k-arc-strong connectivity of D is defined as

$$\lambda_k(D) = \min\{\lambda_{S,r}(D) \mid S \subseteq V(D), |S| = k, r \in S\}.$$

The generalized k-vertex-strong connectivity and generalized k-arc-strong connectivity are also called directed tree connectivity which extends the well-established tree connectivity on undirected graphs to directed graphs and could be seen as a generalization of classical connectivity of digraphs.

In this paper, we completely determine the complexity for both $\kappa_{S,r}(D)$ and $\lambda_{S,r}(D)$ on general digraphs, symmetric digraphs and Eulerian digraphs. In particular, among our results, we prove and use the NP-completeness of 2-linkage problem restricted to Eulerian digraphs. We also give sharp bounds and equalities for the two parameters $\kappa_k(D)$ and $\lambda_k(D)$.

Keywords: Directed Steiner tree packing; Directed tree connectivity; Out-tree; Directed k-linkage; Symmetric digraph; Eulerian digraph; Semicomplete digraph.

AMS subject classification (2020): 05C05, 05C20, 05C40, 05C45, 05C70, 05C75, 05C85, 68Q25.

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1 Introduction

We refer the readers to [2, 4, 5] for graph theoretical notation and terminology not given here. Note that all digraphs considered in this paper have no parallel arcs or loops. For a graph $G = (V, E)$ and a set $S \subseteq V$ of at least two vertices, an *S-Steiner tree* or, simply, an *S-tree* is a tree T of G with $S \subseteq V(T)$. Two S-trees T_1 and T_2 are said to be *edge-disjoint* if $E(T_1) \cap E(T_2) = \emptyset$. Two edge-disjoint S-trees T_1 and T_2 are said to be *internally disjoint* if $V(T_1) \cap V(T_2) = S$. The basic problem of Steiner Tree Packing is defined as follows: the input consists of an undirected graph G and a subset of vertices $S \subseteq V(D)$, the goal is to find a largest collection of edge-disjoint S-Steiner trees. The Steiner tree packing problem has applications in VLSI circuit design [11, 22]. In this application, a Steiner tree is needed to share an electronic signal by a set of terminal nodes. Another application, arises in the Internet Domain [18]. Imagine that a given graph G represents a network. We choose arbitrary k vertices as nodes. Suppose one of them is a *broadcaster*, and all other nodes are either *users* or *routers* (also called *switches*). The broadcaster wants to broadcast as many streams of movies as possible, so that the users have the maximum number of choices. Each stream of movie is broadcasted via a tree connecting all the users and the broadcaster. In essence we need to find the maximum number Steiner trees connecting all the users and the broadcaster. Clearly, it is a Steiner tree packing problem. Besides the classical version, people also study some other variations, such as packing internally disjoint Steiner trees, packing directed Steiner trees and packing strong subgraphs [7, 8, 14, 15, 23, 24, 26].

The generalized k-connectivity $\kappa_k(G)$ of a graph $G = (V, E)$ which is related to internally disjoint Steiner tree packing problem was introduced by Hager [12] in 1985 ($2 \le k \le |V|$). The *generalized local connectivity* $\kappa_S(G)$ is the maximum number of internally disjoint S-trees in G, and the *generalized k-connectivity* of G is defined as

$$\kappa_k(G) = \min\{\kappa_S(G) \mid S \subseteq V(G), |S| = k\}.$$

Li, Mao and Sun [19] introduced the following concept of generalized k-edge-connectivity which is related to basic problem of Steiner tree packing. The *generalized local edge-connectivity* $\lambda_S(G)$ is the maximum number of edge-disjoint S-trees in G and the *generalized k-edge-connectivity* is defined as

$$\lambda_k(G) = \min\{\lambda_S(G) \mid S \subseteq V(G), |S| = k\}.$$

Observe that $\kappa_2(G) = \kappa(G)$ and $\lambda_2(G) = \lambda(G)$. Hence, these two parameters could be seen as natural generalizations of connectivity and edge connectivity of a graph, respectively. This type of generalized connectivity is also called *tree connectivity* in the literature, and has become an established area in graph theory, see a recent monograph [18] by Li and Mao on this topic.

Below we summarize the known complexity for determining whether $\kappa_S(G) \ge \ell$ and for determining whether $\lambda_S(G) \ge \ell$ for any k and ℓ. Note that if $k = 2$ or if $\ell = 1$, then the problem is equivalent to normal connectivity and therefore polynomial. In all other cases the following hold.

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Table A: $\lambda_S(G)$ for undirected graphs			
$\lambda_S(G) \geq \ell$? $\lvert S\rvert = k$	$k = 3$	$k \geq 4$ constant	k part of input
$\ell = 2$	Polynomial [6]	Polynomial [6]	NP-complete [6]
$\ell \geq 3$ constant	Polynomial [6]	Polynomial [6]	NP-complete [6]
ℓ part of input	NP-complete [6]	NP-complete [6]	NP-complete [6]

Table B: $\kappa_S(G)$ for undirected graphs			
$\kappa_S(G) \geq \ell$? $\lvert S\rvert = k$	$k = 3$	$k \geq 4$ constant	k part of input
$\ell = 2$	Polynomial [17]	Polynomial [16]	NP-complete [16]
$\ell \geq 3$ constant	Polynomial [17]	Polynomial [16]	NP-complete [16]
ℓ part of input	NP-complete [6]	NP-complete [16]	NP-complete [16]

We will now prove similar results for directed graphs. An *out-tree (respectively, in-tree)* is an oriented tree in which every vertex except one, called the *root*, has in-degree (respectively, out-degree) one. An *out-branching (respectively, in-branching)* of D is a spanning out-tree (respectively, in-tree) in D. For a digraph $D = (V(D), A(D))$, and a set $S \subseteq V(D)$ with $r \in S$ and $|S| \geq 2$, a *directed* (S, r)*-Steiner tree* or, simply, an (S, r)*-tree* is an out-tree T rooted at r with $S \subseteq V(T)$ [7]. Two (S, r)-trees T_1 and T_2 are said to be *arc-disjoint* if $A(T_1) \cap A(T_2) = \emptyset$. Two arc-disjoint (S, r)-trees T_1 and T_2 are said to be *internally disjoint* if $V(T_1) \cap V(T_2) = S$.

Cheriyan and Salavatipour [7] introduced and studied the following two directed Steiner tree packing problems. Arc-disjoint directed Steiner tree packing (ADSTP): The input consists of a digraph D and a subset of vertices $S \subseteq V(D)$ with a root r, the goal is to find a largest collection of arc-disjoint (S, r)-trees. Internally-disjoint directed Steiner tree packing (IDSTP): The input consists of a digraph D and a subset of vertices $S \subseteq V(D)$ with a root r, the goal is to find a largest collection of internally disjoint (S, r)-trees.

The following concept of directed tree connectivity is related to directed Steiner tree packing problem and is a natural extension of tree connectivity of undirected graphs to directed graphs. Let $\kappa_{S,r}(D)$ (respectively, $\lambda_{S,r}(D)$) be the maximum number of internally disjoint (respectively, arc-disjoint) (S, r)-trees in D. The *generalized* k*-vertex-strong connectivity* of D is defined as

$$\kappa_k(D) = \min\{\kappa_{S,r}(D) \mid S \subseteq V(D), |S| = k, r \in S\}.$$

Similarly, the *generalized* k*-arc-strong connectivity* of D is defined as

$$\lambda_k(D) = \min\{\lambda_{S,r}(D) \mid S \subseteq V(D), |S| = k, r \in S\}.$$

By definition, when $k = 2$, $\kappa_2(D) = \kappa(D)$ and $\lambda_2(D) = \lambda(D)$. Hence, these two parameters could be seen as generalizations of vertex-strong connectivity and arc-strong connectivity of a digraph. The generalized k-vertex-strong connectivity and k-arc-strong connectivity are also called *directed tree connectivity.*

We will in this paper prove the non-cited results of the following tables.

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Table 1: $\lambda_{S,r}(D)$ for directed graphs

$\lambda_{S,r}(D) \geq \ell$? $\|S\| = k$	$k = 3$	$k \geq 4$ constant	k part of input
$\ell = 2$	NP-complete [7]	NP-complete	NP-complete
$\ell \geq 3$ constant	NP-complete	NP-complete	NP-complete
ℓ part of input	NP-complete	NP-complete	NP-complete

Table 2: $\kappa_{S,r}(D)$ for directed graphs

$\kappa_{S,r}(D) \geq \ell$? $\|S\| = k$	$k = 3$	$k \geq 4$ constant	k part of input
$\ell = 2$	NP-complete [7]	NP-complete	NP-complete
$\ell \geq 3$ constant	NP-complete	NP-complete	NP-complete
ℓ part of input	NP-complete	NP-complete	NP-complete

A digraph D is *symmetric* if every arc in D belongs to a 2-cycle. That is, if $xy \in A(D)$ then $yx \in A(D)$. In other words, a symmetric digraph D can be obtained from its underlying undirected graph G by replacing each edge of G with the corresponding arcs of both directions, that is, $D = \overleftrightarrow{G}$. For a digraph D, its *reverse* D^{rev} is a digraph with the same vertex set such that $xy \in A(D^{\text{rev}})$ if and only if $yx \in A(D)$. Note that if a digraph D is *symmetric* then $D^{\text{rev}} = D$. We will in this paper also prove the results in the following tables.

Table 3: $\lambda_{S,r}(D)$ for symmetric digraphs

$\lambda_{S,r}(D) \geq \ell$? $\|S\| = k$	$k = 3$	$k \geq 4$ constant	k part of input
$\ell = 2$	Polynomial	Polynomial	Polynomial
$\ell \geq 3$ constant	Polynomial	Polynomial	Polynomial
ℓ part of input	Polynomial	Polynomial	Polynomial

Table 4: $\kappa_{S,r}(D)$ for symmetric digraphs

$\kappa_{S,r}(D) \geq \ell$? $\|S\| = k$	$k = 3$	$k \geq 4$ constant	k part of input
$\ell = 2$	Polynomial	Polynomial	NP-complete
$\ell \geq 3$ constant	Polynomial	Polynomial	NP-complete
ℓ part of input	NP-complete	NP-complete	NP-complete

In a digraph D, the maximum number of arc-disjoint (x, y)-paths is denoted by $\lambda_D(x, y)$, or just by $\lambda(x, y)$ if D is clear from the context. A digraph D is *Eulerian* if D is connected and $d^+(x) = d^-(x)$ for every vertex $x \in V(D)$. In order to determine the complexities of Table 3 we will actually prove the following stronger result.

Theorem 1.1 *If D is an Eulerian digraph and $S \subseteq V(D)$ and $r \in S$, then $\lambda_{S,r}(D) \geq \ell$ if and only if $\lambda_D(r, s) \geq \ell$ for all $s \in S \setminus \{r\}$.*

As one can determine $\lambda_D(r, s)$ in polynomial time for any r and s in D we note that Theorem 1.1 implies that we can extend the results in Table 3 from symmetric digraphs to Eulerian digraphs implying that the following also holds.

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Table 5: $\lambda_{S,r}(D)$ for Eulerian digraphs

$\lambda_{S,r}(D) \geq \ell$? $\lvert S\rvert = k$	$k = 3$	$k \geq 4$ constant	k part of input
$\ell = 2$	Polynomial	Polynomial	Polynomial
$\ell \geq 3$ constant	Polynomial	Polynomial	Polynomial
ℓ part of input	Polynomial	Polynomial	Polynomial

If we consider $\kappa_{S,r}$ instead of $\lambda_{S,r}$ for Eulerian digraphs then we will prove the following results.

Table 6: $\kappa_{S,r}(D)$ for Eulerian digraphs

$\kappa_{S,r}(D) \geq \ell$? $\lvert S\rvert = k$	$k = 3$	$k \geq 4$ constant	k part of input
$\ell = 2$	NP-complete	NP-complete	NP-complete
$\ell \geq 3$ constant	NP-complete	NP-complete	NP-complete
ℓ part of input	NP-complete	NP-complete	NP-complete

It may be slightly surprising that for Eulerian digraphs the complexity of deciding if $\lambda_{S,r}(D) \geq \ell$ is always polynomial, while the complexity of deciding if $\kappa_{S,r}(D) \geq \ell$ is always NP-complete.

In this paper we furthermore prove some equalities and inequalities for directed tree connectivity. We study the relation between the directed tree connectivity and classical connectivity of digraphs by showing that $\kappa_k(D) \leq \kappa(D)$ (when $n \geq k + \kappa(D)$) and $\lambda_k(D) = \lambda(D)$ (Theorem 6.3). Furthermore, the upper bound for $\kappa_k(D)$ is sharp. Let D be a strong digraph of order n. For $2 \leq k \leq n$, we prove that $1 \leq \kappa_k(D) \leq n-1$ and $1 \leq \lambda_k(D) \leq n-1$ (Theorem 6.4). All bounds are sharp, and we also characterize those digraphs D for which $\kappa_k(D)$ (respectively, $\lambda_k(D)$) attains the upper bound. In Section 6, sharp Nordhaus-Gaddum type bounds for $\lambda_k(D)$ are also given; moreover, we characterize the extremal digraphs for the lower bounds (Theorem 6.6).

Additional Terminology and Notation. A digraph D is *semicomplete* if for every distinct $x, y \in V(D)$ at least one of the arcs xy, yx is in D. For a digraph $D = (V(D), A(D))$, the *complement digraph*, denoted by D^c, is a digraph with vertex set $V(D^c) = V(D)$ such that $xy \in A(D^c)$ if and only if $xy \notin A(D)$.

2 Complexity for packing directed Steiner trees

Let D be a digraph and $S \subseteq V(D)$ with $|S| = k$. It is natural to consider the following problem: what is the complexity of deciding whether $\kappa_{S,r}(D) \geq \ell$ (respectively, $\lambda_{S,r}(D) \geq \ell$)? where $r \in S$ is a root. If $k = 2$, say $S = \{r, x\}$, then the problem of deciding whether $\kappa_{S,r}(D) \geq \ell$ (respectively, $\lambda_{S,r}(D) \geq \ell$) is equivalent to deciding whether $\kappa(r, x) \geq \ell$ (respectively, $\lambda(r, x) \geq \ell$), and so is polynomial-time solvable (see [2]), where $\kappa(r, x)$ (respectively, $\lambda(r, x)$) is the local vertex-strong (respectively, arc-strong) connectivity from r to x. If $\ell = 1$, then the above problem is also polynomial-time solvable by

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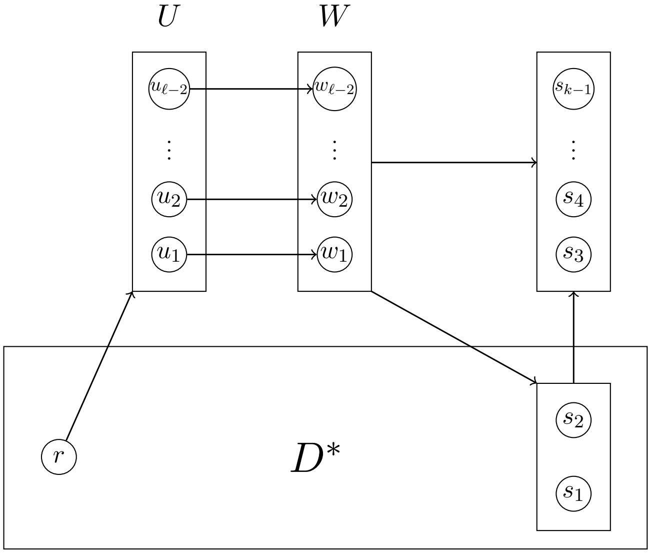

Figure 1: In the above digraph, D, the following statements hold. (1): $\kappa_{\{r,s_1,s_2,\ldots,s_{k-1}\},r}(D) \geq \ell$ if and only if $\kappa_{\{r,s_1,s_2\},r}(D^*) \geq 2$.
(2): $\lambda_{\{r,s_1,s_2,\ldots,s_{k-1}\},r}(D) \geq \ell$ if and only if $\lambda_{\{r,s_1,s_2\},r}(D^*) \geq 2$.

the well-known fact that every strong digraph has an out- and in-branching rooted at any vertex, and these branchings can be found in polynomial-time. Hence, it remains to consider the case that $k \geq 3, \ell \geq 2$. That is, we will prove the results in Tables 1 and 2. The known parts of Table 1 and 2 are the following.

Theorem 2.1 *[7] Let D be a digraph and $S \subseteq V(D)$ with $|S| = 3$. The problem of deciding whether $\kappa_{S,r}(D) \geq 2$ is NP-hard, where $r \in S$.*

Theorem 2.2 *[7] Let D be a digraph and $S \subseteq V(D)$ with $|S| = 3$. The problem of deciding whether $\lambda_{S,r}(D) \geq 2$ is NP-hard, where $r \in S$.*

We will now extend the above results to the following.

Theorem 2.3 *Let $k \geq 3$ and $\ell \geq 2$ be fixed integers (considered as constants). Let D be a digraph and $S \subseteq V(D)$ with $|S| = k$ and $r \in S$. Both the following problems are NP-complete.*

- *Is $\kappa_{S,r}(D) \geq \ell$?*
- *Is $\lambda_{S,r}(D) \geq \ell$?*

Proof: We first consider the problem of determining if $\kappa_{S,r}(D) \geq \ell$. It is easy to see that this problem belongs to NP. To show it is NP-hard, we reduce from the case when $k = 3$ and $\ell = 2$. That is let D^* be a digraph such that $S^* \subseteq V(D^*)$ with $|S^*| = 3$ and $r \in S^*$ where we want to determine if $\kappa_{S^*,r}(D) \geq 2$. This problem is NP-hard by Theorem 2.1.

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We will construct a new digraph D containing a set $S \subseteq V(D)$ with $|S| = k$ and $r \in S$, such that $\kappa_{S,r}(D) \geq \ell$ if and only if $\kappa_{S^*,r}(D^*) \geq 2$. This would complete the proof for the problem of determining if $\kappa_{S,r}(D) \geq \ell$. Assume that $S^* = \{r, s_1, s_2\}$ and let $V(D) = V(D^*) \cup U \cup W \cup \{s_3, s_4, \ldots, s_{k-1}\}$, where $U = \{u_1, u_2, \ldots, u_{\ell-2}\}$ and $W = \{w_1, w_2, \ldots, w_{\ell-2}\}$.

Let $S = \{r, s_1, s_2, s_3, \ldots, s_{k-1}\}$ and let the arc-set of D be the following (see Figure 1).

$$\begin{array}{rcll} A(D) & = & A(D^*) & \cup \ \{ru_i, u_iw_i, w_is \mid i = 1, 2, \ldots, \ell-2 \text{ and } s \in S - r\} \\ & & & \cup \ \{s_is_j \mid i = 1, 2 \text{ and } j = 3, 4, \ldots, k-1\} \end{array}$$

First assume that $\kappa_{S^*,r}(D^*) \geq 2$ and let T_1^* and T_2^* be the two internally disjoint (S^*, r)-trees in D^*. Add all arcs from s_1 to $\{s_3, s_4, \ldots, s_{k-1}\}$ to T_1^* and add all arcs from s_2 to $\{s_3, s_4, \ldots, s_{k-1}\}$ to T_2^* in order to obtain two internally disjoint (S, r)-trees in D. Furthermore, for $i = 1, 2, \ldots, \ell - 2$, we note that the path ru_iw_i together with all arcs from w_i to $S - r$ gives us an (S, r)-tree in D. It is not difficult to see that the ℓ constructed (S, r)-trees are all internally disjoint and therefore $\kappa_{S,r}(D) \geq \ell$.

Conversely, if $\kappa_{S,r}(D) \geq \ell$, then let $T_1, T_2, \ldots, T_\ell$ be the ℓ internally disjoint (S, r)-trees in D. If we remove all the T_i's containing a vertex from U then we are left with at least two trees, not containing any vertex from U or W (as after removing U the vertices in W have no arc into them). Removing all vertices from $\{s_3, s_4, \ldots, s_{k-1}\}$ from the two remaining T_i's gives us two internally disjoint (S^*, r)-trees in D^* and therefore $\kappa_{S^*,r}(D^*) \geq 2$.

This completes the proof for determining if $\kappa_{S,r}(D) \geq \ell$. The case when we want to determine if $\lambda_{S,r}(D) \geq \ell$ is similar. We use exactly the same reduction, except in D^* we look at the problem of determining if $\lambda_{S^*,r}(D^*) \geq 2$, which is also NP-hard by Theorem 2.2. Otherwise the reduction is identical (but delete the arcs u_iw_i instead of the vertices in U in the above proof). □

The above results imply the entries in Tables 1 and 2.

3 Symmetric digraphs

Now we turn our attention to symmetric digraphs. We first need the following theorem by Robertson and Seymour.

Theorem 3.1 *[21] Let G be a graph and let $s_1, s_2, \ldots, s_r, t_1, t_2, \ldots, t_r$ be $2r$ disjoint vertices in G. We can in $O(|V(G)|^3)$ time decide if there exists an (s_i, t_i)-path, P_i, such that all $P_1, P_2, \ldots, P_r$ are vertex disjoint.*

Corollary 3.2 *Let D be a symmetric digraph and let $s_1, s_2, \ldots, s_r, t_1, t_2, \ldots, t_r$ be vertices in D (not necessarily disjoint) and let $S \subseteq V(D)$. We can in $O(|V(G)|^3)$ time decide if there for all $i = 1, 2, \ldots, r$ exists an (s_i, t_i)-path, P_i, such that no internal vertex of any P_i belongs to S or to any path P_j with $j \neq i$ (the end-points of P_j can also not be internal vertices of P_i).*

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Proof: Let D and S and $s_1, s_2, \ldots, s_r, t_1, t_2, \ldots, t_r$ be defined as in the corollary. If some vertex, x, appears q times in the sequence $s_1, s_2, \ldots, s_r, t_1, t_2, \ldots, t_r$, where $q \geq 1$, then we make q copies, $x_1, x_2, \ldots, x_q$ of x and replace x with these q copies. All arcs in D that entered x now enter every copy of x and all arcs out of x in D now go out of every copy of x. We then replace the q copies of x in the sequence by a separate copy of x. Finally, every vertex in S, that does not appear in the sequence $s_1, s_2, \ldots, s_r, t_1, t_2, \ldots, t_r$ gets deleted. This results in a new symmetric digraph D' and a sequence $s'_1, s'_2, \ldots, s'_r, t'_1, t'_2, \ldots, t'_r$, of $2r$ disjoint vertices in D'. Note that every vertex in S only appears as vertices in the sequence $s'_1, s'_2, \ldots, s'_r, t'_1, t'_2, \ldots, t'_r$ (or gets removed completely if it is not in this sequence). Let G' be the underlying graph of D'.

If P' is an (s'_i, t'_i)-path in G', then P' corresponds to an (s'_i, t'_i)-path in D' and therefore also to an (s_i, t_i)-path in D. If $P'_1, P'_2, \ldots, P'_r$ are vertex-disjoint paths in G', where P'_i is an (s'_i, t'_i)-path in G', then the corresponding paths in D are the desired paths we were looking for.

Conversely, if for all $i = 1, 2, \ldots, r$ there exists an (s_i, t_i)-path, P_i, in D, such that no internal vertex of any P_i belongs to S or to any path P_j with $j \neq i$, then the corresponding paths in G' are all vertex-disjoint (s'_i, t'_i)-paths. We are now done by Theorem 3.1. □

Theorem 3.3 *Let $k \geq 3$ and $\ell \geq 2$ be fixed integers and let D be a symmetric digraph. Let $S \subseteq V(D)$ with $|S| = k$ and let r be an arbitrary vertex in S. Let $A_0, A_1, A_2, \ldots, A_\ell$ be a partition of the arcs in $D[S]$.*

We can in time $O(n^{\ell k - 2\ell + 3} \cdot (2k-3)^{\ell(2k-3)})$ decide if there exist ℓ internally disjoint (S, r)-trees, $T_1, T_2, \ldots, T_\ell$, with $A(T_i) \cap A[S] = A_i$ for all $i = 1, 2, \ldots, \ell$ (note that A_0 are the arcs in $D[S]$ not used in any of the trees).

Proof: Let T be any (S, r)-tree in D. Let R be all vertices in T with degree (in the underlying graph of T) at least three and let L be all the leaves of T (in the underlying graph of T). It is well-known that $|R| \leq |L| - 2$. Furthermore we will assume that all (S, r)-trees considered are minimal (i.e. if we delete a vertex in the tree it is not an (S, r)-tree anymore), which implies that all leaves are in S. Under this assumption we have $|R| \leq |L| - 2 \leq |S| - 2$. Note that every vertex $x \in V(T) \setminus (R \cup S)$ has $d^+_T(x) = d^-_T(x) = 1$. We now define the *skeleton* of T as the tree we obtain from T by by-passing all vertices in $V(T) \setminus (R \cup S)$ (i.e. if $ux, xv \in A(T)$ and $x \in V(T) \setminus (R \cup S)$ then replace ux and xv by uv).

Let T^s be a skeleton of an (S, r)-tree in D and recall that $V(T^s)$ consists of S as well as at most $|S| - 2$ $(= k - 2)$ other vertices. Therefore there are less than n^{k-2} possibilities for $V(T^s)$. Once $V(T^s)$ is known there are at most $(2k-3)^{2k-3}$ possibilities for the arcs of T^s (as for each $y \in V(T^s) \setminus \{r\}$ we need to pick the vertex with an arc into y and there are at most $2k - 3$ possibilities for picking this vertex). This implies that there are at most $n^{k-2} \cdot (2k-3)^{2k-3}$ different skeletons of minimal (S, r)-trees in D (as there are at most n^{k-2} possibilities for $V(T^s)$ and at most $(2k-3)^{2k-3}$ possibilities for $E(T^s)$ for each given $V(T^s)$).

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Our algorithm will try all possible ℓ-tuples, $\mathcal{T}^s = (T_1^s, T_2^s, \ldots, T_\ell^s)$, of skeletons of (S,r)-trees and determine if there are ℓ internally disjoint (S,r)-trees, $T_1, T_2, \ldots, T_\ell$, with $A(T_i) \cap A[S] = A_i$ for all $i = 1, 2, \ldots, \ell$ and such that T_i^s is the skeleton of T_i. If such a set of trees exists for any $\mathcal{T}^s$, then we return this solution and if no such set of trees exists for any $\mathcal{T}^s$ we return that no solution exists. We will first prove that this algorithm gives the correct answer and then compute its time complexity.

If our algorithm returns a solution, then clearly a solution exists. So now assume that a solution exists and let $T_1, T_2, \ldots, T_\ell$ be the desired set of internally disjoint (S,r)-trees. When we consider $\mathcal{T}^s = (T_1^s, T_2^s, \ldots, T_\ell^s)$, where T_i^s is the skeleton of T_i, our algorithm will find a solution, so the algorithm always returns a solution if one exists.

We will now analyse the time complexity. The number of different ℓ-tuples, $\mathcal{T}^s$, that we need to consider is bounded by the following.

$$\left(n^{k-2} \cdot (2k-3)^{2k-3}\right)^{\ell} = n^{\ell(k-2)} \cdot (2k-3)^{\ell(2k-3)}$$

Given such a ℓ-tuples, $\mathcal{T}^s$, we need to determine if there exist ℓ internally disjoint (S,r)-trees, $T_1, T_2, \ldots, T_\ell$, with $A(T_i) \cap A[S] = A_i$ for all $i = 1, 2, \ldots, \ell$ and such that T_i^s is the skeleton of T_i. We first check that the arcs in A_i belong to the skeleton T_i^s and that no vertex in $V(D) \setminus S$ belongs to more than one skeleton. If the above does not hold then the desired trees do not exist, so assume the above holds. We will now use Corollary 3.2. In fact, for every arc $uv \notin A(D[S])$ that belongs to some skeleton T_i^s we want to find a (u,v)-path in $D - A(D[S])$, such that no internal vertex on any path belongs to S or to a different path. By Corollary 3.2 this can be done in $O(n^3)$ time. If such paths exist, then substituting the arcs uv by the (u,v)-paths we obtain the desired (S,r)-trees. And if the paths do not exist the desired (S,r)-trees do not exist. Therefore the algorithm works correctly and has complexity $O(n^{\ell k - 2\ell + 3} \cdot (2k-3)^{\ell(2k-3)})$. □

Corollary 3.4 *Let $k \geq 3$ and $\ell \geq 2$ be fixed integers. We can in polynomial time decide if $\kappa_{S,r}(D) \geq \ell$ for any symmetric digraph, D, with $S \subseteq V(D)$, with $|S| = k$ and $r \in S$.*

Proof: Let D be any symmetric digraph with $S \subseteq V(D)$, with $|S| = k$ and $r \in S$. Let $\mathcal{A} = (A_0, A_1, A_2, \ldots, A_\ell)$ be a partition of the arcs in $D[S]$. By Theorem 3.3 we can decide if there exist ℓ internally disjoint (S,r)-trees, $T_1, T_2, \ldots, T_\ell$, with $A(T_i) \cap A[S] = A_i$ for all $i = 1, 2, \ldots, \ell$. If such ℓ internally disjoint (S,r)-trees exist then clearly $\kappa_{S,r}(D) \geq \ell$. We will do the above for all possible partitions $\mathcal{A} = (A_0, A_1, A_2, \ldots, A_\ell)$ and if we find ℓ internally disjoint (S,r)-trees for any such partition then we return "$\kappa_{S,r}(D) \geq \ell$" and otherwise we return "$\kappa_{S,r}(D) < \ell$".

If $\kappa_{S,r}(D) \geq \ell$ then we note that we will correctly determine that $\kappa_{S,r}(D) \geq \ell$, when we consider the correct partition $\mathcal{A}$, which proves that the above algorithms will always return the correct answer.

Furthermore since the number of partitions, $\mathcal{A}$, of $A(D[S])$ is bounded by $(\ell+1)^{|A(D[S])|} \leq (\ell+1)^{k^2/2}$ we note that the above algorithm runs in

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polynomial time (as ℓ and k are considered constants). □

Note that Corollary 3.4 implies all the polynomial entries in Table 4.

We now turn our attention to the NP-complete cases in Table 4. Chen, Li, Liu and Mao [6] introduced the following problem, which turned out to be NP-complete.

CLLM Problem: Given a tripartite graph $G = (V, E)$ with a 3-partition (A, B, C) such that $|A| = |B| = |C| = q$, decide whether there is a partition of V into q disjoint 3-sets $V_1, \dots, V_q$ such that for every $V_i = \{a_{i_1}, b_{i_2}, c_{i_3}\}$ $a_{i_1} \in A, b_{i_2} \in B, c_{i_3} \in C$ and $G[V_i]$ is connected.

Lemma 3.5 *[6] The CLLM Problem is NP-complete.*

In the following theorem we will show the following. Restricted to symmetric digraphs D, for any fixed integer $k \geq 3$, the problem of deciding whether $\kappa_{S,r}(D) \geq \ell$ $(\ell \geq 1)$ is NP-complete for $S \subseteq V(D)$ with $|S| = k$ and $r \in S$.

Theorem 3.6 *Let $k \geq 3$ be a fixed integer. The problem of deciding if a symmetric digraph D, with a k-subset S of $V(D)$ with $r \in S$ satisfies $\kappa_{S,r}(D) \geq \ell$ (ℓ is part of the input), is NP-complete.*

Proof: It is easy to see that this problem is in NP. Let G be a tripartite graph with 3-partition (A, B, C) such that $|A| = |B| = |C| = q$. We will construct a symmetric digraph D, a k-subset $S \subseteq V(D)$ with $r \in S$ and an integer $\ell(= q)$ such that there are ℓ internally disjoint (S, r)-trees in D if and only if G is a positive instance of the CLLM Problem.

Let D be obtained from G by replacing every edge with a 2-cycle and adding the vertices $S = \{r, s_1, s_2, \dots, s_{k-1}\}$ and all arcs between r and A and all arcs between s_1 and B and all arcs between $\{s_2, s_3, \dots, s_{k-1}\}$ and C. This completes the construction of D. Note that the construction of D is equivalent to a construction in [24] and clearly can be done in polynomial time.

First consider the case when there are ℓ internally disjoint (S, r)-trees in D, say T_i $(1 \leq i \leq \ell)$. Each tree must clearly contain at least one vertex from A (connected to r in the tree), at least one vertex from B (connected to s_1 in the tree) and at least one vertex from C (connected to s_2 in the tree). As $|A| = |B| = |C| = q$ and the trees are internally disjoint, we note that every tree, T_i, contains exactly one vertex from A, say a_i, and one vertex from B, say b_i, and exactly one vertex from C, say c_i. However now $G[a_i, b_i, c_i]$ is connected for all $i = 1, 2, \dots, q$, and G is a positive instance of CLLM.

Conversely if G is a positive instance of CLLM, then there is a partition of $V(G)$ into $q = \ell$ disjoint sets $V_1, V_2, \cdots, V_q$ each having three vertices, such that for every $V_i = \{a_{i_1}, b_{i_2}, c_{i_3}\}$ we have $a_{i_1} \in A$, $b_{i_2} \in B$ and $c_{i_3} \in C$, and $G[V_i]$ is connected. Let T_i be an (S, r)-tree with vertex-set $V_i \cup S$. Note that all T_i are internally disjoint (S, r)-trees in D. By the above argument

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and Lemma 3.5, we are done. □

Theorem 3.6 together with the below theorem implies all the NP-completness results in Table 4.

Theorem 3.7 *Let $\ell \geq 2$ be a fixed integer. The problem of deciding if a symmetric digraph D, with an $S \subseteq V(D)$ and $r \in S$ satisfies $\kappa_{S,r}(D) \geq \ell$ ($k = |S|$ is part of the input), is NP-complete.*

Proof: We will reduce from the problem of 2-coloring hypergraphs (see [20]). That is, we are given a hypergraph, H, with vertex set $V(H)$ and edge set $E(H)$, and want to determine if we can 2-colour the vertices $V(H)$ such that every hyperedge in $E(H)$ contains vertices of both colours. This problem is known to be NP-hard (see [20]).

Define a symmetric digraph, D, as follows. Let $U = \{u_1, u_2, \ldots, u_{\ell-2}\}$ and let $V(D) = V(H) \cup E(H) \cup U \cup \{r\}$ and let the arc-set of D be defined as follows.

$$\begin{aligned} A(D) \;\; = \;\; & \{xe, ex \mid x \in V(H),\; e \in e(H) \text{ and } x \in V(e)\} \\ & \cup \; \{ru_i, u_ir, u_ie, eu_i \mid u_i \in U \text{ and } e \in E(H)\} \\ & \cup \; \{rx, xr \mid x \in V(H)\} \end{aligned}$$

Let $S = E(H) \cup \{r\}$. This completes the construction of D, S and r. We will show that $\kappa_{S,r}(D) \geq \ell$ if and only if H is 2-colourable, which will complete the proof.

First assume that H is 2-colourable and let R be the red vertices in H and B be the blue vertices in H in a proper 2-colouring of H. Let T_i contain the arc ru_i and all arcs from u_i to $S \setminus \{r\}$ for $i = 1, 2, \ldots, \ell - 2$. Let $T_{\ell-1}$ contain all arcs from r to R and for each edge $e \in E(H)$ we add an arc from R to e in D to $T_{\ell-1}$ (this is possible as every edge in H contains a red vertex). Analogously, let T_ℓ contain all arcs from r to B and for each edge $e \in E(H)$ we add an arc from B to e in D to T_ℓ (again, this is possible as every edge in H contains a blue vertex). We now note that $T_1, T_2, \ldots, T_\ell$ are internally disjoint (S, r)-trees in D, so $\kappa_{S,r}(D) \geq \ell$.

Conversely assume that $\kappa_{S,r}(D) \geq \ell$ and let $T'_1, T'_2, \ldots, T'_\ell$ be ℓ internally disjoint (S, r)-trees in D. At least two of these trees contain no vertex from U (as $|U| = \ell - 2$). Without loss of generality assume that T'_1 and T'_2 don't contain any vertex from U. Let B' be all out-neighbours of r in T'_1 (i.e. $B' = N^+_{T'_1}(r)$) and let R' be all out-neighbours of r in T'_2 (i.e. $R' = N^+_{T'_2}(r)$). For every $e \in E(H)$ it has an arc into it in T'_1 and an arc into it in T'_2 which implies that in H the edge e contains a vertex from B' and a vertex from R'. Therefore H is 2-colourable (any vertex in H that is not in either R' or B' can be assigned randomly to either B' or R'). This completes the proof. □

4 Proof of Theorem 1.1 (Eulerian digraphs)

We will in this section give a proof for Theorem 1.1. However we first need the following theorem by Bang-Jensen, Frank and Jackson.

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Theorem 4.1 *[3] Let $k \geq 1$ and let $D = (V, A)$ be a directed multigraph with a special vertex z. Let $T' = \{x \mid x \in V \setminus \{z\}$ and $d^-(x) < d^+(x)\}$. If $\lambda(z, x) \geq k$ for every $x \in T'$, then there exists a family $\mathcal{F}$ of k arc-disjoint out-trees rooted at z so that every vertex $x \in V$ belongs to at least* $\min\{k, \lambda(z, x)\}$ *members of $\mathcal{F}$.*

In the case when D is Eulerian, then $d^+(x) = d^-(x)$ for all $x \in V(D)$ and therefore $T' = \emptyset$ in the above theorem. Therefore the following corollary holds.

Corollary 4.2 *Let $k \geq 1$ and let $D = (V, A)$ be an Eulerian digraph with a special vertex z. Then there exists a family $\mathcal{F}$ of k arc-disjoint out-trees rooted at z so that every vertex $x \in V$ belongs to at least* $\min\{k, \lambda(z, x)\}$ *members of $\mathcal{F}$.*

We are now in a position to prove Theorem 1.1. Recall the theorem.

Theorem 1.1: *If D is an Eulerian digraph and $S \subseteq V(D)$ and $r \in S$, then $\lambda_{S,r}(D) \geq \ell$ if and only if $\lambda_D(r, s) \geq \ell$ for all $s \in S \setminus \{r\}$.*

Proof: Let D be an Eulerian digraph and let $S \subseteq V(D)$ and $r \in S$ be arbitrary. First assume that $\lambda_{S,r}(D) \geq \ell$. This implies that $\lambda_D(r, s) \geq \ell$ for all $s \in S \setminus \{r\}$ as there is a path from r to s in each of the ℓ arc-disjoint (S, r)-trees and these ℓ paths are therefore also arc-disjoint.

Now assume that $\lambda_D(r, s) \geq \ell$ for all $s \in S \setminus \{r\}$. By Corollary 4.2 there exists a family $\mathcal{F}$ of ℓ arc-disjoint out-trees rooted at r so that every vertex $x \in V$ belongs to at least $\min\{\ell, \lambda(r, x)\}$ members of $\mathcal{F}$. As $\lambda(r, s) \geq \ell$ for all $s \in S \setminus \{r\}$ we note that every vertex in S belongs to all ℓ out-trees in $\mathcal{F}$. Therefore $\lambda_{S,r}(D) \geq \ell$, which completes the proof of this theorem. □

As one can determine $\lambda_D(r, s)$ in polynomial time for any r and s in D we note that Theorem 1.1 implies all the entries in Tables 3 and 5.

In the proof of Theorem 3.3 we saw how the fact that the k-linkage problem in symmetric graphs is polynomial was used to prove the polynomial entries in Table 4. It is maybe therefore slightly surprising that even though we prove that all entries in Table 5 are polynomial, this could not be proved using a similar linkage result, due to the following.

Theorem 4.3 *[13] The weak k-linkage problem is NP-hard in Eulerian digraphs (where k is part of the input).*

5 Entries of Table 6 (Eulerian digraphs)

We will in this section prove the NP-completeness results given in Table 6. In our argument, we will use the constructions from the proof for the following theorem by Fortune, Hopcroft and Wyllie. And in fact, we will use Figures 10.1-10.3 in the arugment of Theorem 10.2.1 of [2].

Theorem 5.1 *[10] The 2-linkage problem is NP-complete.*

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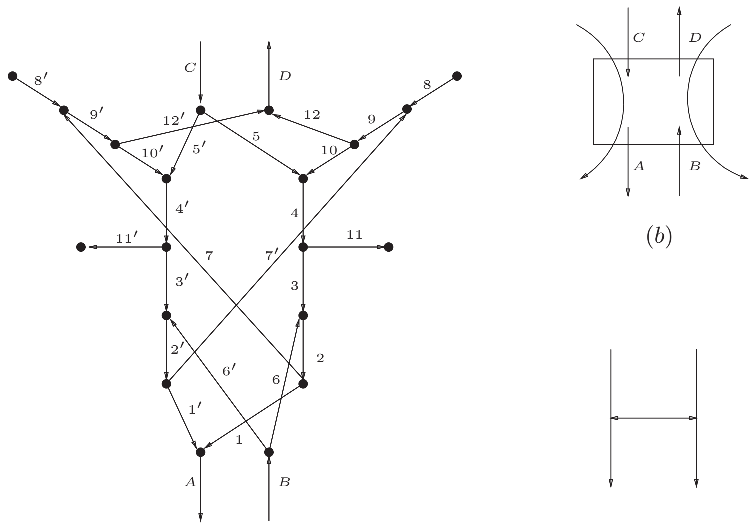
In their argument, Fortune, Hopcroft and Wyllie used the concept of *switch* which is shown in Figure 2(a). Note that in (c) the two vertical arcs correspond to the paths (8, 9, 10, 4, 11), respectively, (8', 9', 10', 4', 11'). For convenience, we label the arcs, rather than the vertices in this figure.



(a) (c)

Figure 2: A switch (a) and its schematic pictures (b) and (c) [2, 10].

By the definition of a switch, the following lemma holds:

Lemma 5.2 *[10] Consider the digraph S shown in Figure 2(a). Suppose there are two vertex-disjoint paths P, Q passing through S such that P leaves S at A and Q enters S at B. Then P must enter S at C and Q must leave S at D. Furthermore, there exists exactly one more path R passing through S which is disjoint from P, Q and this is either (8, 9, 10, 4, 11) or (8', 9', 10', 4', 11'), depending on the actual routing of P.*

As shown in [2, 10], we can stack arbitrarily many switches on top of each other and still have the conclusion on Lemma 5.2 holding for each switch. The way we stack is simply by identifying the C and D arcs of one switch with the A and B arcs of the next (see Figure 3).

Now we can prove the NP-completeness of 2-linkage problem for Eulerian digraphs.

Theorem 5.3 *The 2-linkage problem restricted to Eulerian digraphs is NP-complete.*

Proof: The reduction of the argument for Theorem 5.1 is from 3-SAT problem. Let $\mathcal{F} = C_1 \wedge C_2 \wedge \cdots \wedge C_r$ be an instance of 3-SAT with varibles $x_1, x_2, \ldots, x_k$ and clauses $C_1, C_2, \ldots, C_r$. For each variable x_i we let H_i be a digraph consisting of two internally disjoint (z_i, w_i)-paths of length r (the

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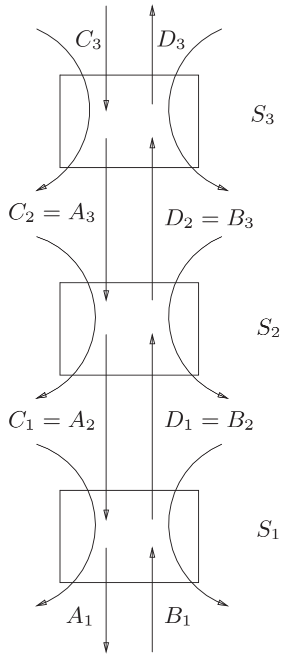

Figure 3: Stacking three switches on top of each other [2, 10].

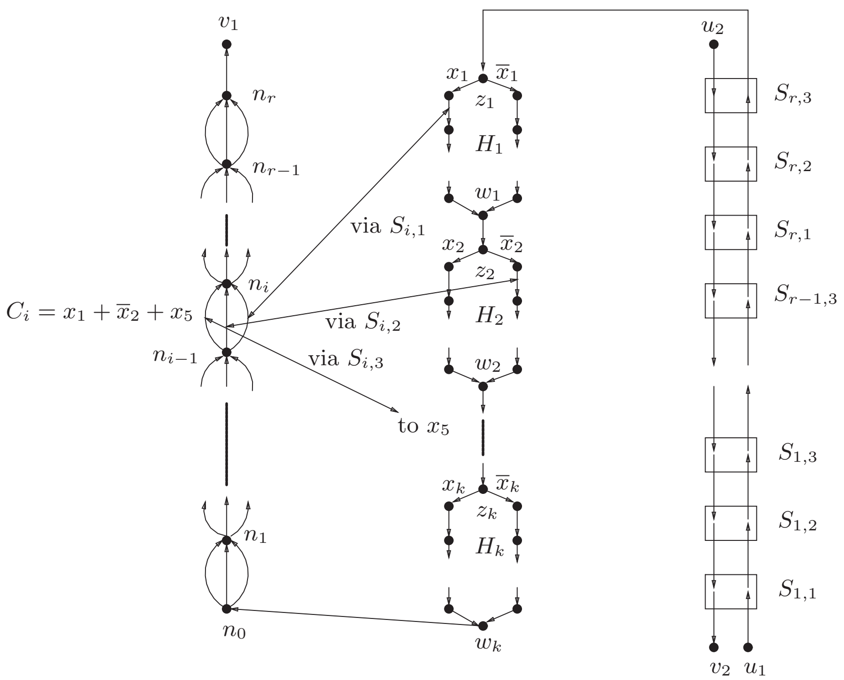

Figure 4: A schematic picture of the digraph $D[\mathcal{F}]$ [2, 10].

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number of clauses in F). See Figure 4 for an illustration. We associate one of these paths with the literal x_i and the other with $\overline{x}_i$.

We will now construct the digraph $D[\mathcal{F}]$ as follows (see Figure 4 and e.g. [2]). Firstly, we form a chain $H_1 \rightarrow H_2 \rightarrow \cdots \rightarrow H_k$ on the subdigraphs corresponding to each variable (see the middle of the figure, H_i corresponds to the variable x_i). Secondly, with each clause C_i we associate three switches, one for each literal it contains, and they are denoted by $S_{i,1}, S_{i,2}, S_{i,3}$, respectively; we then stack these switches in the order $S_{1,1}, S_{1,2}, S_{1,3}, S_{2,1}, \ldots, S_{r,1}, S_{r,2}, S_{r,3}$ as shown in the right part of the figure.

Thirdly, we create a path $n_0, n_1, \ldots, n_r$, such that there are three arcs from n_{i-1} to n_i for all $i = 1, 2, \ldots, r$, each corresponding to a literal in C_i. And if x_a is the b'th literal in C_i, then we substitute the b'th arc from n_{i-1} to n_i with the "left" path of $S_{i,b}$ (that is, the path with arcs $8', 9', 10', 4', 11'$ in $S_{i,b}$) and we substitute a (private) arc of H_a, such that the arc is taken from the path which corresponds to x_a if the literal is x_a and from the path which corresponds to $\overline{x}_a$ if the literal is $\overline{x}_a$, with the "right" path of $S_{i,b}$ (that is, the path with arcs $8, 9, 10, 4, 11$ in $S_{i,b}$). For example in Figure 4 this would imply that the right-most arc from n_{i-1} to n_i actually is the path $8', 9', 10', 4', 11'$ in $S_{i,1}$ (as x_1 is the first literal in C_i) and the arc indicated in H_1 is actually the path $8, 9, 10, 4, 11$ in $S_{i,1}$. In this way one can show that only one of the paths $8', 9', 10', 4', 11'$ and $8, 9, 10, 4, 11$ in each $S_{i,b}$ will ever be used in a solution to the 2-linkage problem. In other words, The double arcs which point to two different arcs indicate that only one of these arcs (which are actually paths) can be used in a solution.

Finally, we join the D arc of the switch $S_{r,3}$ to the vertex z_1 of H_1, add an arc from w_k in H_k to n_0 and choose vertices u_1, u_2, v_1, v_2 as shown in the figure.

We now construct an Eulerian digraph D' from $D[\mathcal{F}]$ as follows: Firstly, for each switch $S_{i,j}$, we duplicate the arcs $A, B, C, D, 9', 9, 4', 4, 2, 2'$ (note that in this procedure there are two parallel arcs between $S_{r,3}$ and the vertex z_1). Secondly, we duplicate the arc $w_k n_0$, and the arc $w_i z_{i+1}$ for each $1 \leq i \leq k-1$, respectively; we also duplicate the arc $n_r v_1$ twice, and add the arc $v_1 n_0$. Observe that now each u_i has exactly two arcs into $U = V(D[\mathcal{F}]) \setminus \{u_1, u_2, v_1, v_2\}$ and has no arc from it, v_1 has three arcs from U and one arc into it, and v_2 has two arcs from U and has no arc into it. Finally, we add a new vertex x and the following new arcs: add the arcs $v_i x$ and $x u_j$ twice for each $1 \leq i, j \leq 2$. To avoid parallel arcs, we could subdivide each new arc.

It is not difficult to show that $D[\mathcal{F}]$ contains a pair of vertex-disjoint (u_1, v_1)-, (u_2, v_2)-paths if and only if D' contains a pair of vertex-disjoint (u_1, v_1)-, (u_2, v_2)-paths. Indeed, if $D[\mathcal{F}]$ contains a pair of vertex-disjoint (u_1, v_1)-, (u_2, v_2)-paths, then clearly these two paths are also vertex-disjoint (u_1, v_1)-, (u_2, v_2)-paths in D'. For the other direction, let P and Q be vertex-disjoint (u_1, v_1)-, (u_2, v_2)-paths in D', then clearly they do not contain arcs incident to x and the arc $v_1 n_0$. If they use other new arcs, then we can replace them by the corresponding parallel original arcs in $D[\mathcal{F}]$, and obtain desired paths in $D[\mathcal{F}]$. It was proved in [10] that $D[\mathcal{F}]$ contains a pair of

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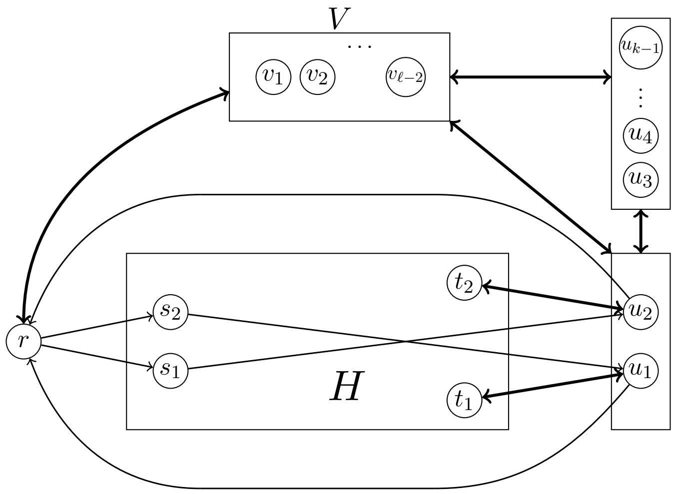

Figure 5: Illustration of D^* in the proof of Theorem 5.4.

vertex-disjoint (u_1, v_1)-, (u_2, v_2)-paths if and only if $\mathcal{F}$ is satisfiable. Hence, the Eulerian digraph D' contains a pair of vertex-disjoint (u_1, v_1)-, (u_2, v_2)-paths if and only if $\mathcal{F}$ is satisfiable, and therefore the 2-linkage problem for Eulerian digraphs is NP-complete. □

Using Theorem 5.3, we can prove the following result, which completes all the entries in Table 6.

Theorem 5.4 *Let $k \geq 3$ and $\ell \geq 2$ be fixed integers. Let D be an Eulerian digraph and $S \subseteq V(D)$ with $|S| = k$ and $r \in S$. Then deciding whether $\kappa_{S,r}(D) \geq \ell$ is NP-complete.*

Proof: It is not difficult to see that the problem belongs to NP. We will show that the problem is NP-hard by reducing from 2-linkage in Eulerian digraphs. Let H be an Eulerian digraph and let s_1, s_2, t_1, t_2 be distinct vertices in H. We now produce a new Eulerian digraph D^* with $V(D^*) = V(H) \cup V \cup U \cup \{r\}$, where $V = \{v_1, v_2, \ldots v_{\ell-2}\}$ and $U = \{u_1, u_2, \ldots, u_{k-1}\}$ (here let $S = \{r, u_1, u_2, \ldots, u_{k-1}\}$). Furthermore let the arcs set of D^* be defined as follows (see Figure 5).

$$\begin{aligned} A(D^*) \quad = \quad & A(H) \cup \{rs_1, rs_2, t_1u_1, u_1t_1, t_2u_2, u_2t_2, s_1u_2, s_2u_1, u_1r, u_2r\} \\ & \cup \{rv, vr, vu, uv \mid v \in V \text{ and } u \in U\} \\ & \cup \{u_iu_j, u_ju_i \mid i = 1, 2 \text{ and } j = 3, 4, \ldots, k-1\} \end{aligned}$$

Note that D^* is Eulerian and let $S = \{r\} \cup U$. We will show that $\kappa_{S,r}(D^*) \geq \ell$ if and only if there exist two vertex-disjoint paths, P_1 and P_2, such that P_i is an (s_i, t_i)-path, for $i = 1, 2$. This will complete the proof.

First assume that there exist two vertex-disjoint paths, P_1 and P_2, such that P_i is an (s_i, t_i)-path, for $i = 1, 2$. Add the arcs, rs_1, s_1u_2, t_1u_1 and all arcs from u_1 to $\{u_3, u_4, \ldots, u_{k-1}\}$ to P_1 and call the resulting (S, r)-tree

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for $T_{\ell-1}$. Analogously, add the arcs, rs_2, s_2u_1, t_2u_2 and all arcs from u_2 to $\{u_3, u_4, \ldots, u_{k-1}\}$ to P_2 and call the resulting (S, r)-tree for T_ℓ. Finally let T_i be the (S, r)-tree containing the arc rv_i and all arcs from v_i to U, for each $i = 1, 2, 3, \ldots, \ell - 2$. The (S, r)-trees, $T_1, T_2, \ldots, T_\ell$ are now internally disjoint, which implies that $\kappa_{S,r}(D^*) \geq \ell$ as desired.

Conversely assume that $\kappa_{S,r}(D^*) \geq \ell$ and let T_1 and T_2 be two (S, r)-trees in D^* that do not use any of the vertices in V (which exist as $|V| = \ell - 2$). Without loss of generality assume that $rs_1 \in A(T_1)$ and $rs_2 \in A(T_2)$. As the arcs into u_1 in T_1 and T_2 is either s_2u_1 or t_1u_1 we note that $s_2u_1 \in A(T_2)$ (as $s_2 \in V(T_2)$) and $t_1u_1 \in A(T_1)$. Analogously, $s_1u_2 \in A(T_1)$ and $t_2u_2 \in A(T_2)$. This implies that there is an (s_1, t_1)-path in T_1 and an (s_2, t_2)-path in T_2, which are vertex-disjoint. This completes the proof of the fact that deciding whether $\kappa_{S,r}(D) \geq \ell$ is NP-complete. □

6 The parameters $\kappa_k(D)$ and $\lambda_k(D)$

We will need the following important theorem by Edmonds (also see Theorem 9.3.1 in [2]), which can be viewed as a generalization of Menger's theorem.

Theorem 6.1 *[9] A directed multigraph $D = (V, A)$ with a special vertex z has k arc-disjoint out-branchings rooted at z if and only if $d^-(X) \geq k$ for any $\emptyset \neq X \subseteq V - z$.*

Furthermore the following proposition can be verified using definitions of $\kappa_k(D)$ and $\lambda_k(D)$.

Proposition 6.2 *Let D be a digraph of order n, and let k be an integer such that $2 \leq k \leq n$. Then the following assertions hold:*

(1): $\lambda_{k+1}(D) \leq \lambda_k(D)$ *when* $k \leq n - 1$.

(2): $\kappa_k(D') \leq \kappa_k(D)$ *and* $\lambda_k(D') \leq \lambda_k(D)$ *where D' is a spanning subdigraph of D.*

(3): $\kappa_k(D) \leq \lambda_k(D) \leq \min\{\delta^+(D), \delta^-(D)\}$.

Note that Proposition 6.2(1) may not hold for $\kappa_k(D)$ due to Theorem 6.3(iii) below, as many digraphs have $\kappa(D) < \lambda(D)$.

Theorem 6.3 *Let $2 \leq k \leq n$ be an integer. The following assertions hold:*

(i) $\kappa_k(D) \leq \kappa(D)$ *when* $n \geq \kappa(D) + k$*, moreover, this bound is sharp.*

(ii) $\kappa_k(D) \leq \lambda(D)$.

(iii) $\kappa_2(D) = \kappa(D)$ *and* $\kappa_n(D) = \lambda(D)$.

(iv) $\lambda_k(D) = \lambda(D)$.

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Proof: We first prove part (i). For $k = 2$, we have $\kappa_2(D) = \kappa(D)$ by definition. In the following argument we therefore consider the case of $k \geq 3$. If $\kappa(D) = 0$, then D is not strong and $\kappa_k(D) = 0$, as can be seen by letting $r, x \in S$ be chosen such that there is no (r, x)-path in D. If $\kappa(D) = n - 1$, then we have $\kappa_k(D) \leq n-1 = \min\{\delta^+(D), \delta^-(D)\}$ by Proposition 6.2(3), so we may assume that $1 \leq \kappa(D) \leq n-2$. There now exists a $\kappa(D)$-vertex cut, say Q, for two vertices u, v in D such that there is no (u, v)-path in $D - Q$. Let $S = \{u, v\} \cup S'$ where $S' \subseteq V(D) \setminus (Q \cup \{u, v\})$ and $|S'| = k-2$. Observe that in each (S, u)-tree, the $u - v$ path must contain a vertex in Q. By the definition of $\kappa_{S,r}(D)$ and $\kappa_k(D)$, we have $\kappa_k(D) \leq \kappa_{S,r}(D) \leq |Q| = \kappa(D)$.

For the sharpness of the bound in (i), consider the following digraph D. Let D be a symmetric digraph whose underlying undirected graph is $K_k \bigvee \overline{K}_{n-k}$ ($n \geq 3k$), i.e. the graph obtained from disjoint graphs K_k and $\overline{K}_{n-k}$ by adding all edges between the vertices in K_k and $\overline{K}_{n-k}$.

Let $V(D) = W \cup U$, where $W = V(K_k) = \{w_i \mid 1 \leq i \leq k\}$ and $U = V(\overline{K}_{n-k}) = \{u_j \mid 1 \leq j \leq n-k\}$. Note that $n - k \geq 2k$ since $n \geq 3k$. Let S be any k-subset of vertices of $V(D)$ such that $|S \cap U| = s$ ($s \leq k$) and $|S \cap W| = k - s$. Without loss of generality, let $w_i \in S$ for $1 \leq i \leq k-s$ and $u_j \in S$ for $1 \leq j \leq s$. For $1 \leq i \leq k-s$, let T_i be a tree with edge set

$$\{w_iu_1, w_iu_2, \ldots, w_iu_s, u_{k+i}w_1, u_{k+i}w_2, \ldots, u_{k+i}w_{k-s}\}.$$

For $k - s + 1 \leq j \leq k$, let T_j be a tree with edge set

$$\{w_ju_1, w_ju_2, \ldots, w_ju_s, w_jw_1, w_jw_2, \ldots, w_jw_{k-s}\}.$$

It is not hard to obtain an (S, r)-tree D_i from T_i by adding appropriate directions to edges of T_i for any $r \in S$. Observe that $\{D_i \mid 1 \leq i \leq k-s\} \cup \{D_j \mid k-s+1 \leq j \leq k\}$ is a set of k internally disjoint (S, r)-trees, so $\kappa_{S,r}(D) \geq k$, and then $\kappa_k(D) \geq k$. Combining this with the bound that $\kappa_k(D) \leq \kappa(D)$ and the fact that $\kappa(D) \leq \min\{\delta^+(D), \delta^-(D)\} = k$, we have $\kappa_k(D) = \kappa(D) = k$.

We now prove part (ii). Let D be any digraph with $\ell = \lambda(D)$. Let $A' \subseteq A(D)$ be defined such that $|A'| = \ell$ and $D - A'$ is not strong. Let r and x be vertices in D such that there is no (r, x)-path in $D - A'$. Let $S \subseteq V(D)$ be chosen such that $\{r, x\} \subseteq S$ and $|S| = k$. Observe that in each (S, r)-tree in D we must use an arc from A', which implies that $\kappa_k(D) \leq \kappa_{S,r}(D) \leq |A'| = \lambda(D)$.

We now prove part (iii). As before let $\ell = \lambda(D)$. For any given vertex $r \in V(D)$, by Menger's Theorem (Theorem 5.4.1 of [2]), the sentence that $d^-(X) \geq \ell$ for any $\emptyset \neq X \subseteq V - r$ is equivalent to the existence of ℓ arc-disjoint paths from r to every other vertex of D. Then by Theorem 6.1, there exist ℓ arc-disjoint out-branchings rooted at r. These ℓ arc-disjoint out-branchings imply that $\kappa_n(D) \geq \lambda(D)$. By part (ii) we therefore must have $\kappa_n(D) = \lambda(D)$. As $\kappa_2(D) = \kappa(D)$ is known to hold this completes the proof of (iii).

We now prove part (iv). The ℓ arc-disjoint out-branchings rooted at r that we obtained in the proof of (iii) imply that $\lambda_k(D) \geq \ell = \lambda(D)$ for

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all $2 \le k \le n$. By Proposition 6.2(1) we have that $\lambda_k(D) \le \lambda(D)$ for all $2 \le k \le n$ which completes the proof of (iv). □

Note that the condition "$n \ge \kappa(D)+k$" (that is, "$k \le n-\kappa(D)$") in Theorem 6.3(i) cannot be removed. As many digraphs, D, have $\lambda(D) > \kappa(D)$ we note that Theorem 6.3(i) would not hold if we removed the condition $n \ge \kappa(D) + k$ (by Theorem 6.3(iii)). It also seems quite interesting that for a general digraph D, computing $\lambda_k(D)$ is polynomial-time solvable (by Theorem 6.3(iv)), but it is NP-complete to determine whether $\lambda_{S,r}(D) \ge \ell$.

Theorem 6.4 *Let D be a strong digraph of order n, and let $k \ge 2$ be an integer. Then*

$$1 \le \kappa_k(D) \le n-1 \tag{1}$$

$$1 \le \lambda_k(D) \le n-1 \tag{2}$$

Moreover, all bounds are sharp, and the upper bounds hold if and only if $D \cong \overleftrightarrow{K}_n$.

Proof: The upper bounds follow from Theorem 6.3(ii) and (iv), as $\lambda(D) \le n-1$ and the lower bound follows from Theorem 6.1 (with $k = 1$).

By Theorem 6.3(ii) and (iv), we note that $\kappa_k(D) < n-1$ and $\lambda_k(D) < n-1$ when D is not equal to $\overleftrightarrow{K}_n$, as in this case $\lambda(D) < n-1$. If $D \cong \overleftrightarrow{K}_n$ then $\lambda_k(D) = \lambda(D) = n-1$ by Theorem 6.3(iv).

So now consider the case when $D \cong \overleftrightarrow{K}_n$ and we want to evaluate $\kappa_k(D)$. Let $S \subseteq V(D)$ with $|S| = k$ and $r \in S$. Let $V(D) = \{u_1, u_2, \dots, u_n\}$ and without loss of generality that $S = \{u_1, u_2, \dots, u_k\}$ and $r = u_1$. For $i = 2, 3, \dots, k$ let T_i be the (S, r)-tree containing the arc ru_i and all arcs from u_i to $S \setminus \{r, u_i\}$. Note that these $k-1$ (S, r)-trees are arc-disjoint and internally disjoint (as they all lie completely within S). For $i = k+1, k+2, \dots, n$, let T_i be the (S, r)-tree containing the arc ru_i and all arcs from u_i to $S \setminus \{r\}$. Now all the (S, r)-trees, $T_2, T_3, \dots, T_n$ are arc-disjoint and internally disjoint. Therefore $\kappa_k(D) = n-1$ in this case. □

We now define a digraph D on eight vertices, such that $\kappa_8(D) = \lambda(D) = \kappa(D) = \kappa_2(D) = 2$, but $\kappa_4(D) = 1$, which shows that the values $\kappa_k(D)$ are neither increasing, nor decreasing over k. Let $V(D) = X \cup Y \cup Z$, where $X = \{x_1, x_2\}$, $Y = \{y_1, y_2, y_3\}$ and $Z = \{z_1, z_2, z_3\}$. Furthermore let D contain all arcs from Z to X and all arcs from X to Y. Finally add all arcs Y to Z, except $y_i z_i$ for $i = 1, 2, 3$. It is not difficult to show that $\lambda(D) = \kappa(D) = 2$, which by Theorem 6.3(ii) implies that $\kappa_8(D) = \kappa_2(D) = 2$.

Let $S = Z \cup \{x_1\}$ and let $r = x_1$. For the sake of contradiction assume that there exist two internally disjoint (S, r)-trees, T_1 and T_2. Either $|V(T_1) \cap Y| \le 1$ or $|V(T_2) \cap Y| \le 1$, as $|Y| = 3$. Without loss of generality assume that $|V(T_1) \cap Y| \le 1$, which implies that there is a vertex in $Z = S \setminus \{r\}$ with no arc into it in T_1, a contradiction. Therefore $\kappa_4(D) \le 1$, and by Theorem 6.4 we have $\kappa_4(D) = 1$, as desired.

Given a graph parameter $f(G)$, the Nordhaus-Gaddum Problem is to determine sharp bounds for (1) $f(G) + f(G^c)$ and (2) $f(G)f(G^c)$, and characterize the extremal graphs. The Nordhaus-Gaddum type relations have

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received wide attention; see a recent survey paper [1] by Aouchiche and Hansen. Theorem 6.6 concerns such type of a problem for the parameter λ_k. To prove the theorem, we will need the following famous result on the Hamiltonian decomposition of complete digraphs.

Theorem 6.5 *(Tillson's decomposition theorem) [25] The arcs of* $\overleftrightarrow{K}_n$ *can be decomposed into Hamiltonian cycles if and only if* $n \neq 4, 6$.

Now we can get our sharp Nordhaus-Gaddum type bounds for the parameter $\lambda_k(D)$.

Theorem 6.6 *For a digraph D with order n, the following assertions hold:*

(i) $0 \leq \lambda_k(D) + \lambda_k(D^c) \leq n-1$. *Moreover, both bounds are sharp. In particular, the lower bound holds if and only if* $\lambda(D) = \lambda(D^c) = 0$.

(ii) $0 \leq \lambda_k(D)\lambda_k(D^c) \leq \lfloor(\frac{n-1}{2})^2\rfloor$. *Moreover, both bounds are sharp. In particular, the lower bound holds if and only if* $\lambda(D) = 0$ *or* $\lambda(D^c) = 0$.

Proof: Part (i). Since $D \cup D^c = \overleftrightarrow{K}_n$, Proposition 6.2(3) implies the following, where $x \in V(D)$ is arbitrary.

$$\lambda_k(D) + \lambda_k(D^c) \leq \delta^+(D) + \delta^+(D^c) \leq d^+_D(x) + d^+_{D^c}(x) = n-1$$

If $H \cong \overleftrightarrow{K}_n$, then we have $\lambda_k(H) = n-1$ and $\lambda_k(H^c) = 0$, so the upper bound is sharp. The lower bound is clear. Furthermore, the lower bound holds, if and only if $\lambda_k(D) = \lambda_k(D^c) = 0$. This is the case if and only if $\lambda(D) = \lambda(D^c) = 0$ by Theorem 6.3(iv). As an example, consider a non-strong tournament, D, in which case it is not difficult to see that D^c is also non-strong.

Part (ii). The lower bound is clear. Furthermore, the lower bound holds, if and only if $\lambda_k(D) = 0$ or $\lambda_k(D^c) = 0$ (i.e. if and only if $\lambda(D) = 0$ or $\lambda(D^c) = 0$ by Theorem 6.3(iv). For the upper bound, we have

$$\lambda_k(D)\lambda_k(D^c) \leq \left(\frac{\lambda_k(D) + \lambda_k(D^c)}{2}\right)^2 \leq \left(\frac{n-1}{2}\right)^2.$$

Since both $\lambda_k(D)$ and $\lambda_k(D^c)$ are integers, the upper bound holds. We now prove the sharpness of the upper bound. By Theorem 6.5, the complete digraph $\overleftrightarrow{K}_n$ can be decomposed into $n-1$ arc-disjoint Hamilton cycles, when n is odd. Let D consist of $(n-1)/2$ of these arc-disjoint Hamilton cycles, which implies that $\lambda(D) = \lambda(D^c) = (n-1)/2$, which shows the sharpness by Theorem 6.3(iv). This completes the proof. □

7 Discussions

In this paper, we completely determine the complexity for both $\kappa_{S,r}(D)$ and $\lambda_{S,r}(D)$ on general digraphs, symmetric digraphs and Eulerian digraphs, as shown in Tables 1-6. We have not considered further classes of digraphs,

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but it would be interesting to determine the complexity for other classes of digraphs, like semicomplete digraphs. For example, one may consider the following question.

Problem 7.1 *What is the complexity of deciding whether $\kappa_{S,r}(D) \geq \ell$ (respectively, $\lambda_{S,r}(D) \geq \ell$) for integers $k \geq 3$ and $\ell \geq 2$, for a semicomplete digraph D?*

Recall that in the argument for the question of deciding whether $\kappa_{S,r}(D) \geq \ell$ for a symmetric digraph D when both k and ℓ are fixed, we use Corollary 3.2 where the $2k$ vertices $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$ are not necessarily distinct. Normally in the k-linkage problem the initial and terminal vertices are considered distinct. However if we allow them to be non-distinct and look for internally disjoint paths instead of vertex-disjoint paths, and the problem remains polynomial, then a similar approach to that of Theorem 3.3 and Corollary 3.4 can be used to show polynomiality of deciding whether $\kappa_{S,r}(D) \geq \ell$.

Data Availability Statement. Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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