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ABSTRACT

The priority model was introduced to capture "greedy-like" algorithms. Motivated by the success of advice complexity in the area of online algorithms, the *fixed* priority model was extended to include advice, and a reduction-based framework was developed for proving lower bounds on the amount of advice required to achieve certain approximation ratios in this rather powerful model. To capture most of the algorithms that are considered greedy-like, the even stronger model of *adaptive* priority algorithms is needed. We extend the adaptive priority model to include advice. We modify the reduction-based framework from the fixed priority case to work with the more powerful adaptive priority algorithms, simplifying the proof of correctness and strengthening all previous lower bounds by a factor of two in the process.

As evidence that adding advice to adaptive priority algorithms extends both adaptive priority algorithms and online algorithms with advice, we present a purely combinatorial adaptive priority algorithm with advice for Minimum Vertex Cover on triangle-free graphs of maximum degree three. Our algorithm achieves optimality and uses at most $7n/22$ bits of advice. No adaptive priority algorithm without advice can achieve optimality without advice, and we prove that an online algorithm with advice needs more than $7n/22$ bits of advice to reach optimality.

We show connections between exact algorithms and priority algorithms with advice. The branching in branch-and-reduce algorithms can be seen as trying all possible advice strings, and all priority algorithms with advice that achieve optimality define corresponding exact algorithms, *priority exact algorithms*. Lower bounds on advice-based adaptive algorithms imply lower bounds on running times of exact algorithms designed in this way.

1. Introduction

Everybody who has studied algorithms has an intuitive notion of a greedy algorithm. In many discrete optimization problems, input can be represented as a sequence of items coming from some infinite universe, and the output of an algorithm can be represented as a sequence of decisions – one decision per item. A decision could, for example, be to accept or reject an item. The quality of such a sequence of decisions is often measured using an objective function that must be maximized (or minimized). Greediness refers to

making the decision that maximizes the objective function at the current point in time. This often means that the algorithm pretends that each input item is the last it is going to receive.[1]

One of the earliest formalizations of a greedy-like notion was in the form of matroids by Whitney [50], more recently extended to greedoids by Korte and Lovász [36–39]. In spite of the profound connection between greedoids and optimization problems admitting optimal greedy algorithms, greedoids do not give a complete characterization of what people usually characterize as greedy algorithms, and there is no consensus in the research community as to a formal definition of greedy algorithms.

Priority algorithms were introduced by Borodin, Nielsen, and Rackoff [11] in an attempt to formalize "greedy-like" or "myopic" algorithms, trying to encompass the algorithm designers' notion of greedy-like that goes beyond the matroid-based framework (earlier works such as [29,33] have discussed the basic idea of using priority functions for scheduling problems as an informal but fairly well understood concept). One of the purposes of this formalization is to prove results giving lower bounds on how well any priority algorithm can approximate an optimal solution, without requiring any assumptions such as P ≠ NP. The priority model has been studied in the context of many combinatorial optimization topics, including classical graph problems [4,6,12,20], scheduling [11,41,44,49], satisfiability [45,46], auctions [10], and general results, present in many of the above contributions as well as in [40]. Many classical greedy algorithms have a simple structure consisting of two components: a sorting, ordering, or priority component and an online, irrevocable decisions component. The second component is where an irrevocable decision is made, while the first component determines the order in which the items are processed by that second component. Priority algorithms have this structure and they come in two flavors: fixed and adaptive. We illustrate these models with two well known examples.

The input to the Minimum Spanning Tree problem is an edge-weighted, undirected, connected graph, and the objective is to select a set of edges forming a spanning tree of minimal total weight. Viewing Kruskal's algorithm for this problem as a *fixed* priority algorithm, we define the universe of input items as $\mathcal{U} = \{(u, v, w) \in \mathbb{N} \times \mathbb{N} \times \mathbb{Q} \mid u \neq v\}$, where (u, v) is an edge between vertices u and v with weight w. An input instance is a finite subset $\mathcal{I} \subset \mathcal{U}$. Kruskal's algorithm can be thought of as defining an ordering on the entire universe $\mathcal{U}$ (by non-decreasing weight w, with arbitrary tie-breaking) *prior to seeing any input items*. The input $\mathcal{I}$ is then given to the algorithm one input item at a time, in the order defined on the universe. When we discuss correctness and quality, we often think of the input as being given by an adversary, but of course still respecting the ordering that may not be total. The algorithm makes an *irrevocable* decision when receiving the next item: accept the edge if it does not form a cycle with the current partial solution (the set of accepted items so far), and reject it otherwise.

Strengthening the model, *adaptive* priority algorithms may change the ordering of the universe after processing each input item. An example of an adaptive priority algorithm is Prim's algorithm for the Minimum Spanning Tree problem. The universe is as above. Prim's algorithm also orders edges by non-decreasing weight, but it has to maintain a single connected component. Thus, the algorithm gives higher priority to edges incident to vertices already added to the solution. Since the set of vertices in the solution keeps growing, the ordering (the priority function) is updated in every step. We emphasize that it is an ordering of the universe, the rest of the input is not known, and the ordering is redefined *before* the next input is given.

Note that online algorithms are usually only used when problems have an online nature, while priority algorithms provide a framework for certain offline algorithms. However, as models, they seem quite similar. Priority algorithms can be seen as either extending the power of online algorithms by allowing a limited ordering of input items, or as limiting the power of an adversary by not allowing it full control over the order of items.

We now discuss *advice*, starting with the online algorithms setting, where advice has been considered for some time. An online algorithm processes a sequence of input items, one at a time, with no knowledge of future input items; an assumption that, even for inherently online problems, is not necessarily realistic. Often some information about the input sequence is known in advance, e.g., its length, the largest weight of an item, etc. The knowledge could be absolute, approximate, or expected from experience. An information-theoretic way of capturing some of this additional knowledge is provided by the *advice tape model*[2] of Hromkovič et al. [32] (further technical development in Böckenhauer et al. [9]). In this model, an all powerful oracle that knows the algorithm and sees the entire input sequence[3] writes bits (referred to as advice bits) on an infinite tape. The algorithm uses the advice tape in processing the online items. The "tape" analogy is used in many other models, but the only important properties are that there are always bits when the algorithm asks for them and there is no detectable end to the collection of bits. The *advice complexity* of an algorithm is the number of bits read. Usually, we are interested in the worst-case number of bits read as a function of the input length. Results for online algorithms with advice are bounds on the number of advice bits necessary and/or sufficient to achieve a given competitive ratio.[4] Often, a few bits of advice improve the competitive ratio dramatically over what is achievable by an online algorithm without advice.

The lower bound results can be interpreted as hardness results for the online problems: if many advice bits are necessary in order to reach optimality (or significantly improve the competitive ratio), the problem is hard. Results can also give strong lower bounds on certain types of semi-online algorithms and inspire algorithm design. See [14] for an extensive list of articles. Of most relevance to us are results concerning graph algorithms [7,22,23,28,30,34,35,43].

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[1] For some problems, in particular many graph problems, the input items received so far may require a certain number of further input items to be given before a well-defined final input is formed; see [12] for a detailed discussion of these issues.

[2] Other advice models have been proposed, including the helper and answerer models of Dobrev et al. [21], the tree exploration model with advice of Fraigniaud et al. [27], and the per request model of Emek et al. [24]. See [14] for a comparison of these models.

[3] In contrast with the online and priority worlds, in the Turing machine world the advice depends only on the input length n and not the input itself.

[4] The competitive ratio is the term used in online algorithms for what is essentially the approximation ratio when considering offline problems.

A superset of the current authors introduced advice into the *fixed* priority model [13]. As for online algorithms in the advice tape model, an oracle knows the algorithm, sees the entire input sequence, and writes advice bits on the tape. The advice is then read by the priority algorithm at its discretion during its execution. Just to emphasize, since the oracle knows the algorithm, the bits always represent what the algorithm expects, so the oracle and the algorithm cooperate. In this model, one is interested in the number of advice bits necessary and/or sufficient to achieve a given approximation ratio. In addition to introducing this model, [13] also developed a general framework for proving lower bounds in this model and applied this framework to several classical problems, including Maximum Independent Set, Maximum Bipartite Matching, Minimum Vertex Cover, etc. That paper left it as an open question whether the ideas can be extended to the (arguably more useful) *adaptive* priority model, and if this would result in useful new paradigms. Our current paper addresses that question.

There are many models that represent computation as a leveled tree (or even more generally as a DAG – directed acyclic graph), such as decision trees, branching programs, small depth formulas/circuits, various proof systems (tree-like and general resolution), pBT algorithms, etc. One can often define a notion for each of these tree/DAG models which intuitively captures the amount of parallelism needed to carry out the computation efficiently. Such a notion can be viewed as being somewhat analogous to the notion of advice in our setting. For example, in the pBT (priority backtracking) model of Alekhnovich et al. [3], an algorithm is represented by a pair of functions: one function allows reordering of the universe of input items, and another function assigns a value to a decision based on already seen input items. The ordering function can be fixed, adaptive, or fully adaptive (we are not discussing this in full here). The execution of such an algorithm on a particular instance can be represented by an ordered leveled tree, where each node corresponds to a partial execution and is labeled by the sequence of input items seen so far and decisions made for those items. The children of a node (in order from left to right) correspond to different input items to be considered next according to the current ordering function. The correctness condition requires that at least one of the leaves contains an optimal choice of decisions. The width of a pBT algorithm is the maximum width of a level of such a tree, where the maximum is taken over all levels and all instances of a given length. The length of the ordered (left-to-right) depth-first search traversal of a pBT tree corresponds to the running time of the natural backtracking algorithm associated with the pBT algorithm. This model captures many backtracking algorithms, but not all of them. For example, early termination as well as choices of which decision to make next can be based on only the already seen portion of the input in pBT, and these choices cannot be made, for example, based on the value of an LP-relaxation of the entire instance (as is often done in real-life backtracking algorithms). The logarithm of the width of a pBT algorithm can be thought of as "advice" length, but there are notable differences between the pBT model and the priority algorithms with advice model. In particular, one can try to simulate the pBT model by a priority algorithm with advice, and vice versa, but one quickly runs into issues of whether priorities and/or decisions are allowed to depend on advice. Establishing precise connections between these models is an interesting open problem. Connections between the fixed pBT algorithms and fixed priority algorithms with advice were previously discussed in [13]. While it is interesting to carry out a comparative study between various tree/DAG-like models and expose informal and formal connections between them and the notion of advice, it is not the goal of the present paper. We discuss only one such connection at length later in this paper, and that is the connection between priority algorithms and branch-and-bound/branch-and-reduce algorithms.

We now briefly list our contributions.

- We introduce the notion of advice in the adaptive priority model and identify four natural models based on how the priority function is allowed to depend on the advice.
- We extend the general lower bound framework of [13] to work in what we call the *oblivious priority function* model. The results automatically apply to the weakest model which does not use advice in the priority functions at all and also to the fixed priority results in [13]. We simplify the proof that the framework from [13] works, and we strengthen the lower bounds implied by the framework by a factor of 2. The framework offers a template for lower bound results: By exhibiting gadget pattern pairs fulfilling a given list of criteria, a lower bound can be computed with fairly limited work.
- As evidence that adding advice to adaptive priority algorithms extends both adaptive priority algorithms and online algorithms with advice, we study the classical Minimum Vertex Cover problem on triangle-free graphs of maximum degree 3. We present an adaptive priority algorithm with advice that achieves optimality. The algorithm works in all but the weakest of our models. Known results imply that adaptive priority algorithms for this problem cannot achieve optimality without advice [12]. We show that online algorithms must use more advice than our algorithm to achieve optimality. Our algorithm is purely combinatorial and requires a somewhat involved analysis.
- Priority algorithms with advice that achieve optimality naturally lead to exact algorithms by trying all possible advice strings of length no more than the upper bound proven. We call exact algorithms designed this way *priority exact algorithms*. We discuss the implications of our lower bounds on priority algorithms with advice for proving lower bounds on the running times of such algorithms.

In [13], the lower bound template is based on an *advice-preserving reduction* (using the same advice) between two problems within the priority framework: it is established that if there exists a fixed priority algorithm with advice for problem A, then there also exists one for Pair Matching (PM) with the same advice length, and it is shown that PM requires a lot of advice. Such a reduction must map each input for PM to an input for A, so that decisions for A can be used for making decisions for PM. The difficulty is that the inputs and the decisions for A and PM must be aligned so that inputs respect priority functions, and decisions are not based on information not available at that point during the execution of the algorithm. This becomes significantly harder when moving to adaptive priority algorithms, since the priorities for the two problems can depend on advice and can change dramatically between

input items. We avoid some of these difficulties by working with an advice-preserving reduction between a problem in an online setting and a problem in the priority setting, removing the difficulties in aligning priority functions, and allowing us to focus more on how priority functions are allowed to depend on advice. Our extension to *adaptive* priority algorithms enables us to define and establish lower bounds for priority exact algorithms.

The remainder of the paper is organized as follows. Section 2 introduces the four adaptive priority models with advice. In Section 3, we discuss connections to exact algorithms. In Section 4, we show the first lower bound, based on a construction from [12], and show that the result is tight for a restricted problem. This first example problem serves as an introduction to some of our lower bound techniques. In Section 5, we present our adaptive priority algorithm for the Minimum Vertex Cover problem on triangle-free graphs of degree at most 3 and analyze its advice complexity. Section 6 presents the extension of the general lower bound framework of [13] to adaptive priority with advice, along with a new framework for algorithms that solve to optimality. Another example problem is considered in Section 7, presenting different lower bounds obtained in two of the different models, along with a matching upper bound in one of the two models. Open problems are discussed in Section 8.

2. Models

A *request-answer game* [5,48] is specified by the universe of input items $\mathcal{U}$, the universe of decisions $\mathcal{D}$, the objective function $\text{OBJ} : \mathcal{U}^n \times \mathcal{D}^n \to \mathbb{R} \cup \{\pm\infty\}$ on inputs of length n, and the type of a problem, which could be either "maximization" or "minimization". An input to a request-answer game is a finite multi-set of items from the universe, i.e., $X = \{x_1, \ldots, x_n\}$ where $x_i \in \mathcal{U}$. We assume that the objective function is invariant under simultaneous permutations of input items and decisions, i.e., for all $x_1, \ldots, x_n$, all $d_1, \ldots, d_n$, and all permutations $\pi : [n] \to [n]$,

$$\text{OBJ}(x_1, \ldots, x_n, d_1, \ldots, d_n) = \text{OBJ}(x_{\pi(1)}, \ldots, x_{\pi(n)}, d_{\pi(1)}, \ldots, d_{\pi(n)}).$$

The values $\pm\infty$ in the objective can be used to specify infeasible input. The setting of request-answer games is very general and includes most problems of interest in the areas of online and priority algorithms.

A function $P : \mathcal{U} \to \mathbb{R}$ is called a *priority function*. We introduce a short-hand notation $\max_P X := \arg\max\{P(x) \mid x \in X\}$ for the element of highest priority in the multi-set X. In case there are multiple elements of highest priority, we assume ties are broken in an adversarial fashion, i.e., we assume the most unfavorable tie-breaking for our algorithms. Thus, all upper bounds we prove will be valid for all input instances, and we can make the simplifying assumption that $\max_P X$ is an element, and not a set of elements.

A priority algorithm ALG is not given all of the input, X, at once. Instead, ALG receives X one item at a time. The priority algorithm has some limited control over the order in which X is given: Each time, before the next input item is given, ALG defines a priority function, P, and the next input item given to ALG is $\max_P X$. Recall that the priority function is defined on the universe, $\mathcal{U}$, and not directly on the remaining part of the input X, which is not known to the algorithm. We use both the terminology that an input item has been *given* to the algorithm and that the algorithm has *received* or *gets* an input item.

What we have described above is the most general version of priority algorithms, called *adaptive*, since the priority function can be adapted based on the input given so far. As the name indicates, *fixed* priority algorithms are those where the priority function cannot be updated during the execution of the algorithm. This simpler class was treated in [13].

We consider priority algorithms in the advice tape model [9,32], and start with a discussion of this model. The setup is exactly as described for online algorithms in the introduction. In the advice tape model, there are two cooperating players – the algorithm and the oracle. The oracle sees the entire input X and writes advice to the algorithm on the infinite advice tape using the binary alphabet. The algorithm can decide to read zero or more bits (for emphasis, often referred to as advice bits) from the advice tape, sequentially from left to right, before making each decision. We use s_i to refer to the prefix of the advice tape that has been read so far by the algorithm. The maximum number of advice bits read, that is, the largest value of $|s_n|$ for any input of size n, is the *advice complexity* of the algorithm (a function of n). See Algorithm 1 for a template illustrating the setup for a priority algorithm with advice.

Algorithm 1 Template: Priority Algorithm with Advice.

1: X is the input
2: read zero or more bits from the advice tape
3: $s_0 \leftarrow$ the prefix of the advice string just read
4: $i \leftarrow 1$
5: **while** $X \neq \emptyset$ **do**
6:     $P_i \leftarrow$ the priority function for iteration i
7:     $x_i \leftarrow \max_{P_i} X$
8:     read zero or more bits from the advice tape
9:     $s_i \leftarrow$ the known content of the advice string
10:     $d_i \leftarrow D_i(x_1, x_2, \ldots, x_i, d_1, d_2, \ldots, d_{i-1}, s_i)$ ▷ the decision for input x_i
11:     $X \leftarrow X \setminus \{x_i\}$
12:     $i \leftarrow i + 1$

A priority algorithm with advice must have this format. A concrete algorithm is defined by specifying three elements for each iteration: the priority functions P_i, how many advice bits to read, and how the decision d_i is made.

The decisions, d_i, and how many bits of advice to read, $|s_{i+1}| - |s_i|$, are always functions of the information seen so far, i.e., the input seen so far, the advice seen so far, and the previous decisions. Of course, one may omit the dependence of d_i on $d_1, \ldots, d_{i-1}$, since these decisions can be reconstructed from $x_1, \ldots, x_{i-1}$ and s_{i-1}. As mentioned in the introduction, priority algorithms with advice can give rise to practical algorithms. However, as a starting point, advice is created by an oracle, and the setup is used to measure some aspect of problem difficulty. Thus, it makes sense to consider how advice may be used by the algorithm. In particular, to what extent do we allow the priority functions to be defined based on the advice obtained by the algorithm at a given time? We make the following distinctions:

Unrestricted priority function model. We allow the priority functions to depend on the input received so far and the advice read so far:

$$P_i(x_1, \ldots, x_{i-1}, s_{i-1}).$$

Oblivious priority function model. We allow the priority function to depend on the input received so far and the advice read so far, as in the unrestricted priority function model, but the priority function must give the same priority to all input items which are indistinguishable, when ignoring names not present in the input items already seen. (For example, for unweighted graph problems, vertices of the same degree, where neither the vertices nor their neighbors have been seen yet, should have the same priority.)

Decision-based priority function model. We allow the priority functions to depend on the input received so far and the decisions made so far:

$$P_i(x_1, \ldots, x_{i-1}, d_1, \ldots, d_{i-1}).$$

Advice-free priority function model. We only allow the priority functions to depend on the input received so far:

$$P_i(x_1, \ldots, x_{i-1}).$$

Similarly to this, in [13] the priority functions were assumed to not depend on the advice (but the priority function was fixed, not adaptive).

Clearly, any algorithm that works in the oblivious priority function model also works in the unrestricted priority function model. Any algorithm that works in the decision-based priority function model also works in the unrestricted priority function model, since the input and advice determine the decisions. Similarly, any algorithm that works in the advice-free priority function model can be simulated by an algorithm in any of the other models, for which reason we refer to this model as the *weakest*. Observe that the unrestricted and decision-based priority functions models coincide when advice encodes the decisions to be made. This sometimes functions as a point of reference, since no more advice than encoding all the decisions is necessary. The oblivious priority function model appears to be incomparable to the decision-based priority function model and its motivation is as follows. Although it seems natural to let decisions depend on the advice in any way and it makes sense to let the priority function depend on advice, it does not seem natural for an algorithm to use, for example, a priority function that prefers input items with certain names that have not been seen yet.

When including advice, one can ask how computationally expensive it is to generate that advice. This could vary significantly from one algorithm/application to the next, but the model allows anything; the priority model does not impose any computational restrictions on priority functions or decisions by the algorithm. This is in line with the information-theoretic nature of the priority model and similar to other areas, such as online algorithms, communication complexity, decision tree complexity, etc. These models sidestep hard computational questions, such as P vs. NP, by introducing informational bottlenecks. The strengths of this information-theoretic modeling are that it makes the proven lower bounds stronger and that it makes it possible to prove results that do not depend on unproven assumptions in complexity theory. The main weakness of this information-theoretic modeling is that the algorithms that are designed might be impractical. However, priority algorithms achieving good approximation ratios tend to have easily computable priority functions and easily computable decisions.

3. Priority exact algorithms

There is a simple, general technique one can use to convert a priority algorithm with advice to an offline algorithm with the same approximation ratio. If the algorithm uses at most ℓ bits of advice for some input length, then, on an input of that length, one can enumerate all 2^ℓ advice strings and execute the algorithm on each of them, keeping track of the best result. We call such algorithms *priority exact algorithms*, since algorithms which solve problems to optimality are generally referred to as exact algorithms.

3.1. Example: Maximum Independent Set

In the textbook *Exact Exponential Algorithms* by Fomin and Krasch [25], in presenting the measure-and-conquer technique, they begin with a simple branching algorithm, mis3 (Algorithm 2), for Maximum Independent Set, the problem of finding the maximum size among subsets of the vertices where no two of the vertices are adjacent. We show how mis3 could be changed to a priority exact algorithm for graphs of bounded degree at most Δ.

As a first intuitive explanation, note that the algorithm gradually decreases the size of the graph until the size of a maximal independent set is found, except that in Line 10 two options are explored recursively. Using advice, one could simply make the correct choice of these two options. A priority exact algorithm could be designed by trying all different sequences of such choices.

Algorithm 2 Maximum Independent Set algorithm `mis3` from [25]. $N[v]$ denotes $\{v\} \cup \{\text{neighbors of } v\}$, $d(v)$ the current degree of v, $\Delta(G)$ the maximum degree in G, and $\alpha(G)$ the size of the maximum independent set.

1: **Algorithm** `mis3` (G)
2: **Input:** A graph $G = (V, E)$.
3: **Output:** A maximum cardinality of an independent set of G.
4: **if** $\exists v \in V$ with $d(v) = 0$ **then**
5:     **return** $1 + \text{mis3}(G \setminus \{v\})$
6: **if** $\exists v \in V$ with $d(v) = 1$ **then**
7:     **return** $1 + \text{mis3}(G \setminus N[v])$
8: **if** $\Delta(G) \geq 3$ **then**
9:     choose a vertex v of maximum degree in G
10:    **return** $\max(1 + \text{mis3}(G \setminus N[v]), \text{mis3}(G \setminus \{v\}))$
11: **if** $\Delta(G) \leq 2$ **then**
12:    compute $\alpha(G)$ using a polynomial time algorithm
13:    **return** $\alpha(G)$

In greater detail, an input item is a vertex, together with a list of all its neighbors. In the priority algorithms framework, we assume that the history is known, i.e., that the algorithm knows the sequence of all earlier input items it has received and which decisions it made for these. Thus, in designing priority functions, we can also talk about the current degree, i.e., the number of neighbors that have not yet been removed, as it is done in `mis3`.

In the priority exact algorithm we design the priority functions, P_i, depending partially on the current degrees of the vertices. Since neighbors of accepted vertices must be rejected, these neighbors are given highest priority ($\Delta + 3$, say). Then, vertices of current degree 0 have the next highest priority, $\Delta + 2$, vertices of current degree 1 have priority $\Delta + 1$, and all other vertices have priority equal to their current degree.

When there are only disjoint cycles remaining, we define priority functions as follows: The lowest priority vertices are those of degree 2, so they are not processed until it is time to start a new cycle. Every time we start the processing of a new cycle (a degree 2 vertex), we accept the vertex (include it in the maximum independent set). The highest priority is given to vertices adjacent to a vertex just processed. If it has current degree 0, it is rejected, because it is adjacent to the first vertex in the cycle. If it has current degree 1, it is accepted if its neighbor was rejected and vice versa. Note that the priority does not alone determine the decision made.

Advice comes into play in the case where the branching occurs, in Line 10. One bit of advice is used to tell which branch gives the better result, and the adaptive priority algorithm with advice takes that branch, i.e., the advice is used to determine if the vertex under consideration should be included into the maximum independent set or not. Note that the algorithm can easily determine when to read a bit of advice, so the maximum amount of advice needed is the number of branches on the shortest (meaning with fewest branches) of the root-to-leaf paths that leads to a maximum independent set. If one has a bound m on that number of branches in the best case, it is never necessary to go through more than all 2^m possible bit strings of length m, and the natural approach is to do the recursive branching with a bit in the advice string indicating which branch to take. In doing so, if one encounters an $(m+1)$st branching, one can simply terminate computation in that direction and move to the next bit string. Thus, `mis3` can be seen as a priority exact algorithm. Since the priority functions depend only on which branches have been taken previously on the current root-to-leaf path, it only depends on decisions made so far, so the defined priority algorithm with advice is in the decision-based priority function model.

The calculation of m is exterior to the algorithm and could, for example, be an upper bound given as a function of $|V|$. By recording accepted vertices, keeping the result with the best $\alpha(G)$, it is simple to return a maximum independent set instead of just the size of it.

By the standard correspondence between Maximum Independent Set and Minimum Vertex Cover, `mis3` can immediately be converted to an algorithm for finding a minimum vertex cover[5] by reversing the decisions made. In Section 5, `mis3` is extended to a priority algorithm with advice, PRIORITYVC, for finding minimum vertex covers in triangle-free graphs of maximum degree 3, adding more priorities, particularly for vertices of degree 3, considering which neighbors are shared with previously processed vertices. PRIORITYVC is shown to require at most $7|V|/22$ bits of advice, which is provably less advice than required by any online algorithm with advice. No adaptive priority algorithm without advice can achieve an approximation ratio for this problem better than 4/3 [12]. Thus, PRIORITYVC is an example of an algorithm with an amount of advice that cannot be matched by any online algorithm and with an approximation ratio that cannot be matched by any priority algorithm without advice. This shows that the class of adaptive priority algorithms with advice is a larger class than either of these related classes of algorithms.

Running the algorithm PRIORITYVC on all possible advice strings of length $7n/22$, we obtain an offline algorithm solving the problem to optimality, a priority exact algorithm, that runs in time[6] $O^*\left(2^{\frac{7n}{22}}\right) \subset O^*(1.247^n)$. This is much better than the naive $O^*(2^n)$ brute-force approach; however, there are other more involved optimal offline algorithms achieving even better running times for the Minimum Vertex Cover problem. The best published exact algorithm for Minimum Vertex Cover restricted to graphs of maximum degree 3 runs in $O^*(1.0836^n)$ [51]. That algorithm is not a priority exact algorithm; in Section 4 and Subsection 6.3, we show that no priority exact algorithm (derived from a priority algorithm with advice in the decision-based or oblivious priority

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[5] Minimum Vertex Cover is the problem of finding a minimum-size subset of the vertices where every edge in the graph is incident to at least one of the vertices.
[6] The notation $O^*()$ is similar to big-Oh, except that it allows ignoring polynomial factors, i.e., $O^*(g(n))$ has the same meaning as $O(g(n)\text{poly}(n))$.

function models) for Minimum Vertex Cover on triangle-free graphs of maximum degree 3 has a running time less than $\Omega(1.142^n)$. We comment further on this in Subsection 6.3.

3.2. Priority exact algorithms, in general

When attacking new NP-hard problems, the priority exact algorithms approach has the potential to deliver a first upper bound that beats the brute force approach, giving an aim for later, more specialized, possible improvements.

A significant motivation for originally introducing and studying priority algorithms was to develop a framework for proving lower bounds for a large collection of algorithms at the same time: Establishing that no fixed (or adaptive) priority algorithm can attain a certain approximation ratio implies that one has to look beyond this fairly broad design pattern to possibly discover an algorithm with a better approximation ratio. We note that this motivation is just as relevant for the design of exact or approximation algorithms using the framework outlined above. A discussion of the lower bound results we obtain is included in Subsection 6.3.

Priority exact algorithms form a subset of the more general branch-and-reduce [18,19] exact algorithms, which find an optimal solution to a problem using a search tree and backtracking. Trying successive possibilities for the advice, setting some decision to accept or reject for example, is essentially the same as a branch operation in the more general algorithms. The restriction that input items be prioritized independently of each other means that there are many possibilities allowed in the general branch-and-reduce algorithms that are not allowed in priority exact algorithms. For the Minimum Vertex Cover problem, for example, priority exact algorithms cannot handle maximal connected components of small size separately (or even handle a vertex of degree 2 differently depending on whether or not it is contained in a triangle); in fact, the lower bounds are proven by considering small connected components.

While there are restrictions, the advantage of priority exact algorithms is that they should be relatively easy to implement and efficient (other than the branching, of course). A straight-forward implementation of a priority exact algorithm as a branching algorithm may lead to many fewer branches than one would obtain by enumerating all bit strings of the maximum length, even in the worst case. In many cases the problem size would reduce by different amounts, depending on whether the decision was accept or reject, for example. One could also apply standard techniques for establishing upper bound results, such as measure and conquer [26] to obtain better upper bounds.

In general, branch-and-reduce algorithms can be considered to have been converted from (usually not priority) algorithms with advice. Advice can be given for each node in the search tree indicating which branch to take to find an optimal solution. If the work done at a node can be handled by a priority algorithm (and all root-to-leaf paths have the same length), then it is essentially a priority exact algorithm. However, for example for Minimum Vertex Cover, most exact algorithms use operations that do not fit in the priority algorithm model.

In [16], the author presents a lower bound that also holds for priority exact algorithms (recursive proofs) for Maximum Independent Set (if one ignores cutting off the length of the root-to-leaf paths considered due to the maximum length of the advice string necessary), but also for more powerful algorithms, and proves that there exists a $c > 1$ such that the running time is at least $\Omega(c^n)$. In fact, this result holds for every graph in a large class.

Exponential lower bounds for other classes of (what can be seen as) branch-and-reduce algorithms exist for other problems as well, for example k-SAT [1,2,47], Maximum Independent Set [16], Graph Coloring [42], and Knapsack [17].

4. Example: Minimum Vertex Cover

We now present an example, mainly illustrating some of our techniques for proving lower bounds for priority algorithms with advice, but also presenting an algorithm showing that the result is tight for the class of inputs given by the adversary. Both the algorithm and lower bound apply to the decision-based priority model.

Given a simple undirected graph $G = (V, E)$, a subset of vertices $S \subseteq V$ is called a *vertex cover* if every edge in G is incident to at least one vertex from S. *Minimum Vertex Cover* is the problem of finding a vertex cover of minimum size. An input item is a vertex together with a complete list of its neighbors (including those vertices that have not even appeared as part of the input yet); this is known as the *vertex arrival, vertex adjacency model*. Thus, for each vertex, when it becomes the highest priority vertex, the priority algorithm must decide whether or not to "accept" or "reject" that vertex, under the condition that at the end, for every edge in the graph, at least one of its endpoints must have been accepted.

Theorem 1. *No adaptive priority algorithm can solve Minimum Vertex Cover optimally with fewer than $|V|/7$ bits of advice in the decision-based priority function model.*

Proof. Within the proof, we have found it beneficial to include intuition and introduce terminology relevant for the general templates, making the style somewhat different from a normal formal proof.

We build on the construction in [12] (which was reused in [13]), showing that for this problem, no adaptive priority algorithm without advice can achieve an approximation ratio better than 4/3. The two graphs in Fig. 1 are used.

In proving lower bounds for adaptive priority algorithms, the adversary chooses the input, first choosing the universe of input items, and then creating an actual input X from that universe. Originally the adversary can set X to the entire universe. Then it (perhaps gradually) removes input items from X as the algorithm selects input items using priority functions and makes irrevocable decisions for them. Thus, the input item selected by the current priority function is always one of the remaining input items in X
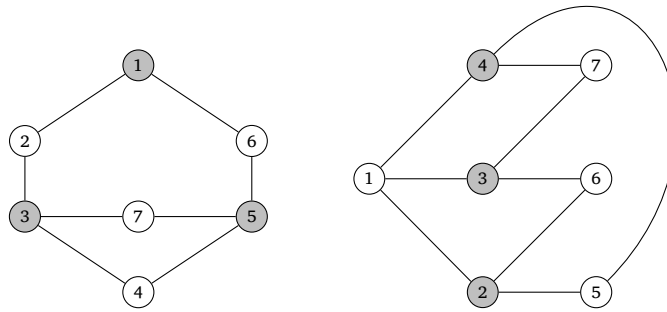
Fig. 1. Topological structures of graphs giving a lower bound for the Minimum Vertex Cover problem. Graph 1 is on the left and Graph 2 is on the right. The unique minimum vertex covers are marked in gray.

with highest priority. When there are ties, the adversary can choose among those with highest priority. (In this proof, the adversary can simply choose an arbitrary item of highest priority, so we may assume that there is always a single input item with highest priority.)

For Minimum Vertex Cover, the adversary, ADV, will select an isomorphic copy of either Graph 1 or Graph 2 from Fig. 1, depending on the algorithm, ALG. Since both graphs have seven vertices, the universe, $\mathcal{U}$, of input items, contains the names of seven vertices (the same names are used for both graphs), and for each of the vertices, all possibilities for input items (names of vertices and lists of neighbors) for degrees two and three. Note that both graphs have unique minimum vertex covers of size 3. The numbers shown in the figure are for our reference only and do not represent actual input items given to an algorithm. The figure represents the *topological* structure of the inputs. The actual input items would be created out of all consistent namings of vertices in such graphs. To illustrate this point, consider vertex 1 in Graph 1. It is adjacent to vertices 2 and 6. The corresponding possible input item could happen to be $(1, \{2, 6\})$, but it could also be $(5, \{2, 3\})$, for example. In the latter case, the actual input vertex 5 would be mapped to vertex 1 in the figure, vertex 2 would be mapped to vertex 2, and vertex 3 would be mapped to vertex 6. In total, there are $7 \times 6 \times 5$ possible input items that could be associated with vertex 1 in Graph 1. After a particular item has been processed, the number of items that could be associated with subsequent vertices is reduced because of consistency requirements.

Given this universe of input items, the first priority function for any algorithm, ALG, for Minimum Vertex Cover must select either a vertex of degree 2 or a vertex of degree 3 as the first vertex to be processed.

In order to obtain a vertex cover of size 3, it is necessary to accept vertex 1 in Graph 1 and reject vertex 2 in Graph 1. Thus, for the case where the first vertex selected by ALG has degree 2, ADV can force ALG to produce a vertex cover of size at least four by choosing vertex 1 from Graph 1 if ALG rejects and choosing vertex 2 from Graph 1 if ALG accepts. Because of how the universe is defined, ADV can do this regardless of which input item with degree 2 ALG chooses.

Similarly, in order to obtain a vertex cover of size 3, it is necessary to accept vertex 3 in Graph 1 and reject vertex 1 in Graph 2. Thus, for the case where the first vertex selected by ALG has degree 3, ADV can force ALG to produce a vertex cover of size at least four by choosing vertex 3 from Graph 1 if ALG rejects and choosing vertex 1 from Graph 2 if ALG accepts. Again, because of how the universe is defined, ADV can do this regardless of which input item with degree 3 ALG chooses.

To define a problem where $k = |V|/7$ bits of advice are necessary for optimality in the decision-based priority model, we consider an algorithm, ALG′, and an adversary, ADV′. We create k disjoint subuniverses, $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_k$, copies of the subuniverse $\mathcal{U}$, with different names for the vertices in each copy, and define the universe, $\mathcal{U}'$, for ALG′ to be the union of these k subuniverses. The input for ALG′ is the union of $H_1, H_2, \ldots, H_k$, where H_i is an isomorphic copy of either Graph 1 or Graph 2.

With its priority functions, ALG′ can choose input items in many different ways, and could, for instance, interleave input items stemming from different copies of $\mathcal{U}$. However, for each $\mathcal{U}_i$, there is always a first vertex in $\mathcal{U}_i$ that ALG′ chooses (from the current subset X of the universe, $\mathcal{U}'$). When ADV′ is not restricted by advice that ALG′ has read, it can force ALG′ to accept a vertex cover of size four for H_i, exactly as ADV forces ALG, depending on whether this first vertex from $\mathcal{U}_i$ has degree 2 or 3.

We now define 2^k sequences of input items for ALG′, by describing how one of these 2^k sequences of input items is defined: ALG′ selects input items one at a time, and ADV′ knows from which of the k subuniverses the input items originate.

In this concrete case of an adaptive priority algorithm (with advice), since we are assuming that ALG′ solves the problem to optimality, the adversary can assume in the decision-based priority model that the current priority function is determined based on ALG′ making the correct accept/reject decisions up to this point. Now, ADV′ does the following: Assume that ALG′ has already received input items originating from i of the subuniverses from which $\mathcal{U}'$ was defined and the adversary has a current subset $X \subseteq \mathcal{U}'$. If that is the case, then X contains exactly enough input items to complete one graph from each of the subuniverses from which ALG′ has received some input item (how this is maintained is explained below). From subuniverses not included in these i subuniverses, X still contains all possible names for vertices in the graphs. Now, ALG′ receives its next input item: the input item in X of the highest priority in this round.

If that next input item, v, is from one of the i subuniverses, nothing further is done. However, if that next input item originates from a subuniverse not among the i, then the following is done.

If v has degree 2, ADV′ can choose that it is vertex 1 in Graph 1 or vertex 2 in Graph 1. If v has degree 3, ADV′ can choose that it is vertex 3 in Graph 1 or vertex 1 in Graph 2. It makes a choice and then removes from X all input items originating from the selected subuniverse of $\mathcal{U}'$, except enough to make up exactly the graph that was chosen (Graph 1 or Graph 2) with the vertex names consistent with the first input item from that graph.

Continuing this inductively defines one of the 2^k distinct input sequences.

If a priority algorithm with advice for Minimum Vertex Cover uses fewer than k bits of advice for instances with $7k$ input items, the same advice must be given for at least two of the sequences, I_1 and I_2, defined above. ALG′ therefore uses the same priorities and makes the same decisions on I_1 and I_2 until some difference is detected. Thus, consider the first time in the processing of I_1 and I_2, where an input item that has current highest priority is the first input item of a graph from some $\mathcal{U}_j$, but the graphs included in I_1 and I_2 from $\mathcal{U}_j$ are different.

Up until (and including) this point, all input items have been the same for the two sets. Thus, ALG′ must make the same decision for v in both I_1 and I_2, but one of those decisions leads to a vertex cover of size four. Thus, ALG′ is not optimal, and k bits of advice are necessary.   □

This lower bound is generalized in Subsection 6.3, giving a template for proving such bounds.

For an algorithm matching the lower bound of the above theorem on these particular types of inputs using Graphs 1 and 2, we begin with the case $k = 1$, i.e., we receive a graph isomorphic to either Graph 1 or 2. Making the correct decision on the first vertex received enables a priority algorithm to obtain a vertex cover of size 3 by giving highest priority after that to neighbors of vertices which are already chosen, accepting if the known neighbor was rejected, and rejecting if the known neighbor was accepted. Continuing in this way until all vertices are processed always produces the minimum vertex cover. Thus, one bit of advice is necessary and sufficient for optimality for these restricted inputs; the one bit indicates whether or not the first vertex should be accepted or rejected.

Extending the algorithm just described for the case $k = 1$ for achieving optimality when one bit of advice is given per subuniverse, one notes that k bits of advice are also sufficient for these very specific types of input. Thus, in this very restricted problem, for every positive integer k, there is an input size where k bits of advice are necessary and sufficient.

Since the results in this section concern exact, rather than approximation algorithms, all results also apply to Maximum Independent Set for graphs of maximum degree 3. Both Graph 1 and Graph 2 are triangle-free graphs, so the lower bound also holds for triangle-free graphs of maximum degree 3, as does the 4/3 lower bound on the approximation ratio for adaptive priority algorithms without advice.

5. Solving Minimum Vertex Cover to optimality for triangle-free graphs of maximum degree 3

We consider the Minimum Vertex Cover problem, as defined in Section 4, on triangle-free graphs of maximum degree 3, in the online and in a priority setting with advice. The vertex arrival, vertex adjacency model is used. (Since the results in this section concern exact, rather than approximation algorithms, all results also apply to Maximum Independent Set for triangle-free graphs of maximum degree 3.) Let n denote the number of vertices in the input graph. As mentioned in Section 4, no adaptive priority algorithm without advice can achieve an approximation ratio for this problem better than 4/3 [12], since graphs used in the construction there were triangle-free with maximum degree 3. In this section, we show that asymptotically this problem requires at least $(n-4)/3$ bits of advice to solve optimally in the online setting, while it can be solved optimally using at most $7n/22 < 0.3182n$ bits of advice in the adaptive priority setting.

We begin with the negative result for the online setting.

Theorem 2. *Asymptotically, for $n \geq 7$, no online algorithm using fewer than $(n-4)/3$ bits of advice can accept a minimum-sized vertex cover for all triangle-free graphs of maximum degree* 3.

Proof. The adversary will use a graph with $n = 6n' + 1$ vertices, where $n' \geq 2$. The set of all vertices is denoted by V.

One way to describe the adversarial input is as if it is being constructed in stages. In the first stage, the adversary creates $2n'$ disconnected paths of length 2 each, or 2-paths, for short (this already gives $6n'$ nodes). In the second stage, the adversary connects endpoints of 2-paths, chaining several paths together into one large cycle. Not all initial 2-paths will necessarily participate in the cycle. Finally, the adversary attaches one more vertex to an appropriately chosen vertex v in the cycle and decides how to present this constructed graph online. An optimal decision to accept or reject a middle vertex of each initial 2-path depends on the answers to these questions: Does this 2-path participate in the large cycle or not, and, if it participates in the cycle, is it located at an even or odd distance from v? When the fully constructed adversarial input is presented to an online algorithm such that middle vertices of initial 2-paths are given first, the algorithm does not yet know the answers to the questions above, so a lot of advice is required to infer correct decisions for these vertices.

More formally, let $S = \{v_1, v_2, \ldots, v_{2n'}\}$ be the first $2n'$ vertices to be given – they form middle vertices of 2-paths, so all vertices in S will have degree 2. Throughout the processing of S, the neighbors will be vertices never seen before. As described above, some neighbors of S will be connected so as to form a cycle, which we denote by C. Then there will be a unique vertex w of degree 1,
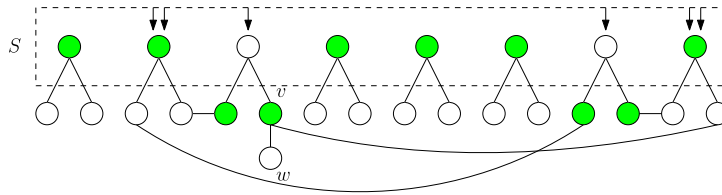
Fig. 2. The construction used in Theorem 2. Here, we have $n' = 4$. The optimal vertex cover is shown in green. Vertices with a single arrow pointing to them are those vertices from S that were selected to be at odd distance from node v. Vertices with two arrows pointing to them are those nodes from S that were selected to be at even distance from node v. Here, the number of vertices in S *not* in the optimal vertex cover is $r = 2$.

connected to one designated neighbor $v \in C \setminus S$. Finally, the set of all other vertices will be denoted I, i.e., $I = V \setminus (C \cup \{w\})$. This set induces isolated 2-paths, with the middle vertices in S. The vertex v will have degree 3. There will be an even number of vertices from S in I and, thus, an even number in C. The construction is illustrated in Fig. 2.

Note that this graph has a unique minimum-size vertex cover: the middle vertex of each path in I and every other vertex in C, starting with v.

For each vertex, $u \in S$, all of which have degree 2, ALG must decide whether to accept or reject this vertex, without knowing if u is in I or C. Of course, within C, ALG will not know if u will be at an even or odd distance from v.

Suppose we want to create a graph G with $0 \le r \le n'$ vertices from S *not* in the optimal vertex cover. We can choose any subset R of r vertices in S to be at odd distances from v in C. Among the other vertices, r can be placed at the even locations in C, and the remaining $2n' - 2r$ vertices from S can be in I. (The placement of v is also arbitrary, but we are fixing a placement in this counting.) For fixed r, there are $\binom{2n'}{r}$ different possibilities for the subset R. In all, there are $\sum_{r=0}^{n'} \binom{2n'}{r}$ different possibilities for the subset R, each with a different optimal vertex cover (note that $r = 0$ is a degenerate case where there is no cycle, v, or w, but the instance is still a possibility, and for $r \ge 1$, the unique cycle C has at least 6 vertices and $n \ge 7$, so the graph is triangle-free). Any online algorithm that gets the same advice for two of them must give a suboptimal cover for at least one of them. Thus, an algorithm that solves the problem to optimality needs at least $\log_2 \sum_{r=0}^{n'} \binom{2n'}{r} > \log_2 2^{2n'-1} = 2n' - 1 = (n-4)/3$ bits of advice.

Just for emphasis, note that all input items in S are fixed to be exactly the same in all instances that we consider, i.e., input items in S do not depend on the choice of R, v, and w. Thus, an online algorithm receiving items from S can only rely on advice to act differently on S from instance to instance. $\square$

Now, we present an *adaptive priority algorithm with advice* that works in both the decision-based and oblivious priority function models, uses fewer than $(n-4)/3$ bits of advice, and achieves optimality.

We present an adaptive priority algorithm PRIORITYVC with advice for the Minimum Vertex Cover problem on triangle-free graphs of maximum degree 3. The main result of this section is the following:

Theorem 3. PRIORITYVC *solves Minimum Vertex Cover on triangle-free graphs with maximum degree* 3 *optimally in both the decision-based and oblivious priority function models and uses at most* $(7/22)n = 0.31\overline{81}n$ *bits of advice, where n is the number of vertices.*

Proof. Follows from Lemmas 1 and 4. $\square$

In order to describe and analyze the algorithm, we have to introduce and define some terminology. We do this in the order from most intuitive to least intuitive. Fortunately, most of the terminology will be self-explanatory, but needs to be stated for the sake of completeness.

Since it is an adaptive priority algorithm, PRIORITYVC works in discrete time steps. Each time step consists of the algorithm updating the priority function, receiving the next input item according to the new priority, potentially reading advice, and then making a decision as to including the vertex corresponding to the input item in the solution or not. We also refer to the decision of including the vertex in the solution as *accepting the vertex* and the opposite decision as *rejecting the vertex*. The decision is called *correct* if it is possible to extend the partial solution obtained after the decision to a minimum vertex cover in the input graph.

In many cases it is possible to make a decision that is guaranteed to be correct without consulting advice at all. Consider, for example, a vertex of degree 1 – it is easy to see that a correct decision is to reject such a vertex and then accept its unique neighbor.

Suppose that at time t vertex v arrives and it is not possible, from the vertices seen so far, to make a decision that can be guaranteed to be correct no matter what happens in the rest of the input. Since we want to solve the problem to optimality, PRIORITYVC then reads a single bit of advice. This bit encodes a correct decision for the algorithm. In other words, if the bit is 1, then the algorithm accepts v and otherwise the algorithm rejects v. In these cases, we say that the advice is to accept or reject the vertex, respectively. We also say that v received advice.

Once a decision has been made for a vertex, this vertex is called *processed*. Vertices that have not been processed are called *unprocessed*. Suppose that the algorithm processes the vertices in the order $v_1, v_2, \ldots, v_n$ – this notation is only for the duration of this paragraph and will have a different meaning in the proofs below. Recall that the input items consist of pairs $(v_i, N(v_i))$, where $N(v_i)$ is the neighborhood of the vertex v_i. Since the priority algorithm is adaptive, it can effectively remove processed vertices from the input graph. Namely, at time i, the algorithm knows $v_1, \ldots, v_{i-1}$. Therefore, in defining the priority function, the algorithm can

ignore vertices in $\{v_1, \ldots, v_{i-1}\}$ when assigning a priority to $(v, N(v))$, which is equivalent to removing vertices $v_1, \ldots, v_{i-1}$ from the rest of the input graph. We refer to $N(v) \setminus \{v_1, \ldots, v_{i-1}\}$ as the *current neighborhood of v* and $|N(v) \setminus \{v_1, \ldots, v_{i-1}\}|$ as the *current degree of v*. We refer to $N(v)$ and $|N(v)|$ as the *original neighborhood* of v and the *original degree* of v, respectively.

The following is less intuitive but useful terminology for vertices:

aa-vertex: a processed vertex that received advice to be accepted.
ar-vertex: a processed vertex that received advice to be rejected.
a-vertex: either an aa-vertex or an ar-vertex.
non-a-vertex: a vertex that was processed without advice.
contributing: an aa-vertex with two rejected and one unprocessed neighbor.
c-neighbor: an unprocessed vertex that is a neighbor of a contributing vertex.
bad-vertex: a vertex that requires advice and all of its neighbors are c-neighbors of other vertices at the time this vertex is processed.
a-sibling: a neighbor of an aa-vertex v such that v has another neighbor that has been accepted.

Observe that the above definitions are with respect to a given time step. In particular, it is possible that a vertex v is processed during some time step and at that point becomes an aa-vertex. At a later time step, it could become a contributing vertex. Also observe that it is possible that a neighbor of an unprocessed vertex is a c-neighbor, that is, a neighbor of some other vertex that is contributing at the time of consideration.

The pseudocode of PRIORITYVC is given in Algorithm 3. Ties that are not broken by PRIORITYVC explicitly can be broken arbitrarily (even by an adaptive adversary).

Algorithm 3 PRIORITYVC algorithm.

---

procedure PRIORITYVC
  while there exist unprocessed vertices **do**
    Define the priority function P as follows
      (listed in order from highest to lowest priorities):
    P1: nodes with a rejected neighbor;
      highest priority is given to those nodes whose neighbor
      was most recently rejected.
    P2: nodes with current degree 0.
    P3: nodes with current degree 1;
      highest priority is given to those nodes with a most recently
      processed neighbor; among those, highest priority is given to
      those nodes that had two neighbors that became aa-vertices.
    P4: nodes with current degree 2 that had a third neighbor in
      common with a previously rejected bad-vertex.
    P5: a-siblings.
    P6: nodes with current degree 3 with 2 or 3 neighbors in common
      with a single aa-vertex that was not a bad-vertex when it
      received advice.
    P7: nodes with current degree 3 that share neighbors with
      a-vertices.
    P8: other nodes with current degree 3.
    P9: nodes with current degree 2

    Receive the next vertex v according to P
    switch priority of v
      case P1 or P6:
        Accept v
      case P2, P3, P4, P5, or P9:
        Reject v
      case P7 or P8:
        Obtain advice to accept or reject and apply it to v

In order to finish the specification of PRIORITYVC, we have to describe how the oracle generates the advice. The oracle sees the entire input beforehand and it knows how PRIORITYVC works. Since PRIORITYVC is deterministic, the oracle can, in effect, simulate PRIORITYVC on the input. Thus, the oracle knows the order in which the vertices are processed and it knows at which time steps PRIORITYVC asks for advice. The oracle supplies the advice in the order in which the advice is requested by PRIORITYVC. Suppose that at some time, PRIORITYVC processes v based on advice. If there is a unique correct decision for v, the oracle provides that decision, either accept or reject, which is one bit of information. If either decision is correct (could be completed to a minimum vertex cover) and v is a bad-vertex, the oracle advises to accept. Finally, if either decision is correct and v is not a bad-vertex, the oracle advises to reject. This tie-breaking condition is particularly important for the analysis.

We mention a few high level features of PRIORITYVC. Vertices that obviously can be handled without advice are those with current degree 0 or 1, and neighbors of rejected (accepted in `mis3`) vertices. The two key observations in the design of `mis3` are the following: First, the vertices just described should receive the highest priorities (as described in Section 3.1). Second, if we
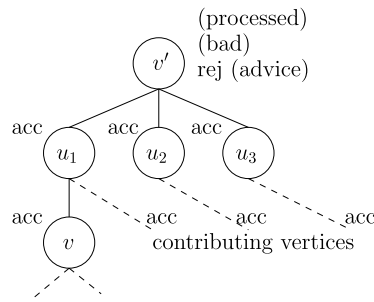
Fig. 3. Illustration of the case of v having priority P4 in Lemma 1. Some edges are omitted so as to avoid clutter.

process vertices of current degree 3 prior to processing vertices of current degree 2 (with a small exception of P4; ignore that for the moment), then, when a vertex of current degree 2 arrives according to P9, we know that all the remaining vertices in the graph have current degree 2. We can conclude that the remaining graph is a collection of disjoint cycles and an optimal vertex cover in such a graph can be computed by a priority algorithm without advice. Therefore, with such an approach, only vertices of current degree 3 may require advice and the goal is to minimize the number of such vertices. This is where cooperation between the oracle and the algorithm becomes crucial – we shall see that the tie-breaking condition of the oracle is chosen so as to create scenarios under which some vertices of current degree 3 may be processed without advice.

Next, we analyze PRIORITYVC formally. We begin with the easier proof of correctness of the algorithm and then establish the sufficient number of bits of advice. Suppose that at time t a vertex arrives according to priority P9 for the first time. Then we refer to the time interval $[1, t-1]$ as *Phase* 1 and to the time interval $[t, n]$ as *Phase* 2. If such t does not exist then we set $t = n + 1$ meaning that the entire time interval $[1, n]$ consists of only Phase 1 and Phase 2 is empty. Correctness of the algorithm follows from the following lemma.

Lemma 1. *Every decision of* PRIORITYVC *is correct.*

Proof. The proof is by induction in the number of input items. For the base case, no input items have been processed and all decisions have trivially been correct. For the inductive step, if all previous decisions have been correct, we need to demonstrate that the decision for the next vertex is also correct. Let v be the newly arriving vertex.

First, suppose that the algorithm is in Phase 1.

Case: v has priority P1, P2, P3, P7, or P8. Decisions of PRIORITYVC are obviously correct.

Case: v has priority P4. PRIORITYVC rejects v, so suppose for the sake of contradiction that v should have been accepted. Let v' denote a bad-vertex and u_1, u_2, and u_3 its three neighbors such that u_1 is also a neighbor of v. This is illustrated in Fig. 3.

Observe that another optimal vertex cover is obtained by accepting the vertices v', u_2, u_3, v and rejecting u_1. Thus, the oracle would have given advice to accept v', since at the time v' was processed, both decisions were correct, and the oracle prefers accepting bad-vertices. This is a contradiction, so the decision of PRIORITYVC to reject v is correct.

Case: v has priority P5. Observe that processing bad-vertices leads to processing of their neighbors prior to any vertex with priority P5 being processed. Therefore, v is a neighbor of an aa-vertex v' and v' was never bad. Denote the neighbors of v' by u_1, u_2, u_3 such that $u_1 = v$. Consider the time when v' received advice to be accepted. We claim that at most one of u_1, u_2, u_3 can be accepted in the future. Suppose, for the sake of contradiction, that at least two nodes, say, u_1 and u_2, must be accepted in the future. Then accepting u_1, u_2, u_3 and rejecting v' would result in a vertex cover of the same size or smaller as accepting v', u_1, u_2 and rejecting/accepting u_3. In this case, since v' was not bad at the time it received advice, the oracle should have given advice to reject v' according to the tie-breaking condition. This is a contradiction, and therefore at most one of u_1, u_2, u_3 can be ever accepted. By definition of an a-sibling, either u_2 or u_3 has been accepted prior to $v = u_1$ being processed, so it is correct to reject v.

Case: v has priority P6. As in the previous case, let v' be the aa-vertex that shares at least two neighbors with v and that was not bad at the time it was processed. As already argued, at most one neighbor of v' can be accepted, therefore at least one of the neighbors of v in common with v' must be rejected. Since each edge must be covered by the solution, we conclude that v must be accepted.

Since the case of P9 cannot happen in Phase 1, we move to the analysis of Phase 2. As discussed prior to this lemma, at the beginning of Phase 2 we know that the remaining graph is a collection of cycles. Once a vertex of current degree 2 arrives according to P9, it is rejected, which creates two vertices of current degree 1 each. They are neighbors of a rejected vertex, so they are processed next according to P1. The degrees of their neighbors on the cycle drop to 1 or 0, so they are processed according to P1–3. This continues until all vertices in this cycle have been processed. Then the next cycle is processed and so on. The correctness of the constructed vertex cover follows from the fact that a minimum vertex cover in every cycle rejects at least one vertex. Thus, by the isomorphisms of a cycle, it is always safe to reject the first vertex from the cycle. After that, correctness follows by the correctness of cases P1–3, as in Phase 1. □

Central to the analysis of the number of bits of advice is the notion of a *component*. A new component starts when a new a-vertex is processed that does not have neighbors in common with a previously processed a-vertex. When a new component is started, any previous component is closed, meaning that it receives no more vertices. A vertex is included in the current component if it is not in any previous component, and one of the following cases applies:

- it is an a-vertex that shares a neighbor with a previously processed a-vertex from the current component,
- it is a neighbor of an a-vertex from the current component,
- it is accepted or rejected before the component is closed.

Note that a component in the above sense is not to be confused with a connected component – it is possible for a connected graph to consist of several components, and it is possible that such a component is not connected.

We let c denote the final number of components created by PRIORITYVC on the given input. For $i \in [c]$, we let $a_i(t)$ denote the number of a-vertices in component i at time t, and we let $s_i(t)$ denote the size of component i at time t. Let $\hat{t}_i$ denote the time component i is closed. We use a shorthand notation $a_i := a_i(\hat{t}_i)$ and $s_i := s_i(\hat{t}_i)$ for the final number of a-vertices in component i and the final size of component i, respectively. We also define $n_i(t) := s_i(t) - a_i(t)$, which is the number of non-a-vertices in component i at time t, and $n_i = s_i - a_i$, which is the number of non-a-vertices in component i.

The high level idea behind bounding the number of advice bits used by PRIORITYVC is to prove two inequalities and then take their linear combination. The first inequality (Lemma 2) is more local in that it is proved for each component independently of other components. The second inequality (Lemma 3) is more global in that it incorporates potential interactions between components. Both inequalities are proved via weight reallocation arguments as explained in the following lemmas.

We begin with the more difficult local lemma.

Lemma 2. *For all $i \in [c]$, we have*

$$s_i \geq 3a_i + 1.$$

Proof. Consider component i.

If $a_i = 1$, then the vertex that received advice and its three neighbors are added to the component by definition, so $s_i \geq 4 = 3a_i + 1$.

If $a_i = 2$, then the two vertices that received advice can share at most one neighbor. If, to the contrary, they had two vertices in common, then if the first of the two vertices is rejected, then its neighbors are accepted, and the second vertex becomes unary and does not need advice due to P3; a contradiction. Similarly, if the first vertex is accepted, it becomes an aa-vertex and the second vertex gets accepted without advice due to P6. So counting the two vertices and their five distinct neighbors gives that $s_i \geq 7 = 3a_i + 1$.

If $a_i \geq 3$, then the situation is more involved. The desired inequality trivially follows if we can establish

$$n_i \geq 2a_i + 1. \tag{1}$$

For $j \in [a_i]$, let t_j denote the time step when the jth a-vertex in component i is processed. Call this vertex v_j. Thus, the component gets started at time t_1 with a-vertex v_1. Denote the three neighbors of v_j by $u_{j,1}, u_{j,2}, u_{j,3}$.

Denote the weight of a vertex v by $w(v)$. Each non-a-vertex v that gets added to this component starts out with weight $w(v) = 1$. Each a-vertex v_j that gets added to this component starts out with weight $w(v_j) = 0$. The weight is reallocated from non-a-vertices to a-vertices, so as to guarantee the properties below at the end of processing the component. We let the index of the a-vertices $v_1, v_2, \ldots, v_\ell$ denote the order in which weight is allocated to them, and establish the following:

- **I1** $w(v_1) = 1.5$;
- **I2** $w(v_2) = 2.5$;
- **I3** $w(v_3) = 2.5$;
- **I4** $w(v_j) = 2$ for $j \in [4, \ell]$.

Note that since we are in the case $a_i \geq 3$, this is well-defined. We check that I1–4 are sufficient to establish the claim. Observe that the total amount of weight allocated to component i is exactly n_i. After reallocating the weight, I1–I4 imply that the total weight in the component is $\geq 1.5 + 2.5 + 2.5 + 2(a_i - 3) = 2a_i + 0.5$. Since the reallocation procedure does not destroy weight or create extra weight, the total amount of weight in the component at the end is n_i. This implies that $n_i \geq 2a_i + 0.5$. Since n_i and a_i are integers, we have $n_i \geq 2a_i + 1$, as desired.

We execute weight reallocation in parallel with PRIORITYVC. The reallocation follows some rules: (a) after sufficient weight is reallocated to an a-vertex, this weight is not reallocated ever again; (b) only the weights of vertices that are in component i can be reallocated (to an a-vertex in component i); (c) at any point in time, the weight of non-a-vertices can be either 0, 0.5, or 1; (d) if the weight of a non-a-vertex is 0, then the vertex has been processed and removed from the graph; (e) the weight of every non-a-vertex can be reallocated twice: 0.5 can be reallocated when its degree goes from 3 to 2 (and not more than 0.5 is reallocated in this scenario) and the remaining 0.5 is reallocated when the degree of the vertex drops down further, when it is processed, or even after it is processed; (f) every unprocessed vertex with weight 0.5 is a neighbor of a processed a-vertex. We do not keep track of each of the above statements explicitly in the following case analysis, since this is rather tedious. It is fairly straightforward to verify that each claim continues to hold in the analysis below.
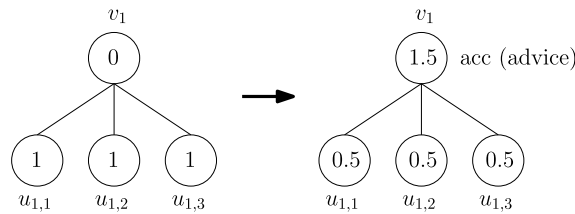
Fig. 4. Illustration of reallocating 0.5 units of weight from each of the neighbors to v_1 in **I1**. In order not to clutter the illustration, we do not show all edges incident to vertices. How a vertex is processed starting at t_1 is indicated next to the vertex. The weight of a vertex is shown inside the vertex.
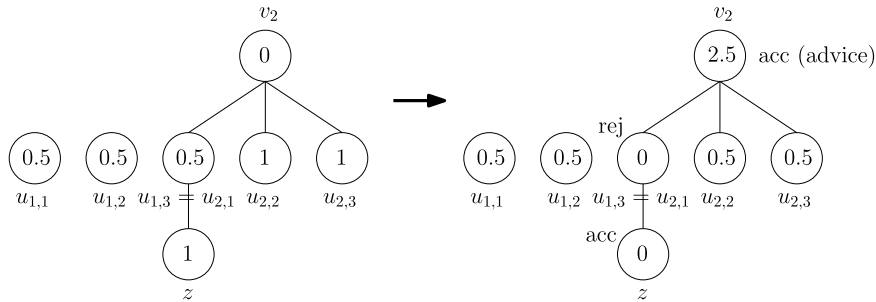

Fig. 5. Illustration of reallocating 1 unit of weight from z and 0.5 units of weight from each of neighbors of v_2 to v_2 in **I2**.

Observation 1: For point (b), we make one observation that is used repeatedly, namely that a certain neighbor of a neighbor cannot belong to an earlier component, which means that we are allowed to reallocate weight from it. Note that the only unprocessed vertices of a closed component are neighbors of aa-vertices. Consider an a-vertex v_j in the current component that shares a neighbor $u_{j,1}$ with a previously processed a-vertex $v_{j'}$, also of component i, for some $j' < j$. Then after processing v_j, the current degree of $u_{j,1}$ drops to 1. Let z be the unique neighbor of $u_{j,1}$ at that point. We claim that z cannot belong to a previous component. If z did belong to a previous component, say $i' < i$, then z would necessarily be a neighbor of an a-vertex in component i'. Suppose that component i' was closed at time t. The degree of $u_{j,1}$ is 3 until $v_{j'}$ is being processed. Thus, at time t, $u_{j,1}$ shared a neighbor, z, with an a-vertex in component i'. This implies that component i' should not have been closed at time t, since it could be extended by considering $u_{j,1}$. Thus, z cannot belong to a previous component, and we are free to allocate weight away from z.

With this additional observation, we are ready to prove I1–4.

I1. Observe that if the first a-vertex is an ar-vertex, then all its neighbors are removed prior to any other vertex receiving advice. Since a-vertices in a component are connected through common neighbors, there can be no other a-vertices in the component, so $a_i = 1$. Therefore, since we assume that $a_i \geq 3$, the first a-vertex must be an aa-vertex. This vertex and its three neighbors are added to the component. We reallocate 0.5 units of weight from each of the neighbors to v_1. This is illustrated in Fig. 4.

I2. The second a-vertex v_2 must have exactly one neighbor in common with v_1: if it had no neighbors in common, a new component would get started; if it had more than one neighbor in common, then it would be processed without advice. Without loss of generality, let that neighbor be $u_{1,3} = u_{2,1}$. Observe that v_2 must have received advice to be accepted. If it received advice to be rejected, then all its neighbors would be accepted and $u_{1,1}$ and $u_{1,2}$ would become a-siblings (if they have not been processed yet), so they would get processed prior to v_3. But this implies that all neighbors of v_1 and v_2 would be eliminated prior to v_3 and v_3 would never be added to the current component, contradicting the assumption that $a_i \geq 3$.

Thus, we assume that v_2 received advice to be accepted. At time t_2, the current degree of $u_{1,3}$ must be 2: if it was higher, then the original degree (which would include v_1) would be more than 3; if it was lower, then $u_{1,3}$ would be processed prior to v_2 and v_2 would not have received advice. One of the vertices contributing to the current degree of $u_{1,3}$ is v_2. Let the other vertex be z. Observe that z is different from all of $u_{1,1}, u_{1,2}, u_{2,2}, u_{2,3}$ since otherwise the input graph would contain a triangle. When v_2 is processed, the current degree of $u_{1,3}$ drops to 1, so it will be rejected and its neighbor accepted. Since z is a new vertex that has been added to the component, we can reallocate one unit of weight from z to v_2. We also reallocate 0.5 units of weight from each of the neighbors of v_2 to v_2. This results in the overall weight of v_2 being 2.5, as desired. It is easy to check that this reallocation satisfies all the rules and the illustration is shown in Fig. 5.

I3. There are several cases for v_3.

Case 1. Consider the case where v_3 shares a single neighbor with a previous aa-vertex (could be either v_1 or v_2). Without loss of generality, let the shared neighbor be $u_{3,1}$. Then $u_{3,2}$ and $u_{3,3}$ are added to the current component for the first time so they start out with weight 1. Since $u_{3,1}$ has not yet been processed at t_3, its weight is 0.5.

Subcase 1(a). Suppose that v_3 receives advice to be rejected. Then the weight of all its neighbors can be reallocated to v_3 resulting in $w(v_3) = 2.5$, as desired. This obeys the reallocation rules, since v_3 and all its neighbors will be removed from the graph prior to t_4. This is illustrated in Fig. 6.
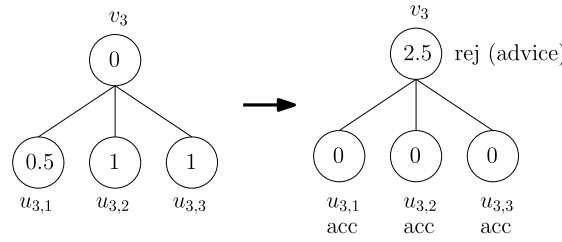
Fig. 6. Illustration of reallocating the weight of all the neighbors of v_3 to v_3 in **I3**, Subcase 1(a).
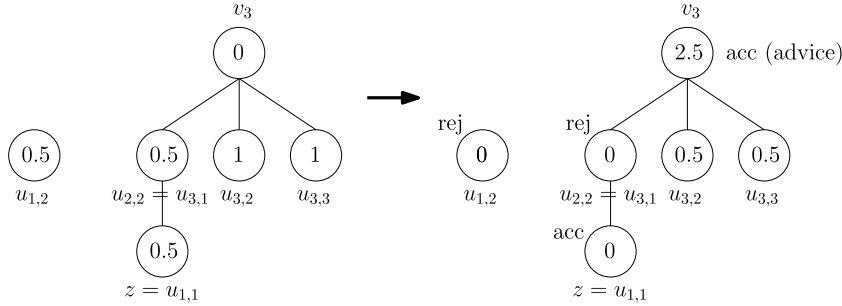

Fig. 7. Illustration of reallocating 0.5 units of weight from each of $u_{1,2}, z = u_{1,1}, u_{2,2} = u_{3,1}, u_{3,2}, u_{3,3}$ to v_3 in **I3**, Subcase 1(b).
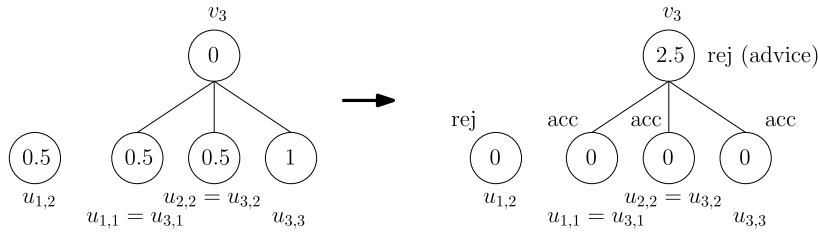

Fig. 8. Illustration of reallocating 1 unit of weight from $u_{3,3}$ and 0.5 units of weight from $u_{1,2}$ and the other neighbors of v_3 to v_3 in **I3**, Subcase 2(a).

Subcase 1(b). Suppose that v_3 receives advice to be accepted. Without loss of generality suppose that $u_{3,1} = u_{2,2}$, i.e., the single shared neighbor is with v_2 (v_2 and v_1 behave symmetrically in the following argument). Arguing similarly to I2, after accepting v_3, the current degree of $u_{3,1}$ would drop to 1. Let z be the unique neighbor of $u_{3,1}$ at that point. Then, by the priority tie breaking in P3, $u_{3,1}$ is rejected and z is accepted. If z has weight 1 at time t_3, then the weight reallocation is done similarly to I2. Otherwise, z has weight 0.5. By Observation 1, z is in the current component. The vertex z cannot be a neighbor of v_3, or there would be a cycle. The only vertices in the current component of weight 0.5 after processing vertices in I1 and I2 and reallocating weights are neighbors of v_1 and v_2. Since z cannot be a neighbor of v_2 (this would create a triangle), it must be a neighbor of v_1. Without loss of generality, assume $z = u_{1,1}$. Since z is accepted, $u_{1,2}$ becomes an a-sibling, unless it was already processed. So, both $u_{1,2}$ and $u_{1,1}$ are processed and removed from the graph prior to t_4. Thus, we can reallocate 0.5 units of weight from each of $u_{1,2}, z = u_{1,1}, u_{2,2} = u_{3,1}, u_{3,2}, u_{3,3}$ to v_3 resulting in $w(v_3) = 2.5$ as desired. This last case is illustrated in Fig. 7.

Case 2. Suppose that v_3 shares two neighbors with previous aa-vertices – one with v_1 and another with v_2. More specifically, without loss of generality suppose that $u_{3,1} = u_{1,1}$ and $u_{3,2} = u_{2,2}$.

Subcase 2(a). If v_3 receives advice to be rejected, then the three neighbors $u_{3,1}, u_{3,2}, u_{3,3}$ are accepted. Their weights are reallocated to v_3. Moreover, $u_{1,2}$ (assuming that $u_{1,3}$ was the neighbor common to v_1 and v_2) was either processed earlier and had 0.5 units of weight remaining, or becomes an a-sibling and is processed prior to t_4. In either case, we can reallocate 0.5 units of weight from $u_{1,2}$ to v_3 for the total amount of weight reallocated to v_3 being 2.5. This is illustrated in Fig. 8.

Subcase 2(b). If v_3 receives advice to be accepted, then the current degrees of $u_{3,1}$ and $u_{3,2}$ drop down to 1 each (same argument as in I2). Let the unique neighbor of $u_{3,1}$ be z_1 and the unique neighbor of $u_{3,2}$ be z_2. Note that z_1 is not a neighbor of v_3 or v_1 in the original graph for otherwise it would contain a triangle. Similarly, z_2 is not a neighbor of v_3 or v_2. Observe that after processing v_3, vertices $u_{3,1}, u_{3,2}, z_1$, and z_2 will be processed prior to t_4. If either z_1 or z_2 (which could be the same vertex) has weight 1 at time t_3, then we can reallocate 0.5 units of weight from each of $u_{3,1}, u_{3,2}, u_{3,3}$ to v_3 and 1 unit of weight from z_1 or z_2 to v_3 for the total weight 2.5 as desired. An example where the weight of z_1 is 1 at time t_3 is illustrated in Fig. 9.

The only remaining scenario is when each of z_1 and z_2 have weight 0.5 at time t_3. Based on I1 and I2 and properties of z_1 and z_2 mentioned above, it must be the case that $z_1 = u_{2,3}$ and $z_2 = u_{1,2}$, since, otherwise, there is a triangle, so $z_1 \neq z_2$. In particular,
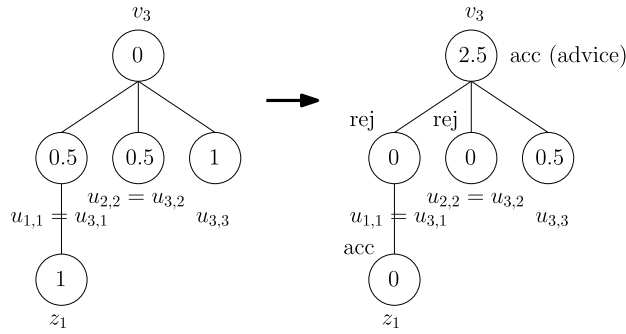
Fig. 9. Illustration of reallocating 0.5 units of weight from each of $u_{3,1}, u_{3,2}, u_{3,3}$ to v_3 and 1 unit of weight from z_1 or z_2 to v_3 in **I3**, Subcase 2(b).
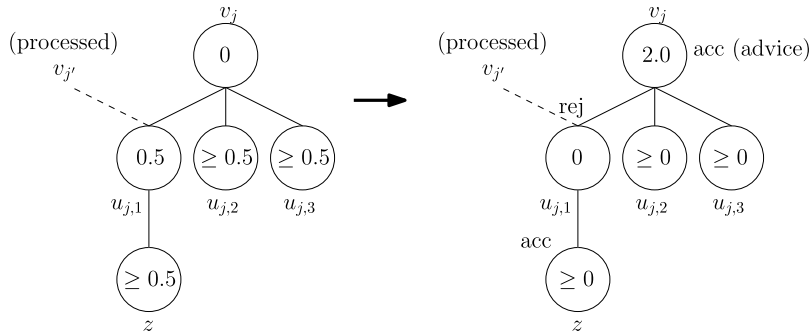

Fig. 10. Illustration of reallocating 0.5 units of weight from each of $u_{j,1}, u_{j,2}, u_{j,3}$ and z to v_j in **I4**, Case 1.

after processing v_3, vertices $u_{1,1}, u_{2,2}, u_{1,2}, u_{2,3}$ will be processed prior to t_4. Thus, we can reallocate 0.5 from each of them, plus 0.5 from $u_{3,3}$.

I4. Let $j \geq 4$ and consider v_j receiving advice at time t_j. Each of the neighbors of v_j has current degree at least 2 (same reason as in I2) and at least one of the neighbors is shared with a previous aa-vertex in the component.

Case 1. Suppose that v_j receives advice to be accepted. Without loss of generality assume that $u_{j,1}$ is a neighbor shared with $v_{j'}$ for some $j' < j$. After processing v_j the degree of $u_{j,1}$ drops to 1, so by the priority tie breaking in P3, it is rejected and its neighbor, call it z, is accepted. We can reallocate 0.5 units of weight from each of $u_{j,1}, u_{j,2}, u_{j,3}$ and z to v_j for the total weight of 2.0, as desired. This is illustrated in Fig. 10.

Case 2. Suppose that v_j receives advice to be rejected. Then the three neighbors are accepted. As argued before, each of the neighbors has degree at least 2 at time t_j, and each of the neighbors has at least 0.5 units of weight available for reallocation. If at least one of the neighbors has 1 unit of weight available, then we can reallocate 2.0 units of weight from the neighbors of v_j to v_j, as desired. If each neighbor has only 0.5 units available, then each neighbor is also a neighbor of a previously processed aa-vertex in this component. Let v'_k be such a processed neighbor of $u_{j,k}$ for $k \in [3]$. Observe that the v'_k are all distinct, since a vertex receiving advice can share at most one neighbor with a previous aa-vertex. If some v'_k is not a contributing vertex at time t_j, then, by accepting $u_{j,k}$, the other remaining neighbor of v'_k becomes an a-sibling and will be processed prior to t_{j+1}. In this case, we can reallocate 0.5 from the a-sibling and each of the $u_{j,k}$ for $k \in [3]$ to v_j for a total weight of 2.0, as desired.

The only remaining subcase is when all of the v'_k are contributing vertices at t_j. This means that v_j is a bad-vertex at time t_j. Consider what happens after processing v_j. The degree of each $u_{j,k}$ drops to exactly 1 and they are accepted. Let z_k be the unique neighbor of some $u_{j,k}$ immediately prior to $u_{j,k}$ being accepted (note that the z_k do not have to be distinct, but this does not matter for the following argument). If, after processing all $u_{j,k}$, the degree of at least one of the z_k drops below 2, then it would be processed prior to t_{j+1}. In this case, we can reallocate 0.5 units of weight from each of $u_{j,k}$ and 0.5 units of weight from the to-be-processed z_k to v_j for a total weight of 2.0. Otherwise, consider z_1, for example. After processing all $u_{j,k}$ the current degree of z_1 is 2. Thus, it can be rejected without advice according to priority P4 and its weight can be reallocated to v_j for the total weight of v_j being at least 2.0 (the other weights coming from the $u_{j,k}$), as desired. $\square$

Next, we prove the second inequality.

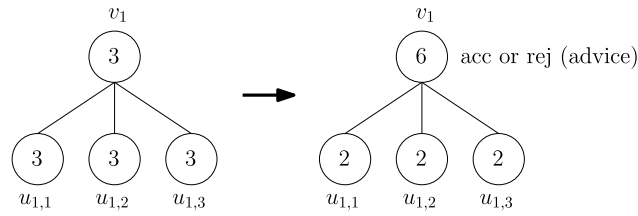**Lemma 3.** *We have*

$$10a - 4c \leq 3n,$$

Fig. 11. Illustration of reallocating one unit of weight from $u_{1,k}$ for $k \in [3]$ to v_1 in **J1**. As in Lemma 2, we do not show all edges incident to vertices so that the illustration does not become cluttered. How a vertex is processed is indicated next to the vertex. The weight of a vertex is shown inside the vertex.

where n is the number of vertices in the graph, a is the number of advice bits read by PRIORITYVC, *and c is the number of components, as defined earlier.*

Proof. We prove this via a weight reallocation argument similar to the one used in Lemma 2. Weight reallocation is done in parallel with PRIORITYVC, so we can describe it one vertex at a time. Weight reallocation is performed each time an input vertex receives advice and may involve vertices that are processed immediately after that without advice. There are several key differences from the weight reallocation done in Lemma 2. First of all, every vertex starts with initial weight 3 – no matter whether the vertex is an a-vertex or non-a-vertex. Secondly, we allow weight to be reallocated even from unprocessed vertices from closed components, since we are not interested in a component-wise inequality, but the inequality for the entire input. The weight reallocation procedure will guarantee the following properties:

J1. The first a-vertex of every component receives 6 units of weight.
J2. Subsequent a-vertices in every component receive 10 units of weight each.

The reallocation procedure satisfies the following additional constraints: (a) no extra weight is created or consumed; (b) the weight of a vertex is at least its current degree; (c) if a vertex has weight 0, then it must have been processed; (d) at any point in time t the weight that could have been reallocated by t comes only from vertices processed by time t or neighbors of a-vertices processed by time t. We will not explicitly check each of these constraints in the cases described below, but it is easy to verify from the arguments.

We first see how J1 and J2 imply the claim and then define the reallocation procedure to satisfy J1 and J2. Observe that after processing the entire input, the total weight in component i is at least $6 + 10(a_i - 1)$. Adding this over all components $i \in [c]$, we see that the total weight in the input graph is at least $6c + 10(a - c)$, since components are vertex disjoint. Without weight reallocation, the total weight would be $3n$ since each vertex starts out with exactly 3 units of weight. Since the weight reallocation procedure does not create extra weight, we have $3n \geq 6c + 10(a - c)$, which implies the statement of the lemma.

Although we are allowed to reallocate weight from unprocessed vertices from closed components, we still define the procedure for each component separately. We use the notation of Lemma 2. More specifically, consider component i. Let a_i denote the total number of a-vertices in the component at the end. For $j \in [a_i]$, let t_j denote the time step when the jth a-vertex v_j in component i was processed. Thus, the component gets started at time t_1 with a-vertex v_1. Denote the three neighbors of v_j by $u_{j,1}, u_{j,2}, u_{j,3}$.

J1. Since v_1 is the first vertex of the component, its neighbors have not been processed and they cannot be neighbors of previous a-vertices. Thus, we have $w(v_1) = w(u_{1,1}) = w(u_{1,2}) = w(u_{1,3}) = 3$. No matter whether v_1 is an aa-vertex or an ar-vertex, after it is processed and removed from the graph, the degrees of the neighbors drop by 1 each. Thus, we can reallocate one unit of weight from $u_{1,k}$ for $k \in [3]$ to v_1, resulting in $w(v_1) = 6$, as desired. This is illustrated in Fig. 11.

J2. Let $j \geq 2$. We consider several cases depending on the type of v_j and its (multi-hop) neighborhood.

Case 1. Suppose that v_j receives advice to be accepted. Since v_j is not the first vertex in the component, it shares a neighbor with a previous aa-vertex $v_{j'}$ in the component for some $j' < j$. Let that neighbor be $u_{j,1}$. As in the proof of Lemma 2, the current degree of $u_{j,1}$ is 2 prior to processing v_j, so its weight is also 2. After processing v_j, we reallocate 1 unit of weight from each $u_{j,1}, u_{j,2}$, and $u_{j,3}$ to v_j and the weight allocated to v_j becomes 6. The current degree of $u_{j,1}$ drops to 1. Let z denote the unique neighbor of $u_{j,1}$ at that moment. Then, by the priority tie breaking in P3, $u_{j,1}$ is rejected and z is accepted. We reallocate one additional unit of weight from $u_{j,1}$ to v_j. Since z was present in the graph prior to v_j being processed, the current degree of z at time t_j must be at least 2. After z is processed, we reallocate its weight to v_j. At this point, the weight allocated to v_j becomes at least 9. Let y be any neighbor of z other than $u_{j,1}$ prior to z being removed. Since processing z decreases the degree of y and we do not care which component y belongs to, we reallocate one unit of weight from y to v_j resulting in total weight allocated to v_j being 10. Observe that the triangle-free condition ensures that z is not $u_{j,2}, u_{j,3}$ and it does not matter for the argument whether y is $u_{j,2}$ or $u_{j,3}$ or any other vertex in the graph. This case is illustrated in Fig. 12.

Case 2. Suppose that v_j receives advice to be rejected. All neighbors of v_j will be accepted after that and we can reallocate the weight from those neighbors to v_j. The current degree of each neighbor of v_j is at least 2 prior to v_j being processed (see arguments in Lemma 2 for why). Thus, if one of the neighbors has current weight 3, then the total weight reallocated to v_j from its neighbors is at least 7. This, together with v_j's initial weight of 3, results in $w(v_j) \geq 10$, as desired.

It only remains to handle the case when neighbors of v_j have degree and weight 2 at the time v_j is processed. Let z be the unique neighbor of $u_{j,1}$. The current degree of z is at least 2 prior to v_j being processed, so its weight is at least 2, as well. Processing
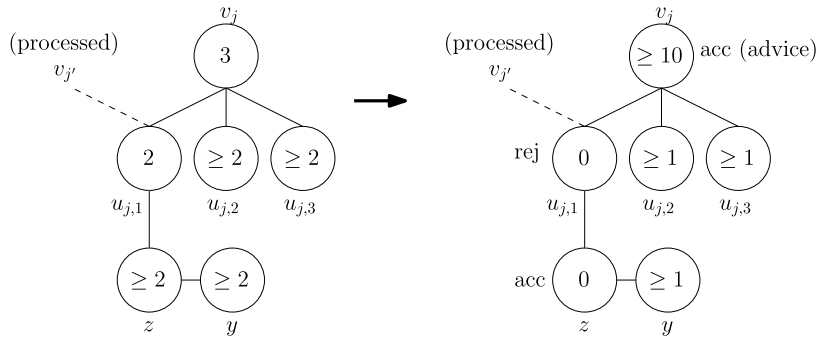
Fig. 12. Illustration of reallocating one unit of weight from y to v_j in **J2**, Case 1.
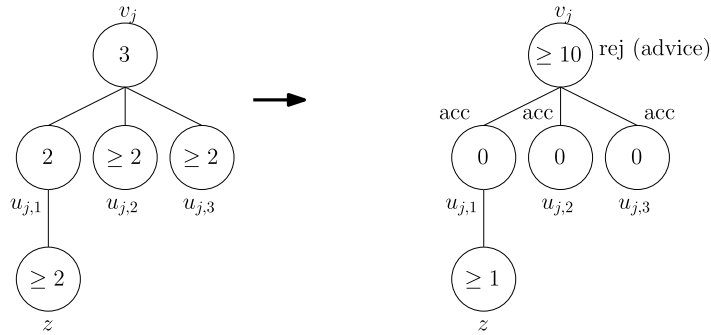

Fig. 13. Illustration of reallocating one unit of weight from z to v_j in **J2**, Case 2.

v_j and its neighbors decreases the degree of z by at least 1 and therefore we may to reallocate one unit of weight from z to v_j. This last case is illustrated in Fig. 13. $\square$

We are now ready to prove the bound on the number of advice bits used by PRIORITYVC.

Lemma 4. PRIORITYVC *uses at most* $(7/22)n = 0.31\overline{81}n$ *bits of advice on triangle-free graphs of maximum degree* 3.

Proof. Lemma 2 says that $1 + 3a_i \leq s_i$ for $i \in [c]$. Since the components are vertex-disjoint, the total number of vertices that received advice is $a = \sum_{i=1}^{c} a_i$ and the total number of vertices is $n = \sum_{i=1}^{c} s_i$. Adding these inequalities over all $i \in [c]$, we obtain

$$3a + c \leq n \tag{2}$$

Lemma 3 says that

$$10a - 4c \leq 3n \tag{3}$$

Adding 4 times Eq. (2) to Eq. (3) results in $22a \leq 7n$, i.e., $a \leq (7/22)n$, as desired. $\square$

Corollary 1. *The priority exact algorithm corresponding to* PRIORITYVC *runs in time*

$$O^*\left(2^{\frac{7n}{22}}\right) \subset O^*(1.247^n).$$

6. Hardness results using templates

In this section, we present templates for proving lower bounds on how much advice is needed for an adaptive priority algorithm to achieve a certain approximation ratio or optimality. The results hold in the oblivious priority function model (and the optimality results also hold in the decision-based priority function model).

The rest of this section is organized as follows: In Subsection 6.1, we introduce the notion of gadget pattern pairs and describe conditions on problems and gadget pattern pairs that are sufficient for proving lower bounds using the templates in the next two subsections. In Subsection 6.2, we present templates for proving trade-offs between the number of advice bits and *approximation* ratios. We finish the section with a table listing the lower bound results that can be obtained for Minimum Vertex Cover with the gadget pattern pairs from Subsection 4 and with known gadget pattern pairs for five other problems. In Subsection 6.3, we present

the template for proving lower bounds on the number of advice bits needed to solve problems to *optimality*. The implications of these results for priority exact algorithms are also discussed.

6.1. Gadget pattern pairs for the templates

In this section, we generalize the construction introduced in Section 4. These types of constructions will be used in our lower bound proofs, some based on reductions and some adversarial. Thus, in some proofs, vertices are given to the priority algorithm with advice by an adversary and, in other proofs, by a reduction (algorithm). In this section, we just use the term "adversary" to represent both of these options.

In Section 4, we presented a lower bound on solving the Minimum Vertex Cover problem to optimality using priority algorithms with advice in the decision-based priority function model. Two graphs, Graph 1 and Graph 2 were used. When a vertex of degree 2 was selected, the adversary chose between two isomorphic copies of Graph 1 to include; these two isomorphic copies constitute an example of the general concept, a *gadget pattern pair*. Similarly, for a vertex of degree 3, the isomorphic copy of Graph 1, along with the isomorphic copy of Graph 2, was another example of a gadget pattern pair. These two gadget pattern pairs constitute our *collection of gadget pattern pairs* for the Minimum Vertex Cover problem.

A gadget G for problem B is simply some constant-sized instance for B, i.e., a collection of input items that satisfy the consistency conditions for problem B. For example, if B is a graph problem in the vertex arrival, vertex adjacency model, G could be a constant-sized graph. In this case, an input item would possibly be a vertex name and a list of neighboring vertex names.

We will define a universe of input items from a union of subuniverses. For this graph problem, in a subuniverse for a collection of gadget pattern pairs, each vertex name exists as many times as the vertex of an input item in the universe, because it can be paired with many different possible lists of neighboring vertex names for the purpose of creating all possible isomorphic instances of the gadget. The effect of this is that when an algorithm receives the first input item of some degree d, it can be any of the degree-d vertices in any of the gadget patterns in the collection. Consistency conditions must apply to the actual given input. For instance, for each vertex name u which is listed as a neighbor of v, it must be the case that v is listed as a neighbor of u. There could of course be further constraints on the input instances; for instance, restricting inputs to graphs of some maximum degree.

In our proofs, the adversary provides multiple gadgets (possibly many isomorphic ones), each coming from some gadget pattern pair in the collection. We need that the sets of possible input items for these multiple gadgets are disjoint, but contain all necessary input items for all gadget patterns in the collection of gadget pattern pairs. To obtain this, we repeat the construction above, creating distinct subuniverses for each gadget the adversary presents. Thus, if, during the execution of an algorithm, the adversary presents m gadgets to the algorithm, the universe consists of m disjoint subuniverses, $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_m$; all of these subuniverses are identical up to renaming of vertices. This implies that an input item identifies which subuniverse it is in. We refer to this property as the *disjoint copies condition*.

We also make an assumption on the objective function related to the gadgets: We say that the objective function for a problem B is *additive with respect to the gadgets* if, for any instance formed from a set of m gadgets from disjoint universes, the objective function value on the instance is the sum of the objective function values on the individual gadgets. This implies that optimality on the instance requires optimality on each gadget. For example, this assumption will hold for many classical graph problems since the gadgets will be maximal connected components and the corresponding objectives are additive with respect to connected components.

Recall that $\max_P R$ denotes the first item in a set R according to the current priority function P, i.e., the highest priority item (possibly after tie-breaking by an adversary). Assume that ALG responds "accept" or "reject" to any possible input item. This captures problems such as Minimum Vertex Cover, Independent Set, Clique, etc.

Each collection of gadget pattern pairs also satisfies the first item condition, and the distinguishing decision condition. The *first item condition* says that the first input item chosen by ALG from the subuniverse $\mathcal{U}_j$, $first(\mathcal{U}_j)$, identifies a gadget pattern pair, (G_j^a, G_j^r), from the collection of gadget pattern pairs, and that the input item itself gives no information about which of the two gadgets G_j^a or G_j^r it is in. For the Vertex Cover example from Section 4, the first item could be a vertex of degree 2 or degree 3, and the two cases lead to different gadget pattern pairs, but the actual input item gives no information as to which of the gadget patterns within the pair it belongs to. Given a priority function P, the first item condition can be written as: $first(\mathcal{U}_j) = \max_P G_j^a = \max_P G_j^r$. The *distinguishing decision condition* says that the decision with regards to item $first(\mathcal{U}_j)$ that results in the optimal value of the objective function in G_j^a is different from the decision that results in the optimal value of the objective function in G_j^r. This first input item is said to be the *distinguishing item*. For accept/reject, we list G_j^a, where the correct decision is to accept, as the first gadget pattern of the pair, and G_j^r as the second.

6.2. Lower bounds on the advice needed for approximation

In this section, we establish two theorems that give general templates for gadget-based reductions from a problem referred to as 2-SGKH, one for maximization problems and one for minimization problems. While it takes some work to establish these results, the theorems are easy to apply to concrete problems once established. One simply has to define a collection of gadget pattern pairs with the required properties and then plug numbers into our formulas. We do this for a number of approximation problems at the end of this section.

The following online problem, while seeming artificial, has been used extensively in proving lower bounds for online algorithms with advice, and we can also use it for adaptive priority algorithms with advice.

Definition 1. The *Binary String Guessing Problem [8] with known history* (2-SGKH) is the following online problem. The input consists of $(n, \sigma = (x_1, \ldots, x_n))$, where $x_i \in \{0, 1\}$. Upon seeing $x_1, \ldots, x_{i-1}$, an algorithm guesses the value of x_i. The actual value of x_i is revealed after the guess. The goal is to maximize the number of correct guesses. $\square$

Böckenhauer et al. [8] provide a trade-off between the number of advice bits and the approximation ratio for the binary string guessing problem. This can be used to show that a linear number of bits of advice is necessary for many online problems.

Theorem 4 (Böckenhauer et al. [8]). *For the 2-SGKH problem and any $\varepsilon \in (0, \frac{1}{2}]$, no online algorithm using fewer than $(1 - H(\varepsilon))n$ advice bits can make fewer than εn mistakes for large enough n, where $H(p) = H(1 - p) = -p \log(p) - (1 - p) \log(1 - p)$ is the binary entropy function.*

To obtain an optimal online algorithm with advice for 2-SGKH, n bits of advice are necessary and sufficient [8].

Results and proofs presented here are somewhat similar to those presented in [13] for fixed priority algorithms with advice. However, there are two major differences. The harder and more interesting one is that we handle adaptive priorities, where the priority functions may depend partially on the advice. In addition, we reduce from string guessing directly instead of going via an intermediate priority algorithm problem. The purpose of this is to avoid losing constant factors with regards to the inapproximability results through intermediate reductions, but this change also made it easier to handle adaptive priorities.

The lower bounds in this section hold in the oblivious priority function model. Recall that in Section 4, we showed a lower bound result for solving Vertex Cover to optimality in the decision-based priority function model. It is an open problem to determine if the approximation lower bounds we prove here also hold in the decision-based priority function model. The problem in proving this when dealing with approximation algorithms is that, theoretically, a priority algorithm with advice could use that advice to encode information in the decisions it makes and then use those decisions in later priority functions. This would allow the priority functions to depend on the advice. For algorithms solving a problem to optimality, this encoding cannot be done since the gadgets in the proof ensure that each decision made by an optimal algorithm is forced.

The template is restricted to binary decision problems since the goal is to derive inapproximability results based on the 2-SGKH problem, where guesses (answers) are either 0 or 1. In our reduction from 2-SGKH to a problem B, we assume that we have a priority algorithm ALG with advice in the oblivious priority function model for problem B. Thus, the priority functions may vary between inputs to ALG, but must assign the same priority to unseen items that are indistinguishable (except for names of items) given the input seen before this point. The current priority function will generally be referred to as P. For the reduction, the inputs to 2-SGKH are $X = \langle x_1, \ldots, x_n \rangle$.

Reduction algorithm Based on ALG, its advice, and its priority functions, we define an online algorithm ALG$'$ with advice (the reduction algorithm) for 2-SGKH. The reduction is *advice-preserving*, since ALG$'$ only uses the advice that ALG does, no more. The input items, $n, x_1, x_2, \ldots, x_n$ with $x_i \in \{0, 1\}$, to 2-SGKH arrive in an online manner, so after n arrives, ALG$'$ must guess x_1, and then the actual value of x_1 is revealed. In the general case, immediately after the value x_i is revealed, ALG$'$ must guess x_{i+1} and then the actual value x_{i+1} is revealed. When x_n is revealed, ALG$'$ knows that this is the end of the input. At the end, there is some post-processing to allow ALG$'$ to complete its computation. ALG$'$ is outlined in Algorithm 4, but we now describe how ALG$'$ provides input to ALG in a consistent manner.

Algorithm 4 The reduction algorithm.

Given: ALG for problem B; the inputs to 2-SGKH are $X = \langle x_1, \ldots, x_n \rangle$

1: $R = \mathcal{U}$ ▷ Use the input to B to give answers for X
2: $i = 0$ ▷ Current index of 2-SGKH input
3: **while** $i < n$ **do**
4:     Let P be the current priority function for ALG
5:     $v = \max_P R$ ▷ Choose v as described under *Consistent choice of input items*
6:     **if** v is the first vertex from universe $\mathcal{U}_{i+1}$ **then**
7:        $i = i + 1$
8:        present v to ALG
9:        answer 0 if ALG answers "accept" and 1 if ALG answers "reject"
10:       receive actual x_i
11:       update R to only contain vertices from one of the two gadgets
12:       make H_i gadget G_i^a if $x_i = 0$ and G_i^R if $x_i = 1$
13:       $R = R \setminus (\mathcal{U}_i \setminus H_i)$
14:     **else**
15:       present v to ALG
16:     $R = R \setminus \{v\}$
17: **while** $R \neq \emptyset$ **do**   ▷ Post-processing to finish inputs for problem B
18:     Let P be the current priority function for ALG
19:     $v = \max_P R$
20:     present v to ALG
21:     $R = R \setminus \{v\}$

Consistent choice of input items

- ALG′ defines the universe $\mathcal{U}$ to be the union of n disjoint gadget pair universes, $\{\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n\}$. It eventually defines an input to problem B, $H_1, H_2, \ldots, H_k$, where H_i is a gadget G_i^a from $\mathcal{U}_i$ if $x_i = 0$; otherwise it is a gadget G_i^r from $\mathcal{U}_i$.

  These H_i can be defined initially, if the input items are isomorphic, in which case a set R is initialized to contain the input items from these gadgets. Otherwise, as in the case of the Vertex Cover gadget patterns from Section 4, the algorithm's priority functions can give subsets of input items with identical priorities due to its oblivious nature. Knowing the inputs to 2-SGKH and using the fact that the first item condition holds, ALG′ can always determine which gadget to actually use for H_i when the first input item from $\mathcal{U}_i$ is selected. The set R initially contains all of the universe $\mathcal{U}$, and ALG′ removes input items from R that are in $\mathcal{U}_i$, but are not in H_i, when H_i has been determined. Other input items are removed as they are processed.

- ALG′ decides which input item to give when the algorithm's priority function designates a set S of size greater than one as those input items having highest priority. If at least one input item from every universe has been processed, the reduction algorithm can make an arbitrary choice, lexicographically, for example. The same holds if the input items contain names of one or more input items that have already appeared in earlier input items (for a graph in the vertex arrival, vertex adjacency model, this means that the input item is a neighbor or a neighbor of a neighbor of some vertex already processed). Otherwise, ALG′ has arranged that ALG has seen or will see input items from the first $i-1$ universes and now presents the first from $\mathcal{U}_i$. From the set of input items with current highest priority, ALG′ chooses which gadget pattern is correct for H_i: G_i^a if $x_i = 0$ or G_i^r if $x_i = 1$, satisfying that the distinguishing item, v, for the gadget is among those in S. ALG′ presents v to the algorithm and chooses the actual gadget H_i consistent with that.

The main challenge is to ensure that the input items to ALG are presented in the order determined by the priority functions, which may change over time. The fact that the priority function does not distinguish between input items that have no known connection to input items already seen allows ALG′ to choose a distinguishing item in a new gadget from a new universe when that is necessary. In this case, by the disjointness of the universes for the gadgets and since we work in the oblivious priority function model, such a distinguishing item will always be in the set of items of highest priority. Thus, the first items in the successive gadgets are chosen in order. The first item chosen from a gadget is one where the distinguishing decision condition holds, i.e., one where one decision is optimal for that gadget and the other leads to a non-optimal solution.

We let ALG(I) denote the value of the objective function for ALG on input I. The size of a gadget pattern G, denoted by $|G|$, is the number of input items specifying a gadget consistent with that gadget pattern. We write OPT(G) to denote the best value of the objective function on G. Recall that we focus on problems where a solution is specified by making an accept/reject decision for each input item. We slightly abuse notation and let *first*(G) denote the input item from gadget G that was presented to ALG first, due to ALG′'s choice among the set of input items with highest priority. We write BAD(G) to denote the best value of the objective function attainable on G after making the wrong decision for that first item, *first*(G), i.e., if there is an optimal solution that accepts (rejects) *first*(G), then BAD(G) denotes the best value of the objective function *given* that *first*(G) was rejected (accepted).

Definition 2. A collection of $k \geq 1$ gadget pattern pairs

$$\left\{ (G_j^a, G_j^r) \mid 1 \leq j \leq k \right\}$$

for an optimization problem B is called (s, ρ)-*reducible* if the following conditions are fulfilled:

- The objective function for B is additive with respect to the gadgets.
- Let $\mathcal{U}_1, \ldots, \mathcal{U}_n$ each be subuniverses for the collection of gadget pattern pairs, and the universe $\mathcal{U}$ be the union of the subuniverses. The following conditions must be satisfied for the gadget pattern pairs with respect to the subuniverses:
  - the consistency condition for B,
  - the first item condition,
  - the distinguishing decision condition,
  - and the disjoint copies condition.
- s is the maximum number of input items in any gadget pattern in the collection.
- The values

  $$\text{OPT}(G_j^a), \text{BAD}(G_j^a), \text{OPT}(G_j^r), \text{BAD}(G_j^r)$$

  must be independent of j, and we denote them by

  $$\text{OPT}(G^a), \text{BAD}(G^a), \text{OPT}(G^r), \text{BAD}(G^r);$$

  - We must have that $\text{OPT}(G^r) \geq \text{OPT}(G^a)$.
  - If B is a minimization problem, $\rho = \min\left\{ \frac{\text{BAD}(G^a)}{\text{OPT}(G^a)}, \frac{\text{BAD}(G^r)}{\text{OPT}(G^r)} \right\}$, and if B is a maximization problem, $\rho = \min\left\{ \frac{\text{OPT}(G^a)}{\text{BAD}(G^a)}, \frac{\text{OPT}(G^r)}{\text{BAD}(G^r)} \right\}$. □

Theorem 5. *Assume that the collection of gadget pattern pairs*

$$\left\{ (G_j^a, G_j^r) \mid 1 \le j \le k \right\}$$

for a minimization problem B is (s, ρ)-reducible. Then for any $\varepsilon \in (0, \frac{1}{2}]$, no adaptive priority algorithm in the oblivious priority function model using fewer than $(1 - H(\varepsilon))n/s$ advice bits can achieve an approximation ratio smaller than

$$1 + \frac{\varepsilon(\rho - 1)\,\mathrm{OPT}(G^a)}{\varepsilon\,\mathrm{OPT}(G^a) + (1 - \varepsilon)\,\mathrm{OPT}(G^r)}.$$

Proof. Consider an adaptive priority algorithm ALG for B in the oblivious priority function model. A reduction from 2-SGKH is specified in Algorithm 4, combined with the definition of ALG'. The set R contains the remaining items which could still be in the input to B and have not yet been presented to ALG. At any point in time, one of the input items with the highest priority among those still available in R is presented to ALG. This item is the first input item from a gadget when (1) there are still gadgets in R, where none of their input items have been seen, and (2) the set of input items with highest priority is the set of input items containing no reference to any input item referenced in any input item already seen. If this item is the first input item from a gadget, H_i, from $first(\mathcal{U}_i)$, it is an input item where the distinguishing decision condition holds. In this case, the next input to 2-SGKH to be processed is x_i, and ALG' guesses 0 for x_i if ALG accepts $first(\mathcal{U}_i)$ and 1 if ALG rejects. Note that ALG' has created $H = \langle H_1, H_2, \dots, H_n \rangle$ such that the answer ALG gives is correct for problem B if and only if the answer ALG' gives is correct for 2-SGKH. The correct answer by ALG is well defined by the distinguishing decision condition.

The amount of advice is the same for both algorithms, so when it is $(1 - H(\varepsilon))n'$ bits for the n' inputs to 2-SGKH, it is at least $(1 - H(\varepsilon))n/s$ bits for the $n \le sn'$ inputs to B.

Now we turn to the approximation ratio obtained. We want to lower-bound the number of incorrect decisions by ALG. We focus on the input items which are $first(\mathcal{U}_i)$ and assume that x_i is the next input to 2-SGKH when $first(\mathcal{U}_i)$ is processed. Assume that ALG answers correctly on all inputs that are not $first(\mathcal{U}_i)$ for any i.

We know from Theorem 4 that for any $\varepsilon \in (0, \frac{1}{2}]$, any online algorithm using fewer than $(1 - H(\varepsilon))n$ advice bits makes at least εn mistakes on 2-SGKH. Since we want to lower-bound the approximation ratio of ALG, and since a ratio larger than one decreases when increasing the numerator and denominator by equal quantities, we can assume that when ALG answers correctly, it is on the gadget pattern pair with the larger OPT-value, G^r. For the same reason, we can assume that the "at least εn" incorrect answers are in fact exactly εn, since classifying some of the incorrect answers as correct just lowers the ratio. For the incorrect answers, assume that the gadget pattern G^a is presented w times, and, thus, the gadget pattern, G^r, $\varepsilon n - w$ times.

Denoting the input created by ALG' for ALG by I, we obtain the following, where we use that $\mathrm{BAD}(G_j^x) \ge \rho\,\mathrm{OPT}(G_j^x)$ for $x \in \{a, r\}$. Since the objective function for problem B is additive,

$$\frac{\mathrm{ALG}(I)}{\mathrm{OPT}(I)} \ge \frac{(1 - \varepsilon)n\,\mathrm{OPT}(G^r) + w\,\mathrm{BAD}(G^a) + (\varepsilon n - w)\,\mathrm{BAD}(G^r)}{(1 - \varepsilon)n\,\mathrm{OPT}(G^r) + w\,\mathrm{OPT}(G^a) + (\varepsilon n - w)\,\mathrm{OPT}(G^r)}$$

$$\ge \frac{(1 - \varepsilon)n\,\mathrm{OPT}(G^r) + w\rho\,\mathrm{OPT}(G^a) + (\varepsilon n - w)\rho\,\mathrm{OPT}(G^r)}{(1 - \varepsilon)n\,\mathrm{OPT}(G^r) + w\,\mathrm{OPT}(G^a) + (\varepsilon n - w)\,\mathrm{OPT}(G^r)}$$

$$= 1 + \frac{w(\rho - 1)\,\mathrm{OPT}(G^a) + (\varepsilon n - w)(\rho - 1)\,\mathrm{OPT}(G^r)}{w\,\mathrm{OPT}(G^a) + (n - w)\,\mathrm{OPT}(G^r)}$$

Taking the derivative with respect to w and setting equal to zero gives no solutions for w, so the extreme values must be found at the endpoints of the range for w which is $[0, \varepsilon n]$.

Inserting $w = 0$, we get $1 + \varepsilon(\rho - 1)$, while $w = \varepsilon n$ gives

$$1 + \frac{\varepsilon(\rho - 1)\,\mathrm{OPT}(G^a)}{\varepsilon\,\mathrm{OPT}(G^a) + (1 - \varepsilon)\,\mathrm{OPT}(G^r)}.$$

The latter is the smaller ratio and thus the lower bound we can provide. $\square$

The following theorem for maximization problems is proved analogously.

Theorem 6. *Assume that the collection of gadget pattern pairs*

$$\left\{ (G_j^a, G_j^r) \mid 1 \le j \le k \right\}$$

for a maximization problem B is (s, ρ)-reducible. Then for any $\varepsilon \in (0, \frac{1}{2}]$, no adaptive priority algorithm in the oblivious priority function model using fewer than $(1 - H(\varepsilon))n/s$ advice bits can achieve an approximation ratio smaller than

$$1 + \frac{\varepsilon(\rho - 1)\,\mathrm{OPT}(G^a)}{\varepsilon\,\mathrm{OPT}(G^a) + (1 - \varepsilon)\rho\,\mathrm{OPT}(G^r)}.$$

Proof. The proof proceeds as for the minimization case in Theorem 5 until the calculation of the lower bound of $\frac{\text{ALG}(I)}{\text{OPT}(I)}$. We continue from that point, using the inverse ratio to get values larger than one.

We use that for $x \in \{a, r\}$, $\text{BAD}(G^x) \leq \text{OPT}(G^x)/\rho$.

$$\frac{\text{OPT}(I)}{\text{ALG}(I)} \geq \frac{(1-\varepsilon)n\,\text{OPT}(G^r) + w\,\text{OPT}(G^a) + (\varepsilon n - w)\,\text{OPT}(G^r)}{(1-\varepsilon)n\,\text{OPT}(G^r) + w\,\text{BAD}(G^a) + (\varepsilon n - w)\,\text{BAD}(G^r)}$$

$$\geq \frac{(1-\varepsilon)n\,\text{OPT}(G^r) + w\,\text{OPT}(G^a) + (\varepsilon n - w)\,\text{OPT}(G^r)}{(1-\varepsilon)n\,\text{OPT}(G^r) + \frac{w}{\rho}\,\text{OPT}(G^a) + \frac{\varepsilon n - w}{\rho}\,\text{OPT}(G^r)}$$

Again, taking the derivative with respect to w gives an always non-positive result. Thus, the smallest value in the range $[0, \varepsilon n]$ for w is found at $w = \varepsilon n$. Inserting this value, we continue the calculations from above:

$$\frac{\text{OPT}(I)}{\text{ALG}(I)} \geq \frac{(1-\varepsilon)n\,\text{OPT}(G^r) + w\,\text{OPT}(G^a) + (\varepsilon n - w)\,\text{OPT}(G^r)}{(1-\varepsilon)n\,\text{OPT}(G^r) + \frac{w}{\rho}\,\text{OPT}(G^a) + \frac{\varepsilon n - w}{\rho}\,\text{OPT}(G^r)}$$

$$= \frac{(1-\varepsilon)n\,\text{OPT}(G^r) + (\varepsilon n)\,\text{OPT}(G^a)}{(1-\varepsilon)n\,\text{OPT}(G^r) + \frac{\varepsilon n}{\rho}\,\text{OPT}(G^a)}$$

$$= \frac{(1-\varepsilon)\rho\,\text{OPT}(G^r) + \varepsilon\rho\,\text{OPT}(G^a)}{(1-\varepsilon)\rho\,\text{OPT}(G^r) + \varepsilon\,\text{OPT}(G^a)}$$

$$= 1 + \frac{\varepsilon(\rho - 1)\,\text{OPT}(G^a)}{(1-\varepsilon)\rho\,\text{OPT}(G^r) + \varepsilon\,\text{OPT}(G^a)}$$

The latter is the smaller ratio and thus the lower bound we can provide. $\quad\square$

We mostly use Theorems 5 and 6 in the following specialized form.

Corollary 2. *Assume we are considering a (s, ρ)-reducible optimization problem.*

For a minimization problem, if $\text{OPT}(G^a) = \text{OPT}(G^r) = \text{BAD}(G^a) - 1 = \text{BAD}(G^r) - 1$, then no adaptive priority algorithm in the oblivious priority function model using fewer than $(1 - H(\varepsilon))n/s$ advice bits can achieve an approximation ratio smaller than $1 + \frac{\varepsilon}{\text{OPT}(G^a)}$.

For a maximization problem, if $\text{OPT}(G^a) = \text{OPT}(G^r) = \text{BAD}(G^a) + 1 = \text{BAD}(G^r) + 1$, then no adaptive priority algorithm in the oblivious priority function model using fewer than $(1 - H(\varepsilon))n/s$ advice bits can achieve an approximation ratio smaller than $1 + \frac{\varepsilon}{\text{OPT}(G^a) - \varepsilon}$.

For the Minimum Vertex Cover problem, for example, we can apply the minimization version of Corollary 2. The size of the gadget patterns is $s = 7$ vertices in all cases. Since $\text{OPT}(G^a) = \text{OPT}(G^r) = 3$, and, when the optimal decision is not made on the first vertex processed, the vertex cover size is at least 4, we obtain the following:

Corollary 3. *For Minimum Vertex Cover and any $\varepsilon \in (0, \frac{1}{2}]$, no adaptive priority algorithm in the oblivious priority function model using fewer than $(1 - H(\varepsilon))n/7$ advice bits can achieve an approximation ratio smaller than $1 + \frac{\varepsilon}{3}$.*

The gadget pattern pairs used in [13] (called gadget patterns in that paper) to prove lower bounds in the fixed priority model also work here in the adaptive priority model; there are no additional restrictions used in the proof here. (These gadget patterns are included for completeness.) The reductions done here are directly from 2-SGKH, as opposed to going through the Pair Matching problem, as in [13]. As mentioned earlier, this makes the proofs simpler in most respects (except for having to take into account changing priority functions), and it means that one does not lose a factor 2 in the amount of advice required. Thus, the results from [13] can be expressed using Table 1 as lower bounds for adaptive priority algorithm with advice. All of the ratios obtained approach one as the amount of advice approaches some fraction of n. The gadget pattern pairs used in [13] can also be used for lower bounds on the amount of advice required for optimality. Thus, those gadget pattern pairs satisfy the conditions of both templates in this paper.

To collect results in one table, we include results for optimality though they are treated in the next section.

Note that the gadget patterns for Maximum Independent Set from [13] have a smaller optimal independent set than the gadget patterns for the equivalent Minimum Vertex Cover, shown in Fig. 1. Thus, there is a trade-off between the lower bound for the approximation ratio one can prove and the lower bound on the amount of advice needed to prove it.

We now go through the constructions that establish the remaining results listed in Table 1. They are based on gadget pattern pairs that were presented in [13] but included here for completeness.

6.2.1. Maximum Independent Set and Maximum Cut

The gadgets are drawn to have vertex 1 be the one with highest priority. (See Fig. 14.)

Maximum Independent Set The optimal decision is to accept in G^1 and reject in G^2. The maximum number s of input items for a gadget is 8, $\text{OPT}(G^1) = \text{OPT}(G^2) = 3$, and $\text{BAD}(G^1) = \text{BAD}(G^2) = 2$.

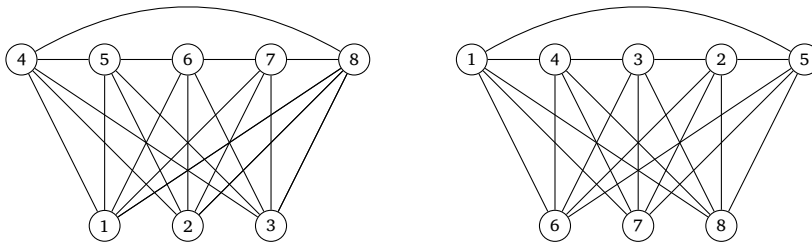| Problem	Advice for Approx.	Ratio	Advice for Opt.
Maximum Independent Set [13]	$(1 - H(\varepsilon))n/8$	$1 + \frac{\varepsilon}{3-\varepsilon}$	$n/8$
Maximum Independent Set [Fig. 1]	$(1 - H(\varepsilon))n/7$	$1 + \frac{\varepsilon}{4-\varepsilon}$	$n/7$
Maximum Bipartite Matching	$(1 - H(\varepsilon))n/3$	$1 + \frac{\varepsilon}{3-\varepsilon}$	$n/3$
Maximum Cut	$(1 - H(\varepsilon))n/8$	$1 + \frac{\varepsilon}{15-\varepsilon}$	$n/8$
Minimum Vertex Cover	$(1 - H(\varepsilon))n/7$	$1 + \frac{\varepsilon}{3}$	$n/7$
Maximum 3-Satisfiability	$(1 - H(\varepsilon))n/3$	$1 + \frac{\varepsilon}{8-\varepsilon}$	$n/3$
Unit Job Sched., Prec. Constraints	$(1 - H(\varepsilon))n/9$	$1 + \frac{\varepsilon}{6-\varepsilon}$	$n/9$

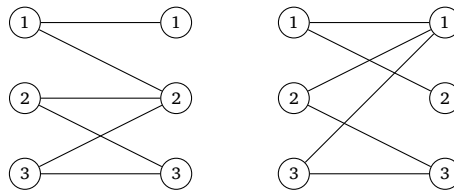**Fig. 14.** Topological structure of the gadgets (G^1, G^2) for independent set.

Fig. 15. Topological structure of the gadgets (G^1, G^2) for bipartite matching.

Maximum Cut   The goal is to partition the vertices into two sets such that the number of edges crossing the two sets is maximized. The partition is specified by the algorithm assigning 0 or 1 to each vertex. In addition, we require that 0 is assigned to vertices belonging to the larger block of the partition. The maximum cut in G^1 (or G^2) puts the upper vertices in the larger set and the lower vertices in the other set. The optimal decision for the first vertex is unique: For G^1, respond 1, and for G^2, respond 0. The maximum number s of input items for a gadget is 8, $\text{OPT}(G^1) = \text{OPT}(G^2) = 15$, and $\text{BAD}(G^1) = \text{BAD}(G^2) = 14$.

6.2.2. Maximum Bipartite Matching

The vertices on the right-hand side are known in advance, and the vertices on the left arrive online. The gadgets are drawn to have vertex 1 be the one with highest priority, and all possible first vertices look identical. The optimal decision is to accept in G^1 and reject in G^2. (See Fig. 15.)

The (maximum) number s of input items (the number of vertices given) for any of the two gadgets is 3, $\text{OPT}(G^1) = \text{OPT}(G^2) = 3$, and $\text{BAD}(G^1) = \text{BAD}(G^2) = 2$.

6.2.3. Maximum Satisfiability (MAX-3-SAT)

An input item (x, S^+, S^-) consists of a variable name x, a set S^+ of *clause information tuples* for those clauses in which x appears positively, and a set S^- of clause information tuples for those clauses where the variable x appears negated. The clause information tuples for a particular clause contain the name of the clause, the total number of literals in that clause, and the names of the other variables in the clause, but no information regarding whether those other variables are negated or not. The goal is to satisfy the maximum number of clauses.

$$G^1 = C_1 \wedge C_2 \wedge C_3 \wedge C_4 \wedge C_5 \wedge C_6 \wedge C_7 \wedge C_8,$$
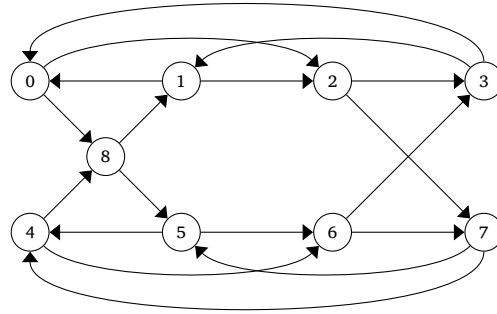
where

Fig. 16. Topological structure of a gadget for job scheduling of unit time jobs with precedence constraints.

$$
\begin{aligned}
C_1 &= (x_1 \vee x_2 \vee x_3) & C_2 &= (x_1 \vee \neg x_2 \vee \neg x_3) \\
C_3 &= (x_1 \vee \neg x_2 \vee x_3) & C_4 &= (x_1 \vee x_2 \vee \neg x_3) \\
C_5 &= (\neg x_1 \vee x_2 \vee x_3) & C_6 &= (\neg x_1 \vee x_2 \vee x_3) \\
C_7 &= (\neg x_1 \vee \neg x_2 \vee \neg x_3) & C_8 &= (\neg x_1 \vee \neg x_2 \vee \neg x_3) \\
G^2 &= C_1 \wedge C_2 \wedge C_3 \wedge C_4 \wedge C_5 \wedge C_6 \wedge C_7 \wedge C_8,
\end{aligned}
$$

where

$$
\begin{aligned}
C_1 &= (\neg x_1 \vee x_2 \vee x_3) & C_2 &= (\neg x_1 \vee \neg x_2 \vee \neg x_3) \\
C_3 &= (\neg x_1 \vee \neg x_2 \vee x_3) & C_4 &= (\neg x_1 \vee x_2 \vee \neg x_3) \\
C_5 &= (x_1 \vee x_2 \vee x_3) & C_6 &= (x_1 \vee x_2 \vee x_3) \\
C_7 &= (x_1 \vee \neg x_2 \vee \neg x_3) & C_8 &= (x_1 \vee \neg x_2 \vee \neg x_3)
\end{aligned}
$$

Suppose without loss of generality that the highest priority input is

$$(x_1, \{(C_1, 3, \{x_2, x_3\}), (C_2, 3, \{x_2, x_3\}), (C_3, 3, \{x_2, x_3\}), (C_4, 3, \{x_2, x_3\})\},$$

$$\{(C_5, 3, \{x_2, x_3\}), (C_6, 3, \{x_2, x_3\}), (C_7, 3, \{x_2, x_3\}), (C_8, 3, \{x_2, x_3\})\}).$$

Note that the optimal decision for x_1 is unique for each of these gadgets and is "True" for G^1 and "False" for G^2. The maximum number s of input items for a gadget is 3, $\text{OPT}(G^1) = \text{OPT}(G^2) = 8$, and $\text{BAD}(G^1) = \text{BAD}(G^2) = 7$.

6.2.4. Unit Job Scheduling with Precedence Constraints

In this problem, we have a single machine and the requests are unit time jobs with precedence constraints, indicating which jobs must be scheduled before which others. There could be a cyclic set of constraints. The goal is to schedule a maximum number of jobs that are compatible. The input items are of the form (J, S^+, S^-), where J is the name of a job, S^+ is the set of jobs such that if they were scheduled together with J they would have to be scheduled before J, and S^- is the set of jobs such that if they were scheduled together with J they would have to be scheduled after J.

The gadget is a directed graph, specifying the precedence constraints. (See Fig. 16.)

This gadget consists only of isomorphic items (each vertex has in-degree 2, out-degree 2, and 4 different neighbors in all). Thus, this gadget can represent both G^1 and G^2 with renaming. Every optimal solution contains Job 0 and excludes Job 8, so G^1 has the job labeled 0 in this gadget as the highest priority item and G^2 has the job labeled 8 in this gadget as the highest priority item. The maximum number s of input items for a gadget is 9, $\text{OPT}(G^1) = \text{OPT}(G^2) = 6$ (for instance, schedule jobs $1, 0, 2, 5, 4, 6$), and $\text{BAD}(G^1) = \text{BAD}(G^2) = 5$.

6.3. Lower bounds on the advice needed for optimality

In this section, we consider adaptive priority algorithms that solve problems to optimality.

Theorem 7. *Assume that the collection of gadget pattern pairs*

$$\left\{ (G_j^a, G_j^r) \mid 1 \leq j \leq k \right\}$$

for a problem B is (s, ρ)-reducible. Then, any optimal adaptive priority algorithm, ALG*, with advice in the oblivious priority function model must use at least $\lfloor n/s \rfloor$ advice bits on worst case instances with n input items.*

Proof. We use the proof of Theorems 5 and 6. Note that the reduction algorithm in Fig. 4 uses the same amount of advice for the algorithm for 2-SGKH as for the algorithm for problem B and makes exactly the same number of errors in guessing bits for 2-SGKH as it makes on first input items of gadgets for problem B. Thus, if it solves B to optimality, it also solves 2-SGKH to optimality. Since

n' bits of advice are required on n'-bit inputs to 2-SGKH [8], n' bits of advice must be required for n' gadgets as input to problem B. If the maximum gadget size is s, then at least $\lfloor n/s \rfloor$ are necessary to achieve optimality. $\square$

In the following, we consider completable problems. A problem B is *completable* if every consistent set S' of $n' < n$ input items can be completed to a consistent set S of n input items in such a way that if $C' \subseteq S'$ is not an optimal solution for S', there is no subset $C = C' \cup E$ of S with E a subset of additional $n - n'$ items such that C is an optimal solution for S. In other words, a problem is completable if there is a way to give the remaining input items without giving an algorithm the opportunity to fix an earlier non-optimal decision. For Minimum Vertex Cover and many other problems, for example, one can complete the set S' to S by adding $n - n'$ isolated vertices.

The result in Section 4 for Vertex Cover in the decision-based priority function model can be generalized to give the same result as above.

Theorem 8. *Assume that the collection of gadget pattern pairs*

$$\left\{ (G_j^a, G_j^r) \mid 1 \le j \le k \right\}$$

for a completable problem B is (s, ρ)-reducible. Then, any optimal adaptive priority algorithm, ALG*, with advice in the decision-based priority function model must use at least $\lfloor n/s \rfloor$ advice bits on worst case instances with n input items.*

Proof. To define a problem where $k = \lfloor n/s \rfloor$ bits of advice are necessary and sufficient for optimality in the decision-based priority function model, we consider an arbitrary algorithm, ALG$'$, for problem B, and an adversary, ADV$'$. We create k disjoint universes, $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_k$, copies of the universe $\mathcal{U}$, with different names for the input items in each copy, and define the universe, $\mathcal{U}'$, for ALG$'$ to be the union of these k universes. The input for ALG$'$ is the union of $H_1, H_2, \ldots, H_k$, where H_i is an isomorphic copy of either G_i^a or G_i^r.

We now define 2^k distinct sequences of input items for ALG$'$, by describing how one of these 2^k sequences of input items is defined: ALG$'$ selects input items one at a time, and ADV$'$ knows from which of the k universes the input items originate.

Since we are assuming that ALG$'$ solves the problem to optimality, the adversary can assume that the current priority function is determined based on ALG$'$ making the correct accept/reject decisions up to this point. Now, ADV$'$ does the following: Assume that ALG$'$ has already received input items originating from i of the universes from which $\mathcal{U}'$ was defined and the adversary has a current subset $X \subseteq U'$. If that is the case, then X contains exactly enough input items to complete one gadget from each of the universes from which ALG$'$ has received some input item (how this is maintained is explained below). From universes not included in these i universes, X still contains all possible namings of vertices from the gadgets.

Now, ALG$'$ receives its next input item which will be the input item in X of the highest priority in this round, and that input item is the next in the input sequence we are defining. This item is determined by the current priority function which only depends on the input items received so far and its decisions so far.

If that next input item, v, is from one of the i universes, nothing further is done. However, if that next input item originates from a universe, $\mathcal{U}_j$, not among the i, then the following is done.

By the first item condition and the disjoint copies condition, the input item v identifies for which gadget pattern pair, (G_j^a, G_j^r), in the collection v is the distinguishing item, ADV$'$ chooses G_j^a or G_j^r, and then removes from X all input items originating from $\mathcal{U}_j$, except enough to make up exactly the gadget that was chosen (consistent with whichever of G_j^a or G_j^r was chosen), with the naming consistent with v being the distinguishing item from that gadget.

Continuing this inductively defines one sequence of the 2^k distinct sequences of input items. The number of input items in each sequence is at most $sk \le n$. If it is less than n, irrelevant input items can be added, since B is completable.

If a priority algorithm with advice for problem B in the decision-based model uses fewer than k bits of advice for instances with sk input items, the same advice must be given for at least two of the sequences, I_1 and I_2, defined above. ALG$'$ therefore uses the same priorities and makes the same decisions on I_1 and I_2 until some difference is detected. Thus, consider the first time in the processing of I_1 and I_2, where an input item v that has current highest priority is the first input item of a gadget from some $\mathcal{U}_i$, but the gadgets included in I_1 and I_2 from $\mathcal{U}_i$ are different.

Up until (and including) this point, all input items have been the same for the two sets. Thus, ALG$'$ must make the same decision for v in both I_1 and I_2, but, by the distinguishing decision condition, one of those decisions leads to a solution which is not optimal, by the additivity of the objective function. Thus, ALG$'$ is not optimal, and k bits of advice are necessary. $\square$

The templates from the theorems in this section are quite similar and general, applying to binary decision problems where collections of gadget pattern pairs satisfying the required conditions can be created. One can check that all of the gadget pattern pairs presented in [13] are appropriate, thus giving immediate lower bounds for several problems.

Recall that an exact algorithm created in the obvious way (trying all advice strings of the maximal required length) from adaptive priority algorithms with advice is called a *priority exact algorithm*. For any problem satisfying the conditions of the previous theorem, any priority exact algorithm obtained for the problem examines at least $2^{n/s}$ possibilities. This can rule out the possibility of improvements using priority exact algorithms for certain problems that already have known complexities better than this. When the size of the gadget patterns is small, this gives larger lower bounds. For example, for Minimum Vertex Cover the size of the gadget patterns is $s = 7$, since all possible gadgets have seven vertices. Thus, the lower bound for Minimum Vertex Cover (on triangle-free graphs
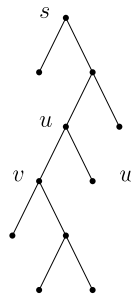
Fig. 17. An example of a 4-thorny path.

with maximum degree 3) is $\Omega(2^{\frac{n}{7}}) \subset \Omega(1.142^n)$, which is larger than the best known exact algorithms for this problem, showing that those algorithms are not priority exact algorithms (derived from a priority algorithm with advice in the decision-based or oblivious priority function models). For Maximum Independent Set, our previous gadget patterns [13] have size $s = 8$, but the gadget patterns for Minimum Vertex Cover also work for Maximum Independent Set (the problems are complements of each other), so the lower bound we obtain for Maximum Independent Set is also $\Omega(2^{\frac{n}{7}})$.

Unfortunately, these lower bound results only apply to priority exact algorithms as defined from priority algorithms with advice in either the decision-based or the oblivious priority function models (obtained from a priority algorithm with advice by running the algorithm on all possible advice strings, all of the same length). As mentioned earlier, there are usually better implementations of these algorithms as branch-and-reduce algorithms, giving the possibility of better analyses of their running times.

In particular, these lower bounds were all proven using constant-sized gadget patterns, each one being a connected component of the entire graph. In practice, though, each connected component (gadget) should be treated independently, each only requiring one bit of advice. Then, if a lower bound of $f(n)$ is proven on the number of advice bits needed for a problem of size n, consisting of s components, instead of running time $\Omega(2^{f(n)})$, only $O^*(2s) = O^*(1)$ time is necessary (trying the advice strings "0" and "1" for each component). Thus, it seems very limited how broadly these lower bounds can be interpreted.

Brahe [15] has a construction for Maximum Independent Set and Minimum Vertex Cover using a connected graph which also gives a linear lower bound on the amount of advice required for optimality in the decision-based priority function model (those specific connected graphs were explicitly designed to have triangles, so they are not triangle-free, but they still have maximum degree 3). Thus, the technique of running the algorithm independently on each connected component fails there, and one obtains an exponential lower bound for exact algorithms based on the adaptive priority algorithms with advice in the decision-based priority function model.

7. The thorny path problem

In this section, we consider another problem using adaptive priority algorithms with advice. Using different techniques, we prove lower bounds for this problem in the unrestricted and decision-based models. We conjecture that the lower bound in the unrestricted model is not tight. We prove matching upper and lower bounds in the decision-based priority function model. These bounds establish a hierarchy, showing that for any positive integer k, there is a problem for which k of bits of advice is necessary and sufficient for optimality.

We call a tree a *thorny path* if it has a root, s, with two children, and at any depth greater than zero and smaller than the maximum depth of the tree, there are exactly two nodes; one with zero and one with two children.

We define the *thorny path problem* as follows. Given a forest G consisting of a number of trees, each of which is a thorny path, as well as a start vertex s of one of the thorny paths of G, the goal is to construct a path from s to one of the two leaves of maximum depth. The universe of input items is $\mathcal{U} = \mathbb{Z}^3$. An input item (u, v, w) is a vertex u with a left child v and a right child w. One can think of u, v, and w as vertex names or object identifiers. The universe of decisions is $D = \{0, 1, \perp\}$. Given an input item (u, v, w), the decision 0 means to include edge (u, v) in the solution, the decision 1 means to include edge (u, w) in the solution, and the decision $\perp$ means to not include any of the two edges in the solution. The thorny path problem is parameterized by a single parameter $k \in \mathbb{N}$, which is one less than the maximum depth in the thorny path containing s. We refer to the parameterized thorny path problem as the k-thorny path problem. An example of a thorny path is shown in Fig. 17.

We begin with a simple observation.

Lemma 5. *In the decision-based priority function model, the k-thorny path problem can be solved by an adaptive priority algorithm with k bits of advice.*

Proof. The first priority function gives highest priority to an input item of the form $(s, \cdot, \cdot)$ and an advice bit is used to select the correct child. Subsequent priority functions give highest priority to items with the most recently selected child as the first entry and an advice bit is used to choose the next child correctly. No advice is necessary at depth k, since including either edge gives a valid solution, a leaf at depth $k + 1$. $\quad\square$

Now we turn to lower bounds, starting with the unrestricted priority function model. We do not give upper bounds. Note, however, that advice giving the name of a leaf in the thorny path can be used to follow parents up to the root, without using additional advice. This advice can be quite large, however, since the universe size is unbounded.

Theorem 9. *In the unrestricted priority function model, the k-thorny path problem cannot be solved by an adaptive priority algorithm with $\log k - 1$ bits of advice.*

Proof. Assume that we have ℓ adaptive priority algorithms without advice, $\text{ALG}_1, \ldots, \text{ALG}_\ell$. We fix m large enough (depending on ℓ, to be specified later) and let $x_1, \ldots, x_m \in \mathbb{Z} \setminus \{1\}$ be distinct. Let $\mathcal{U}$ be the input universe, consisting of all triples with distinct items formed from $\{s, x_1, \ldots, x_m\}$, with the only exception being that s only appears as a first element of any triple. We construct a thorny path instance $\mathcal{I}$ (that is, a subset of the input universe that will be used as input) with one thorny path such that each algorithm $\text{ALG}_1, \ldots, \text{ALG}_\ell$ makes a mistake on $\mathcal{I}$. We construct $\mathcal{I}$ iteratively. In step j, we construct a subinstance $\mathcal{I}_j$ that guarantees that algorithm ALG_j makes a mistake. The thorny path of $\mathcal{I}_j$ starts at vertex s and ends in two leaves. In addition to $\mathcal{I}_j$, we keep track of a leaf v_j that is going to be extended in step $j + 1$. We also keep track of a set of input items $S_j \subseteq S$ that can be used to extend our instance beyond $\mathcal{I}_j$. S_j will not contain any input items where the first entry is currently a non-leaf element of $\mathcal{I}_j$. The condition that ALG_j makes a mistake on $\mathcal{I}_j$ also continues to hold no matter how $\mathcal{I}_j$ is extended with elements from S_j.

For the base case, $\mathcal{I}_0$ is empty, and none of the algorithms have made a mistake yet. We set $v_0 = s$ and $S_0 = \mathcal{U}$.

Assume that we have constructed a thorny path $\mathcal{I}_j$ and the leaf of $\mathcal{I}_j$ to be extended using items from S_j is v_j. Moreover each of $\text{ALG}_1, \ldots, \text{ALG}_j$ makes a mistake on $\mathcal{I}_j$ and continues to make that mistake no matter how $\mathcal{I}_j$ is extended by elements from S_j. Consider running ALG_{j+1} on input $\mathcal{I}_j \cup S_j$ (in spite of it being an invalid input). In each iteration, the algorithm gives highest priority to an input item from $\mathcal{I}_j$ or from S_j. Consider the first time ALG_{j+1} selects an input item from S_j.

If ALG_{j+1} has already made a mistake on an input item from $\mathcal{I}_j$, then we can simply take $\mathcal{I}_{j+1} = \mathcal{I}_j$, $v_{j+1} = v_j$, and $S_{j+1} = S_j$. All the properties are easy to verify in this case.

Otherwise, let (x, y, z) be the first element from S_j that is requested by ALG_{j+1}. Without loss of generality, assume that the decision of ALG_{j+1} is to accept edge (x, y) and not (x, z). If $x = v_j$, then we extend $\mathcal{I}_{j+1} = \mathcal{I}_j \cup \{(x, y, z)\}$ and S_{j+1} is S_j with all items involving y or x removed, as well as those items that have z as second or third coordinate. Observe that this ensures that ALG_{j+1} makes a mistake on item (x, y, z) and this fact is unaffected by further extensions of $\mathcal{I}_{j+1}$. In this case, we have $v_{j+1} = z$.

The last case to consider is when ALG_{j+1} requests (x, y, z) from S_j and $x \neq v_j$. In this case, we also consider an item $(v_j, x, w) \in S_j$ for some w that is different from any other value appearing in the construction so far. By the way S_j is constructed, and taking m large enough, such a w is guaranteed to exist. We extend $\mathcal{I}_{j+1} = \mathcal{I}_j \cup \{(v_j, x, w), (x, y, z)\}$. Again, without loss of generality, assume that ALG_{j+1} accepts (x, y) rather than (x, z). We again set $v_{j+1} = z$ and S_{j+1} to be the set S_j with all items involving x, y, w, or v_j removed, as well as those items that have z as the second or third coordinate. This guarantees that ALG_{j+1} makes a mistake on item (x, y, z) and continues to make a mistake on this item no matter how $\mathcal{I}_{j+1}$ is extended with elements from S_{j+1}.

After all ℓ algorithms have made a mistake, leaving a final v_j and S_j, an input item from S_j with v_j as the first coordinate is moved from S_j to $\mathcal{I}_j$, finishing the construction.

Observe that each S_j can be defined by some subset $F \subseteq \{s, x_1, \ldots, x_m\}$. Namely, S_j consists of all triples formed from F, as well as triples formed by having the first coordinate equal to v_j and the remaining two coordinates coming from F. In each iteration going from j to $j + 1$, at most 4 elements are removed from F. At the end, three additional elements from F are used for the last item. Therefore, $m = 3 + 4\ell$ is sufficient to guarantee a universe large enough that the construction terminates only after all algorithms are fooled by the instance.

Finally, assume that b advice bits are used by an adaptive priority algorithm with advice with the above construction as input. We determine a lower bound on b. Running an algorithm in the unrestricted priority function with b bits of advice is equivalent to running 2^b algorithms in parallel. Thus, we have $\ell = 2^b$ algorithms that can all be fooled simultaneously by a k-thorny path problem, where $k \leq 2\ell$, since the last case above uses two layers to fool the algorithm in question. Since b bits are insufficient and $2^{b+1} = 2\ell \geq k$, it follows that $\log k - 1$ bits are insufficient. $\square$

The following theorem shows that the upper bound in Lemma 5 is tight for the decision-based priority function model. The proof uses the same ideas as the proof of the lower bound in Section 4. In that proof, 2^k different input sequences were created, and using fewer than k bits of advice led to at least two of them getting the same advice and an error being made on one of those two. Those sequences can be seen as forming a binary tree, with inputs at the nodes in the tree and the two possible decisions leading to the two subtrees. Thus, sequences that are the same up until input m share the same path from the root to that input. This is not quite the case for the thorny path problem, since it is possible for the adaptive priority algorithm to select an input item that is not connected to the last one seen. However, the tree determines 2^k root-to-leaf paths, which naturally define 2^k thorny paths and their inputs.

Theorem 10. *An adaptive priority algorithm with advice in the decision-based priority function model must use at least k bits of advice to solve the k-thorny path problem.*

Proof. Let ALG be an adaptive priority algorithm with advice in the decision-based priority function model. We consider ALG's computation on the k-thorny path problem.

We construct 2^k input instances of the k-thorny path problem. To explain the construction, we use a binary tree with 2^{k+1} leaves and s as the root. The leaves will be the leaves in the thorny path problems, and each root-to-leaf path, along with the siblings of

the vertices on the path, will be the thorny paths that should be followed by ALG to get to a leaf. The 2^k different paths one can take from s to a parent of a leaf will represent the 2^k input instances we are constructing. They will not *be* the input instances since input instances could have further input items that are discarded by ALG. Each node in the tree has an associated ordered list of input items, which are all the ones for which the algorithm chooses ⊥ (discard) until the next time it chooses 0 or 1 (left or right). Thus, a path in the tree defines an input sequence consisting of the input items forming the path with the associated ordered lists of input items added. More precisely, the ordered list of input items associated with a node u appears in the input sequence just prior to the input item with u as root (of that input item; recall that an input item consists of three nodes, two of which are the children of the first, the root).

The tree represents all execution paths ALG can take based on different advice. Along the way, we will also explain how the adversary will change the input universe as an execution proceeds. In an execution (that follows one path), the universe is decreased gradually as execution progresses down the path, and the universe that is used at a given point varies depending on which path was chosen by ALG (based on its advice).

In constructing the tree, we start with s in the root and we add nodes to the tree gradually by adding two children to a currently childless node. Let u be such a node of depth at most k. We consider ALG's execution on the partial input defined by the path from s to u (including the input items associated with nodes on the path). The path, together with the associated lists, defines the decisions ALG must make on this partial input.

Naturally, ALG just follows one path in the tree, making decisions to go left or right or discard based on the advice it gets. However, if it is at u, then the next input item ALG's priority function selects is only based on the partial input and its decisions. Now, for an input item, (x, y, z), either $x = u$ or x is not on the s-to-u path (this follows from how we treat the universe; see later).

If $x = u$, we add the leaves y and z to the tree as children of u. All input items remaining containing u or its sibling are removed from the universe.

If $x \neq u$, the adversary removes all input items remaining that contain x, y, or z from the universe. Thus, x can never become part of any root-to-leaf path that currently ends at u. The input item (x, y, z) is then appended to the ordered list of discarded input items associated with u.

There are no more input items added after there are 2^{k+1} leaves at depth $k + 1$. If the tree is never completed, there is a path where ALG never finds a leaf, so ALG fails. Otherwise, the 2^k different input sequences defined by the paths in the tree must have distinct advice strings. Thus, at least k advice bits are necessary. □

Note that this proof does not appear to work in the unrestricted priority function model, since it is not clear that the tree can be defined in that case. For example, if advice (in addition to the decisions made) is used to determine which input item is chosen next, an input that we placed off of a thorny path might actually only be chosen if it is on the path.

8. Open problems

The extension of the adaptive priority model to the advice tape model leads to many new research directions. We consider the following open problems to be of particular interest:

- Design and analyze new adaptive priority algorithms with advice for (special cases of) classical optimization problems and convert them to offline algorithms, by trying all possibilities for the advice as with priority exact algorithms or by implementing them as branch-and-reduce algorithms. In particular, are there priority algorithms with advice that lead to faster (in terms of the base of the exponent) exact exponential time offline algorithms than the best known?
- The previous question also applies to approximation algorithms, when the best known offline approximation algorithm is exponential in terms of running time.
- Suggest how to extend the lower bound results to the unrestricted priority function model. A first example of such a lower bound for an artificial problem was given in Section 7 for the thorny path problem.
- Suggest and investigate other extensions of the adaptive priority framework besides the information-theoretic advice tape extension. For instance, one could consider a class of adaptive priority algorithms where advice is given by an AC^0 circuit. What can be said about the power and limitations of such algorithms?
- More generally, study the structural complexity of priority algorithms with advice. What reasonable complexity classes can be defined based on advice complexity and approximation ratio?
- The lower bounds implied by our reduction-based framework are of the form "constant inapproximability even given linear advice." Can this framework be extended to handle super-constant inapproximability with sublinear advice? More generally, the goal is to design some framework that could work in this other realm of parameters. A good starting point would be to show that we cannot obtain an approximation ratio of $n^{1-\epsilon}$ for Maximum Independent Set with $O(\log n)$ bits of advice for any fixed $\epsilon \in (0, 1]$. Note that under the assumption P ≠ NP, this lower bound follows from the famous result of Håstad [31]. The goal here is to prove this lower bound unconditionally for the restricted class of priority algorithms with advice.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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