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ARTICLE

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Making a tournament k-strong

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Abstract

A digraph is **k**-strong if it has $n \geq k + 1$ vertices and every induced subdigraph on at least $n - k + 1$ vertices is strongly connected. A tournament is a digraph with no pair of nonadjacent vertices. We prove that every tournament on $n \geq k + 1$ vertices can be made k-strong by adding no more than $\binom{k+1}{2}$ arcs. This solves a conjecture from 1994. A digraph is semicomplete if there is at least one arc between any pair of distinct vertices x, y. Since every semicomplete digraph contains a spanning tournament, the result above also holds for semicomplete digraphs. Our result also implies that for every $k \geq 2$, every semicomplete digraph on at least $3k - 1$ vertices can be made k-strong by reversing no more than $\binom{k+1}{2}$ arcs.

KEYWORDS

directed vertex-connectivity augmentation, increasing connectivity, one-way pairs, semicomplete digraph, tournament

1 | INTRODUCTION

We follow the notation in [1] and just recall a few definitions here. A digraph $D = (V, A)$ is **k**-strong if D has at least $k + 1$ vertices and every subdigraph D' that we can obtain from D by deleting any set of at most $k - 1$ vertices is strongly connected. For an arbitrary digraph on n vertices which is not k-strong we may have to add nk new arcs to obtain a k-strong digraph. One example attaining this bound is the digraph on n vertices and no arcs which needs nk new arcs (and this suffices). For a given digraph D on at least $k + 1$ vertices we denote by $a_k(D)$ the minimum number of new arcs that must be added to D to obtain a k-strong digraph.

By the remark above, $a_k(D) \leq nk$ for every digraph on $n \geq k+1$ vertices. Frank and Jordán [5, 6] solved the problem of characterizing $a_k(D)$ for a given digraph D and also gave a polynomial algorithm to find a set of arcs A' of minimum cardinality to add to a given digraph $D = (V, A)$ so that the resulting digraph $\hat{D} = (V, A \cup A')$ is k-strong (see Theorems 2.2 and 2.3 below).

The out-degree of a vertex v in a digraph $D = (V, A)$ is the number of arcs of the kind $vw \in A$ (having v as their tail). The transitive tournament TT_n on n vertices is the tournament with vertex set $\{v_1, v_2, ..., v_n\}$ and arc set $\{v_iv_j \mid i < j\}$. It is easy to see that $a_k(TT_n) \geq \binom{k+1}{2}$ for each $n \geq k+1$: for each $i \in [k]$ the vertex v_{n-i+1} needs $k-i+1$ new arcs going out of it just to have out-degree at least k. It is also easy to check that if we add $k-i+1$ arcs from v_{n+1-i} to $v_1, ..., v_{k-i+1}$ for $i \in [k]$, then the resulting semicomplete digraph is, in fact, k-strong so $a_k(TT_n) = \binom{k+1}{2}$ when $n \geq k+1$.

It is easy to show that for every tournament on at least $k+1$ vertices $a_k(D)$ is upper bounded by some function depending only on k. We will just sketch the argument here. We use the following two facts (see e.g., [2]):

(i) Every tournament on at least $4k-1$ vertices has a vertex v with both in-degree and out-degree at least k.
(ii) If $H = (V, A)$ is a k-strong digraph and we add a new vertex v along with an arc from v to each of k distinct vertices of V and from a set of k distinct vertices of V to v, then the resulting digraph is k-strong (see [1, exercise 14.8]).

These facts imply that for every tournament D on more than $4k-2$ vertices we have $a_k(D) \leq a_{k(D')}$ for some subtournament D' of D on at most $4k-2$ vertices. For such a tournament we clearly have that $a_{k(D')} \leq \binom{4k-2}{2}$ and hence $a_{k(D')}$ is bounded by a function of k. Observations like this made the first author conjecture in 1994 (see e.g., [1, conjecture 14.3.12]) that $a_k(D)$ is at most $\binom{k+1}{2}$ for every tournament D on at least $k+1$ vertices. The purpose of this paper is to give a short proof of this conjecture. We do so in Section 3.

Theorem 1.1. *For every tournament D on at least $k+1$ vertices and every positive integer k we have $a_k(D) \leq \binom{k+1}{2}$.*

2 | ONE-WAY PAIRS AND VERTEX CONNECTIVITY AUGMENTATION

In this section, we relate the value of $a_k(D)$ for a given digraph $D = (V, A)$ to so-called one-way pairs. Let H, T be disjoint nonempty proper subsets of V. The ordered pair (H, T) is a one-way pair in $D = (V, A)$ if D has no arc with tail in T and head in H. This definition is due to Frank and Jordán [5] but we use a slightly different notation here. For such a pair (H, T) we refer to H (T) as the head (tail) of the pair. Let $h(H, T) = |V - H - T|$. The deficiency $\eta_k(H, T)$ of a one-way pair (H, T) with respect to k-strong connectivity is defined as

$$\eta_k(H, T) = \max\{0, k - h(H, T)\}. \tag{1}$$

For instance, if $N^+[X] \neq V$ then the pair $(X, V - N^+[X])$ is a one-way pair with deficiency $\eta_k(X, V - N^+[X]) = \max\{0, k - |N^+(X)|\}$. Here $N^+[X]$ is the closed out-neighborhood of the set $X \subset V$, that is, $N^+[X] = \{v \in V | \exists\, x \in X, (x = v \vee xv \in A)\}$. One-way pairs are closely related to k-strong connectivity.

Lemma 2.1 (Frank and Jordán [5, claim 4.1]). *A digraph $D = (V, A)$ is k-strong if and only if we have $h(H, T) \geq k$ for every one-way pair (H, T) in D.*

Two one-way pairs $(H_1, T_1), (H_2, T_2)$ are independent if either their heads or their tails are disjoint. By Lemma 2.1, to obtain a k-strong superdigraph of D, we must add enough new arcs to cover all one-way pairs with $\eta_k(H, T) > 0$: we must add at least $\eta_k(H, T)$ arcs from T to H. This is the reason why H (T) is called the head (tail) of the one-way pair (H, T). Clearly, if $(H_1, T_1), (H_2, T_2)$ are independent one-way pairs, then no new arc can decrease both $\eta_k(H_1, T_1)$ and $\eta_k(H_2, T_2)$. This implies that, if $\mathcal{F}$ is any family of pairwise independent one-way pairs in D, then we must add at least

$$\eta_k(\mathcal{F}) = \sum_{(H,T)\in\mathcal{F}} \eta_k(H, T). \tag{2}$$

new arcs to D to obtain a k-strong digraph. We call the number $\eta_k(\mathcal{F})$ the deficiency of $\mathcal{F}$ with respect to k-strong connectivity.

The following theorem, due to Frank and Jordán, shows that the maximum deficiency over families of independent one-way pairs gives the right lower bound for the vertex-strong connectivity augmentation problem.

Theorem 2.2 (Frank and Jordán [5, theorem 4.3]). *For every digraph D on at least $k + 1$ vertices we have*

$$a_k(D) = \max\{\eta_k(\mathcal{F}) : \mathcal{F} \text{ is a family of independent one-way pairs in } D\}. \tag{3}$$

Frank and Jordán also gave a polynomial algorithm for finding an optimal augmentation (set of new arcs to add) for any given input digraph D.

Theorem 2.3 (Frank and Jordán [5, section 7]). *There exists a polynomial algorithm which, given a digraph $D = (V, A)$ and a natural number k, finds a minimum cardinality set F of new arcs to add to D so that the resulting digraph is k-strong.*

In the proof of Theorem 1.1 below we will use the notation (H, S, T) to denote a one-way pair. Here H, T are as above and $S = V - H - T$ so $\eta_k(H, S, T) = \max\{0, k - |S|\}$. For a one-way pair (H, S, T) with $|H| = 1$ ($|T| = 1$) we call H (T) a singleton head (tail) and say that (H, S, T) is a singleton one-way pair.

3 | PROOF OF THEOREM 1.1

Proof. Let D be a tournament such that $a_k(D)$ is maximum and for all tournaments D' with $a_{k(D')} = a_k(D)$ we have $n = |V(D)| \leq |V(D')|$. We will prove that $n = k + 1$ from which the theorem follows as we can make any tournament D on $k + 1$ vertices k-strong by adding all the arcs of the converse of D and this has $\binom{k+1}{2}$ arcs. Here we used the easy fact that the complete digraph on $k + 1$ vertices is k-strong. □

Assume below that $n \geq k + 2$ and let $\mathcal{F} = \{(H_1, S_1, T_1), \ldots, (H_p, S_p, T_p)\}$ be an independent family of one-way pairs of D achieving the value $a_k(D)$, that is, by Theorem 2.2, $\eta_k(\mathcal{F}) = a_k(D)$.

Claim. Every vertex of D is either a singleton head or a singleton tail of some one-way pair in $\mathcal{F}$.

Proof of Claim. Suppose x is neither a singleton head nor a singleton tail. Then we consider the tournament $D' = D - x$ and the family $\mathcal{F} - x = \left\{\left(H_1', S_1', T_1'\right), \ldots, \left(H_p', S_p', T_p'\right)\right\}$ of one-ways pair in D', where $H_i' = H_i - x$, $S_i' = S_i - x$ and $T_i' = T_i - x$ (precisely one of H_i, S_i, T_i contains x). As x is neither a singleton head nor a singleton tail, each (H_i', S_i', T_i') is a one-way pair in D' and $\eta_{k(\mathcal{F}')} \geq \eta_k(\mathcal{F})$ with strict inequality if x belongs to at least one $S_i, i \in [p]$. This contradicts the choice of D. □

By the claim, $\mathcal{F}$ contains at least n singleton one-way pairs and we can assume that the one-way pairs of $\mathcal{F}$ are ordered so that the first $n + r$ one-way pairs $(H_1, S_1, T_1), \ldots, (H_{n+r}, S_{n+r}, T_{n+r})$ are singleton one-way pairs. Here $r \geq 0$ and it may be larger than 0 as some vertex x may occur both as a singleton head and as a singleton tail in the list.

By the choice of D, whenever we remove a vertex x from D the resulting tournament $D - x$ will have $a_k(D - x) < a_k(D)$. Let us consider just the family $\mathcal{F}$ and see how removing x affects this family:

- For each one-way pair (H_f, S_f, T_f) of $\mathcal{F}$ such that $x \in S_f$ the deficiency of that one-way pair increases by one. Let $S(x) = \{j \in [p] : x \in S_j\}$ and let $g(x) = |S(x)|$.
- For each one-way pair (H_q, S_q, T_q) of $\mathcal{F}$ such that either $H_q = \{x\}$ or $T_q = \{x\}$ this one-way pair of D is no longer a one-way pair of $D - x$ (since either the head or the tail is empty) so we loose the deficiency of such a one-way pair. For each such one-way pair (H_q, S_q, T_q) the deficiency of the family without this one-way pair goes down by $k - |S_q| = k - (n - 1 - \theta_q)$, where $\theta_q = |T_q|$ if $H_q = \{x\}$ and $\theta_q = |H_q|$ if $T_q = \{x\}$. Note that we cannot have $|H_q| = |T_q| = 1$ for some q as we assumed that $n \geq k + 2$ (here we used that $|S_q| < k$). Denote by $\ell(x)$ the total loss in deficiency of the family $\mathcal{F}(x)$ that results from deleting x and removing singleton one-ways pairs as above. Note that x may be both a singleton head of one one-way pair (H_q, S_q, T_q) and a singleton tail of another pair (H_a, S_a, T_a) in which case the loss in deficiency is $k - (n - 1 - \theta_q) + k - (n - 1 - \theta_a)$.

By the choice of D, we cannot remove any vertex x and still have the same or higher deficiency for the resulting family of one-way pairs. Thus we have that $g(x) < \ell(x)$ for every vertex $x \in V(D)$. Now we get

$$\sum_{x \in V(D)} g(x) < \sum_{x \in V(D)} \ell(x).$$

Using that $g(x) = |S(x)|$ for each $x \in V(D)$ and that each singleton one-way pair (H_i, S_i, T_i) among $(H_1, S_1, T_1), \ldots (H_{n+r}, S_{n+r}, T_{n+r})$ contributes to the right hand side with $k - (n - 1 - \theta_i)$ we conclude that

$$\sum_{x \in V(D)} |S(x)| < \sum_{i=1}^{n+r} [k - (n - 1 - \theta_i)]. \tag{4}$$

By the definition of $S(x)$ we have

$$\sum_{x \in V(D)} |S(x)| = \sum_{i=1}^{p} |S_i| \geq \sum_{i=1}^{n+r} |S_i|.$$

Inserting this into (4) we get.

$$\sum_{i=1}^{n+r} |S_i| \leq \sum_{x \in V(D)} |S(x)| < \sum_{i=1}^{n+r} [k - n + 1 + \theta_i].$$

Now using that $|S_i| = n - 1 - \theta_i$ for $i \in [n + r]$ we get

$$\sum_{i=1}^{n+r} n - 1 - \theta_i < \sum_{i=1}^{n+r} [k - n + 1 + \theta_i].$$

Rearranging this we get

$$\sum_{i=1}^{n+r} (2n - k - 2) < \sum_{i=1}^{n+r} 2\theta_i.$$

Recall that since D is a tournament, for every one-way pair (H, S, T) D contains all arcs ht with $h \in H, t \in T$, so each pair (H_i, S_i, T_i), where (precisely) one of H_i, T_i is a singleton reserves θ_i arcs that cannot belong to any other one-way pair in $\mathcal{F}$ as these are all independent. This implies that $\sum_{i=1}^{n+r} \theta_i$ is less than or equal to the total number of arcs in D. Using this and the fact that the left-hand summands do not depend on i we see that

$$(n + r)(2n - k - 2) < 2\binom{n}{2} = n(n - 1).$$

As $r \geq 0$ this implies that $n < k + 1$, contradicting the assumption that $n \geq k + 2$. □

Recall that a digraph is semicomplete if it has no pair of nonadjacent vertices. As every semicomplete digraph H contains a spanning tournament D, Theorem 1.1 implies the following.

Theorem 3.1. *For every semicomplete digraph H on $n \geq k + 1$ vertices we have $a_k(H) \leq \binom{k+1}{2}$.*

4 | REVERSING ARCS TO ACHIEVE HIGH CONNECTIVITY

For a given digraph D let $r_k(D)$ denote the minimum number of arcs one needs to reverse to obtain a k-strong reorientation of D. If no such reversal exists we set $r_k(D) = \infty$. It is an open problem whether $r_k(D)$ can be determined efficiently already when $k = 2$. For $k = 1$ the problem can be solved in polynomial time [4] (see also [1, section 13.1]).

We now consider the function r_k for tournaments and semicomplete digraphs. Using the fact that every tournament T on n vertices has a Hamiltonian path $v_1v_2...v_n$ it is easy to see that $r_1(T) \leq 1$, because either v_nv_1 is an arc and T is already strong or v_1v_n is an arc and we can reverse that arc to obtain a strong tournament. The kth power C_q^k of a directed cycle on $q \geq 2k + 1$ vertices $C_q = v_1v_2...v_qv_1$ is the digraph we obtain from C_q by adding all arcs v_iv_j, where the distance from v_i to v_j on C_q is at least 2 and at most k. It is easy to check that C_q^k is k-strong when $q \geq 2k + 1$.

Using this observation as well as (i) and (ii) of Section 1 as we did when we studied the function a_k it is easy to check that for every tournament T on at least $2k + 1$ vertices we have $r_k(T) \leq \binom{4k-2}{2}/2$ (at most half of the arcs need to be reversed before the new tournament contains a copy of C_n^k) so r_k is bounded by a function of k for every tournament T on at least $2k + 1$ vertices.

Since adding a copy of an existing arc uv cannot increase the vertex connectivity it is easy to see that no optimal reversal will reverse an arc of a directed 2-cycle. This implies that we have $r_k(D) \geq a_k(D)$ for every digraph. Our earlier arguments for TT_n imply that when $n \geq 2k + 1$ we have $r_k(TT_n) = \binom{k+1}{2}$. This made the first author conjecture the following stronger version of Theorem 1.1.

Conjecture 4.1 (Bang-Jensen, 1994). *For every tournament T on at least $2k + 1$ vertices we have $r_k(T) \leq \binom{k+1}{2}$.*

Some support for Conjecture 4.1 was provided in [3], where it was shown that $r_k^{\text{arc-strong}}(T) = \text{deor}_k^{\text{arc-strong}}(T)$ for every tournament T on at least $2k + 1$ vertices. Here $r_k^{\text{arc-strong}}(T)$ is the minimum number of arcs one must reverse in T to obtain a k-arc-strong tournament and $\text{deor}_k^{\text{arc-strong}}(T)$ is the minimum number of arcs of T we must replace by a directed 2-cycle (same as adding the opposite arc) to obtain a k-arc-strong semicomplete digraph. A digraph $D = (V, A)$ is k-arc-strong if the number of arcs from X to $V - X$ is at least k for every subset $\varnothing \neq X \neq V$ of V.

Theorem 4.2 (Bang-Jensen and Jordan [2]). *For every integer $k \geq 2$ and every semicomplete digraph on $n \geq 3k - 1$ vertices we have $a_k(D) = r_k(D)$. For every $k \geq 3$, there exists a semicomplete digraph D on $3k - 2$ vertices for which we have $a_k(D) < r_k(D)$.*

Combining this with Theorem 3.1 we see that Conjecture 4.1 holds when $n \geq 3k - 1$.

Corollary 4.3. *For every integer $k \geq 2$ and every semicomplete digraph D on $n \geq 3k - 1$ vertices we have $r_k(D) \leq \binom{k+1}{2}$.*

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DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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