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A complete description of convex sets associated with matchings and edge-connectivity in graphs

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Abstract

Let $L_{k,\lambda}$ be the set of pairs (γ, β) of real numbers with the property there exists a constant $C(k, \lambda)$, depending only on k and λ, such that $\alpha'(G) \ge \gamma n + \beta m - C(k, \lambda)$ for every connected graph G of order n, size m, with maximum degree at most k and edge-connectivity at least $\lambda \ge 1$. In a recent paper [J. Graph Theory 89(2) (2018), 115–149] the authors give a complete description of the set $L_{k,1}$. In this paper we raise the problem to a higher level, and give a complete description of the convex set $L_{k,\lambda}$ for all $k \ge \lambda \ge 2$, and determine its extreme points.

Keywords: Convex set; Matching number; Maximum degree; Edge-connectivity.
AMS subject classification: 05C40, 05C70

1 Introduction

Two fundamental concepts in graph theory are those of matchings and edge-connectivity. In this paper, we determine new tight lower bounds on the matching number in a graph with given maximum degree and edge-connectivity in terms of its order and size. The new bounds we obtain, together with earlier known bounds of the authors [5, 6], suffice to give a complete description of convex sets associated with matchings and edge-connectivity in graphs.

We shall adopt the following graph theory notation for matchings and edge-connectivity in graphs. Two edges in a graph G are *independent* if they are not adjacent in G. A set of pairwise independent edges of G is called a *matching* in G, while a matching of maximum cardinality is a *maximum matching*. The number of edges in a maximum matching of G is called the *matching number* of G which we denote by $\alpha'(G)$. Matchings in graphs are extensively studied in the literature (see, for example, the classical book on matchings my Lovász and Plummer [8]) with over 8,000 publications according to MathSciNet.

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Results on convex sets and matching can be found, for example, in [3, 5], while results on connectivity and matchings can be found, for example, in [6, 7, 9, 10, 11].

The *edge-connectivity* of G, denoted $\lambda(G)$, is the minimum number of edges whose removal from G produces a disconnected graph. In particular, if G is disconnected, then $\lambda(G) = 0$, while if G is a connected graph that contains a bridge, then $\lambda(G) = 1$.

For graph theory and terminology, we generally follow [4]. In particular, we denote the *degree* of a vertex v in the graph G by $d_G(v)$. The maximum degree among the vertices of G is denoted by $\Delta(G)$. For a subset S of vertices of a graph G, we let $G[S]$ denote the subgraph induced by S. The number of odd components of a graph G we denote by $\mathrm{oc}(G)$. For subsets X and Y of vertices in G, we denote by $[X, Y]$ the set of all edges joining the set X and the set Y in G. For a set X of vertices in G, we let $\overline{X} = V(G) \setminus X$ denote the complement of the set X. Thus, $[X, \overline{X}]$ is the set of all edges joining the set X and its complement $\overline{X}$ in G. We use the standard notation $[k] = \{1, 2, \ldots, k\}$.

2 Main Results

Let $\mathcal{G}_{k,\lambda}(n, m)$ denote the class of graphs G of order n and size m and with maximum degree at most k and edge-connectivity at least λ, where $k \geq 2$ and $\lambda \geq 1$. For $k \geq 2$ and $\lambda \geq 1$, let $L_{k,\lambda}$ be the set of all pairs (γ, β) of real numbers for which there exists a constant $C(k, \lambda)$, depending only on k and λ, such that

$$\alpha'(G) \geq \gamma n + \beta m - C(k, \lambda)$$

holds for every graph G in the family $\mathcal{G}_{k,\lambda}(n, m)$. Our aim in this paper is to give a complete description of the set $L_{k,\lambda}$. For this purpose, let $a_{k,\lambda}$, $b_{k,\lambda}$, $c_{k,\lambda}$, $d_{k,\lambda}$, $e_{k,\lambda}$ and $f_{k,\lambda}$ be the constants (depending only on k and λ) defined in Section 3. Let $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5$ be the following five closed half-spaces over the reals γ and β:

$$\begin{array}{lllrcl}
\ell_1 : & \beta & \leq & -\left(\frac{2}{k}\right)\gamma & + & b_{k,\lambda} + \frac{2a_{k,\lambda}}{k} \\
\ell_2 : & \beta & \leq & -\frac{1}{\lambda}\left(\frac{1+\lambda b_{k,\lambda}}{1-a_{k,\lambda}}\right)\gamma & + & b_{k,\lambda} + \frac{a_{k,\lambda}}{\lambda}\left(\frac{1+\lambda b_{k,\lambda}}{1-a_{k,\lambda}}\right) \\
\ell_3 : & \beta & \leq & -\left(\frac{2}{\lambda}\right)\gamma & + & \frac{1}{\lambda} \\
\ell_4 : & \beta & \leq & -\left(\frac{2}{k}\right)\gamma & + & d_{k,\lambda} - \frac{2e_{k,\lambda}}{k} \\
\ell_5 : & \beta & \leq & -\left(\frac{b_{k,\lambda}-d_{k,\lambda}}{a_{k,\lambda}+e_{k,\lambda}}\right)\gamma & + & d_{k,\lambda} + e_{k,\lambda}\left(\frac{b_{k,\lambda}-d_{k,\lambda}}{a_{k,\lambda}+e_{k,\lambda}}\right)
\end{array}$$

We are now in a position to state our main result.

Theorem 1 *For $k \geq \lambda \geq 2$ the set $L_{k,\lambda}$ is a convex set. Further, the following holds.*

(a) *If k and λ have the same parity, then $L_{k,\lambda}$ is the intersection of the closed half-spaces ℓ_1, ℓ_2 and ℓ_3, and there are precisely two extreme points of $L_{k,\lambda}$, namely $(a_{k,\lambda}, b_{k,\lambda})$ and $(1, -\frac{1}{\lambda})$.*

(b) *If k and λ have different parity, then $L_{k,\lambda}$ is the intersection of the closed half-spaces ℓ_2, ℓ_3, ℓ_4 and ℓ_5, and there are precisely three extreme points of $L_{k,\lambda}$, namely $(-e_{k,\lambda}, d_{k,\lambda})$, $(a_{k,\lambda}, b_{k,\lambda})$ and $(1, -\frac{1}{\lambda})$.*

Theorem 1 is illustrated in Figure 1(a) when k and λ have the same parities, and is illustrated in Figure 1(b) when k and λ have different parities, where the convex set $L_{k,\lambda}$ corresponds to the grey area in the pictures.

For small values of k and λ, namely $(k, \lambda) \in \{(5, 2), (5, 3), (6, 3), (6, 4)\}$, the convex set $L_{k,\lambda}$ corresponds to the grey area in the pictures illustrated in Figure 2.

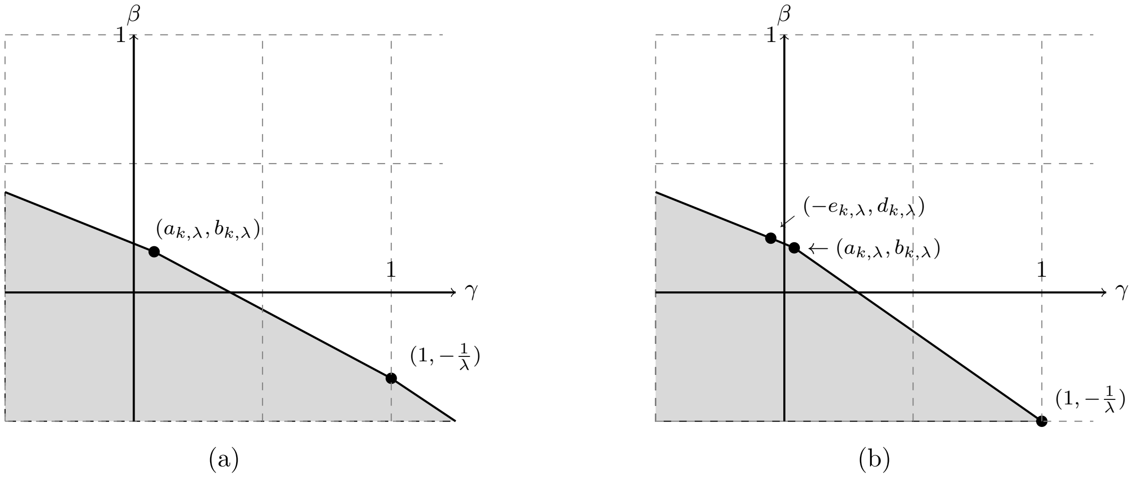

Figure 1: The convex sets $L_{k,\lambda}$ depending on the parity of k and λ

We proceed as follows. In Section 3 we define the constants used in our main result, namely Theorem 1. Thereafter we present known results in Section 4.

3 The Constants used in the Main Result

In this section, we define the constants used in our main results presented in Section 2. The constants $a_{k,\lambda}$ and $b_{k,\lambda}$ (depending only on k and λ) are defined as given in Table 1.

	k even	k odd
λ even	$a_{k,\lambda} = \dfrac{\lambda}{k^2+k+\lambda}$ $b_{k,\lambda} = \dfrac{k}{k^2+k+\lambda}$	$a_{k,\lambda} = \dfrac{\lambda(k-\lambda-1)}{k^3-k^2\lambda+2k^2-k\lambda-k-\lambda^2-\lambda}$ $b_{k,\lambda} = \dfrac{k^2-k\lambda-\lambda+k}{k^3-k^2\lambda+2k^2-k\lambda-k-\lambda^2-\lambda}$
λ odd	$a_{k,\lambda} = \dfrac{\lambda(k-\lambda-1)}{k^3-k^2\lambda+k^2-k-\lambda^2-\lambda}$ $b_{k,\lambda} = \dfrac{k(k-\lambda)}{k^3-k^2\lambda+k^2-k-\lambda^2-\lambda}$	$a_{k,\lambda} = \dfrac{\lambda}{k^2+2k+\lambda}$ $b_{k,\lambda} = \dfrac{k+1}{k^2+2k+\lambda}$

Table 1. The constants $a_{k,\lambda}$ and $b_{k,\lambda}$.

We note that the constants $a_{k,\lambda}$ and $b_{k,\lambda}$ are related by the equation

$$a_{k,\lambda}\left(\frac{k+\lambda}{\lambda}\right) + k\,b_{k,\lambda} = 1. \tag{1}$$

Further, let the constant $c_{k,\lambda}$ used in Corollary 1 be defined as in Table 2. We remark that the constant $c_{k,\lambda}$ depends only on k and λ.

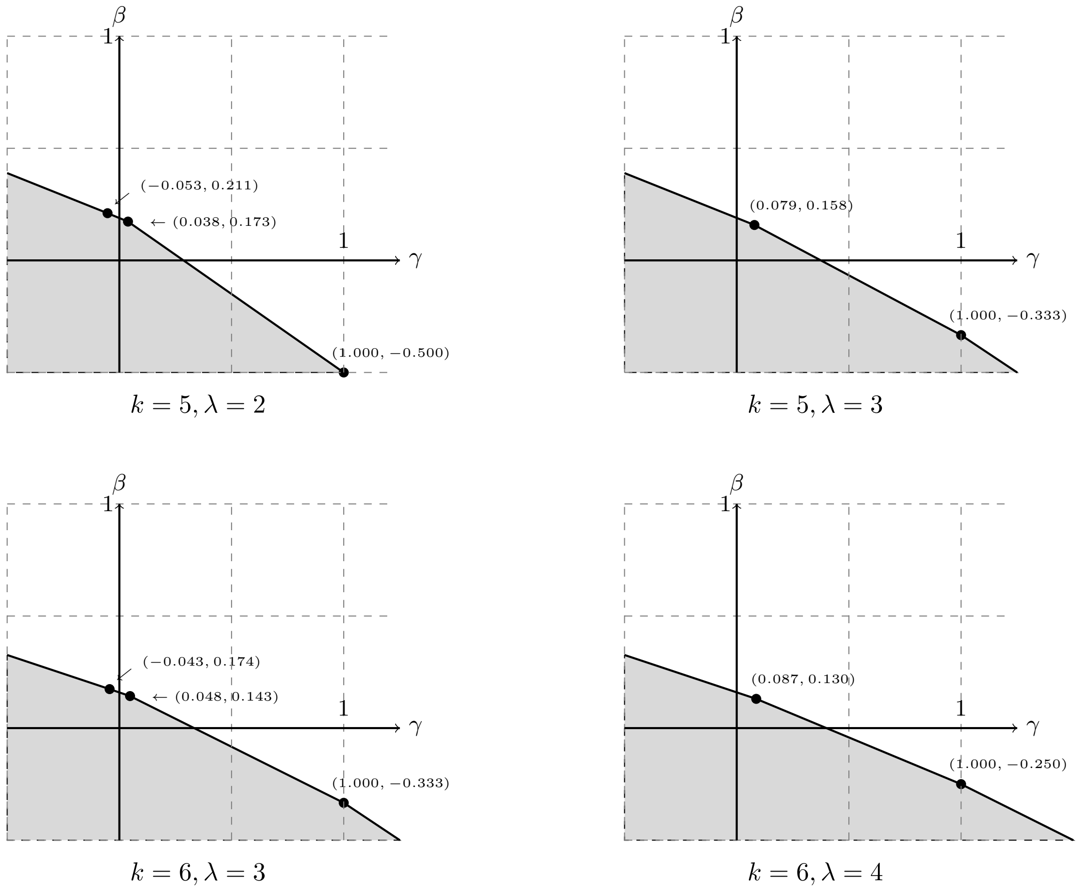

Figure 2: The convex set $L_{k,\lambda}$ for small values of k and λ

	$c_{k,\lambda}$
k even	$\frac{1}{2} - \frac{a_{k,\lambda}(k+1)(k-\lambda)}{2\lambda}$
k odd	$\frac{1-b_{k,\lambda}}{2} - \frac{a_{k,\lambda}(k+2)(k-\lambda)}{2\lambda}$

Table 2. The constant $c_{k,\lambda}$.

Substituting the values for $a_{k,\lambda}$ and $b_{k,\lambda}$ (which depends on the parity of k and the parity of λ) given in Table 1, gives the associated values of $c_{k,\lambda}$ as a function of k and λ.

Let $d_{k,\lambda}$ and $e_{k,\lambda}$ be the constants (depending only on k and λ) given in Table 3.

	$d_{k,\lambda}$	$e_{k,\lambda}$
k even, λ odd	$\frac{k+2}{k^2+k+\lambda+1}$	$\frac{k-\lambda-1}{k^2+k+\lambda+1}$
k odd, λ even	$\frac{k+3}{k^2+2k+\lambda+1}$	$\frac{k-\lambda-1}{k^2+2k+\lambda+1}$

Table 3. The constants $d_{k,\lambda}$ and $e_{k,\lambda}$.

We note that the constants $d_{k,\lambda}$ and $e_{k,\lambda}$ are related by the equation

$$k \cdot d_{k,\lambda} - e_{k,\lambda} = 1. \tag{2}$$

Further, let the constant $f_{k,\lambda}$ (depending only on k and λ) be defined as in Table 4.

	$f_{k,\lambda}$
k even, λ odd	$\frac{1}{2}(1-(k+1)e_{k,\lambda})$
k odd, λ even	$\frac{1}{2}(1-d_{k,\lambda}-(k+2)e_{k,\lambda})$

Table 4. The constant $f_{k,\lambda}$.

Substituting the values for $d_{k,\lambda}$ and $e_{k,\lambda}$ (which depends on the parity of k and the parity of λ) given in Table 3, gives the associated values of $f_{k,\lambda}$ as a function of k and λ.

4 Known Matching Results

In this section, we present known results that we will need in order to prove our main result, namely Theorem 1.

One of the earliest results in graph theory is Petersen's theorem in 1891 that every 2-edge-connected 3-regular graph contains a perfect matching. The following result in 1938 due to Bäbler [1] generalizes Petersen's theorem.

Theorem 2 ([1]) *For all $k \geq 2$, every $(k-1)$-edge-connected k-regular graph of even order contains a perfect matching.*

In order to prove our main results, we rely heavily on the following theorem of Berge [2] about the matching number of a graph, which is sometimes referred to as the Tutte-Berge formulation for the matching number.

Theorem 3 (Tutte-Berge Formula) *For every graph G,*

$$\alpha'(G) = \min_{X \subseteq V(G)} \frac{1}{2} \left(|V(G)| + |X| - \text{oc}(G - X) \right).$$

In 2018 the authors [5] determined tight lower bounds on the matching number of a connected graph in terms of its order and size with given maximum degree.

Theorem 4 ([5]) *For $k \ge 3$ is an integer, if $G \in \mathcal{G}_{k,1}(n,m)$, then*

$$\alpha'(G) \ge \begin{cases} \left(\frac{k-1}{k(k^2-3)}\right) n + \left(\frac{k^2-k-2}{k(k^2-3)}\right) m - \frac{k-1}{k(k^2-3)} & \text{if } k \text{ is odd} \\ \left(\frac{1}{k(k+1)}\right) n + \left(\frac{1}{k+1}\right) m - \frac{1}{k} & \text{if } k \text{ is even} \end{cases}$$

Recall that $\mathcal{G}_{k,1}(n,m)$ denote the class of connected graphs G with order n and size m and with maximum degree at most k. A complete description of the set $L_{k,1}$ of pairs (γ, β) of real numbers with the property that there exists a constant K such that $\alpha'(G) \ge \gamma n + \beta m - K$ holds for every graph $G \in \mathcal{G}_{k,1}(n,m)$ is given in [5]. In this paper we extend these results in [5] to graphs of higher edge connectivity. For this purpose, we shall need the following recent results of the authors [6].

Theorem 5 ([6]) *For $k \ge \lambda \ge 2$ integers, if $G \in \mathcal{G}_{k,\lambda}(n,m)$, then $\alpha'(G) = \frac{1}{2}(n-1)$ or*

$$\alpha'(G) \ge a_{k,\lambda} \cdot n + b_{k,\lambda} \cdot m.$$

As a consequence of Theorem 5, we have the following result.

Corollary 1 ([6]) *For $k \ge \lambda \ge 2$ integers, if $G \in \mathcal{G}_{k,\lambda}(n,m)$, then $\alpha'(G) \ge a_{k,\lambda} \cdot n + b_{k,\lambda} \cdot m - c_{k,\lambda}$.*

5 New Matching Results

In this section, we determine new tight lower bounds on the matching number in a graph that belongs to the class $\mathcal{G}_{k,\lambda}(n,m)$ for all $k \ge \lambda \ge 2$. In order to prove our main result, namely Theorem 1, we shall prove the following lower bound on the matching number.

Theorem 6 *For $k \ge \lambda \ge 2$ integers, if $G \in \mathcal{G}_{k,\lambda}(n,m)$, then $\alpha'(G) \ge n - \frac{1}{\lambda}m$ unless $\alpha'(G) = \frac{1}{2}(n-1)$ in which case $\alpha'(G) \ge n - \frac{1}{\lambda}m - \frac{1}{2}$.*

Let $d_{k,\lambda}$, $e_{k,\lambda}$ and $f_{k,\lambda}$ be the constants (depending only on k and λ) defined in Section 3. A proof of the following result is presented in Section 5.1.

Theorem 7 *For $k > \lambda \ge 2$ integers where k and λ have different parities, if $G \in \mathcal{G}_{k,\lambda}(n,m)$, then $\alpha'(G) = \frac{1}{2}(n-1)$ or*

$$\alpha'(G) \ge d_{k,\lambda} \cdot m - e_{k,\lambda} \cdot n.$$

As a consequence of Theorem 7, we have the following corollary, a proof of which is presented in Section 5.2.

Corollary 2 *For $k > \lambda \geq 2$ integers where k and λ have different parities, if $G \in \mathcal{G}_{k,\lambda}(n,m)$, then $\alpha'(G) \geq d_{k,\lambda} \cdot m - e_{k,\lambda} \cdot n - f_{k,\lambda}$.*

In Section 5.4, we show that the lower bounds on the matching number presented in Theorem 6, Theorem 7 and Corollary 2 are all tight. As a consequence of these new results and earlier results established in [6], we have the following result where the families $\mathcal{F}^a_{k,\lambda}$, $\mathcal{F}^b_{k,\lambda}$, $\mathcal{F}^c_{k,\lambda}$, $\mathcal{F}^d_{k,\lambda}$ and $\mathcal{F}^e_{k,\lambda}$ are defined in Section 5.5.

Theorem 8 ([6]) *For integers $k \geq \lambda \geq 2$, the following holds.*

(a) *If k and λ have the same parity, then there exists an infinite family, $\mathcal{F}^a_{k,\lambda}$, of k-regular graphs G of order n and size m with edge-connectivity λ, such that $\alpha'(G) = a_{k,\lambda} \cdot n + b_{k,\lambda} \cdot m$ for all $G \in \mathcal{F}^a_{k,\lambda}$.*

(b) *If k and λ have different parity, then there exists an infinite family, $\mathcal{F}^b_{k,\lambda}$, of k-regular graphs G of order n and size m with edge-connectivity λ, such that $\alpha'(G) = -e_{k,\lambda} \cdot n + d_{k,\lambda} \cdot m$ for all $G \in \mathcal{F}^b_{k,\lambda}$.*

(c) *If k and λ have different parity, then there exists an infinite family, $\mathcal{F}^c_{k,\lambda}$, of graphs G of order n and size m with edge-connectivity λ, such that $\alpha'(G) = a_{k,\lambda} \cdot n + b_{k,\lambda} \cdot m$ and $\alpha'(G) = -e_{k,\lambda} \cdot n + d_{k,\lambda} \cdot m$ for all $G \in \mathcal{F}^c_{k,\lambda}$.*

(d) *There exists an infinite family, $\mathcal{F}^d_{k,\lambda}$, of graphs G of order n and size m with edge-connectivity λ, such that $\alpha'(G) = a_{k,\lambda} \cdot n + b_{k,\lambda} \cdot m$ and $\alpha'(G) = n - \frac{1}{\lambda}m$ for all $G \in \mathcal{F}^d_{k,\lambda}$.*

(e) *There exists an infinite family, $\mathcal{F}^e_{k,\lambda}$, of λ-regular graphs G of order n and size m with edge-connectivity λ, such that $\alpha'(G) = n - \frac{1}{\lambda}m$ for all $G \in \mathcal{F}^e_{k,\lambda}$.*

5.1 Proof of Theorem 6

In this section, we prove Theorem 6. Recall its statement.

Theorem 6. *For $k \geq \lambda \geq 2$, if $G \in \mathcal{G}_{k,\lambda}(n,m)$, then $\alpha'(G) \geq n - \frac{1}{\lambda}m$ unless $\alpha'(G) = \frac{1}{2}(n-1)$ in which case $\alpha'(G) \geq n - \frac{1}{\lambda}m - \frac{1}{2}$.*

Proof. By the Tutte-Berge formula in Theorem 3, there is a subset X of vertices in G satisfying

$$\alpha'(G) = \frac{1}{2}\left(|V(G)| + |X| - \mathrm{oc}(G-X)\right).$$

Suppose that $X \neq \emptyset$. Since the edge-connectivity of G is at least λ, each component of $G - X$ is joined in G to X by at least λ edges, implying that

$$\begin{aligned} 2m &= \sum_{v \in X} d_G(v) + \sum_{v \in \overline{X}} d_G(v) \\ &\geq |[X,\overline{X}]| + |\overline{X}| \cdot \lambda \\ &\geq \lambda \cdot \mathrm{oc}(G-X) + \lambda(n - |X|). \end{aligned}$$

Thus,

$$n - \frac{1}{\lambda}m \leq n - \frac{1}{2}\mathrm{oc}(G-X) - \frac{1}{2}(n-|X|) = \frac{1}{2}\left(n + |X| - \mathrm{oc}(G-X)\right) = \alpha'(G).$$

Hence if $X \neq \emptyset$, then $\alpha'(G) \geq n - \frac{1}{\lambda}m$. Hence we may assume that $X = \emptyset$, for otherwise the desired result holds. With this assumption, $\alpha'(G) = \frac{1}{2}(|V(G)| - \mathrm{oc}(G)) \geq \frac{1}{2}(n-1)$ since G is connected. Since the graph G has minimum degree at least λ, we note that $m \geq \frac{1}{2}\lambda n$, which implies $n - \frac{1}{\lambda}m \leq \frac{n}{2}$. This implies that if $\alpha'(G) = \frac{1}{2}n$, then $\alpha'(G) \geq n - \frac{1}{\lambda}m$. Hence the only possibly exception to the lower bound $\alpha'(G) \geq n - \frac{1}{\lambda}m$ on the matching number is if $\alpha'(G) = \frac{1}{2}(n-1)$. In this case, $n - \frac{1}{\lambda}m - \frac{1}{2} \leq \frac{n}{2} - \frac{1}{2}$ (and n is odd). This implies that if $\alpha'(G) = \frac{1}{2}(n-1)$, then $\alpha'(G) \geq n - \frac{1}{\lambda}m - \frac{1}{2}$. □

5.2 Proof of Theorem 7

In this section, we present a proof of Theorem 7 and Corollary 2. For this purpose, we consider two cases depending on the parity of k and λ. Recall that the constants $d_{k,\lambda}$ and $e_{k,\lambda}$ are defined in Table 1, while the constant $f_{k,\lambda}$ is defined in Table 2.

5.2.1 λ Odd and k Even

In this section we prove the following result.

Theorem 9 *Let $\lambda \geq 3$ be an odd integer and $k \geq 4$ an even integer where $k > \lambda$. If $G \in \mathcal{G}_{k,\lambda}(n,m)$, then $\alpha'(G) \geq \frac{1}{2}(n-1)$ or $\alpha'(G) \geq d_{k,\lambda} \cdot m - e_{k,\lambda} \cdot n$.*

Proof. By the Tutte-Berge formula in Theorem 3, there is a subset X of vertices in G satisfying

$$\alpha'(G) = \frac{1}{2}\left(|V(G)| + |X| - \mathrm{oc}(G-X)\right).$$

If $X = \emptyset$, then $\alpha'(G) \geq (n-1)/2$ as desired. Hence we may assume that $X \neq \emptyset$. Let n_i be the number of components in $G - X$ of order i. Recall that in this case when λ is odd and k is even,

$$d_{k,\lambda} = \frac{k+2}{k^2+k+\lambda+1} \qquad \text{and} \qquad e_{k,\lambda} = \frac{k-\lambda-1}{k^2+k+\lambda+1}.$$

As, by the Tutte-Berge formula, any maximum matching in G contains $\lceil \frac{1}{2}(|C|-1) \rceil$ edges in each component, C, in $G-X$ and an additional X edges (each with exactly one endpoint in X), it suffices for us to show that the following inequality holds.

$$|X| + \sum_{i=2}^{\infty} \left\lceil \frac{i-1}{2} \right\rceil \cdot n_i \geq d_{k,\lambda} \cdot m \; - \; e_{k,\lambda} \cdot n. \tag{3}$$

By the definition of n_i we note that

$$n = |X| + \sum_{i=1}^{\infty} i \cdot n_i. \tag{4}$$

If C is a component of $G-X$ of order i for some $i \geq k+1$, and $m(C)$ is the size of C, then by the edge-connectivity of G there are at least λ edges with one end in X and the other end in $V(C)$, implying that

$$2m(C) = \sum_{v \in V(C)} d_C(v) \leq i \cdot k - \lambda.$$

As observed in [6] every component of $G-X$ of order $i \geq k+1$ therefore has size at most $(i \cdot k - \lambda)/2$. Further since every vertex in X has degree at most k and since every component of $G-X$ of order $i \leq k$ has size at most $\binom{i}{2}$, the size of G is bounded above by

$$m \leq k|X| + \sum_{i=2}^{k} \binom{i}{2} n_i + \sum_{i=k+1}^{\infty} \left(\frac{i \cdot k}{2} - \frac{\lambda}{2}\right) n_i. \tag{5}$$

By the maximum degree condition, every vertex in X has at most k neighbors in $\overline{X} = V(G) \setminus X$. Since $\lambda(G) \geq \lambda$, there are at least λ edges joining every component in $G-X$ to vertices in X. Since λ is odd and k is even, we note that every component of $G-X$ of order $i \geq k+1$ has size at most $(i\cdot k-(\lambda+1))/2$, implying that the size of G is bounded above by

$$m \leq k|X| + \sum_{i=2}^{k} \binom{i}{2} n_i + \sum_{i=k+1}^{\infty} \left(\frac{i\cdot k}{2} - \frac{\lambda+1}{2} \right) n_i. \tag{6}$$

We point out that the inequality

$$\frac{(i^2-i)(k+2) - 2i(k-\lambda-1)}{2(k^2+k+\lambda+1)} \leq \frac{i-1}{2} \tag{7}$$

holds for all i where $1 \leq i \leq k$, since the above inequality is equivalent to

$$0 \leq (i-1)((k-i)(k+2) + k - \lambda - 1) + 2(k-\lambda-1),$$

which clearly holds under the conditions of $1 \leq i \leq k$ and $\lambda+1 \leq k$. Moreover, the inequality

$$\frac{(ik-\lambda-1)(k+2) - 2i(k-\lambda-1)}{2(k^2+k+\lambda+1)} \leq \frac{i-1}{2} \tag{8}$$

holds for all i where $i \geq k+1$, since the above inequality is equivalent to

$$i(k-\lambda-1) - k^2 + \lambda k + \lambda + 1 \geq (k+1)(k-\lambda-1) - k^2 + \lambda k + \lambda + 1 = 0,$$

which clearly holds under the conditions of $i \geq k+1$ and $\lambda+1 \leq k$. Therefore by Equation (4) and Inequality (6), and by definition of $e_{k,\lambda}$ and $d_{k,\lambda}$, the following holds.

$$\begin{aligned}
& d_{k,\lambda}\cdot m \;-\; e_{k,\lambda}\cdot n \\
\leq\; & \left(\frac{k+2}{k^2+k+\lambda+1}\right)\left(k|X| + \sum_{i=1}^{k}\binom{i}{2} n_i + \sum_{i=k+1}^{\infty}\left(\frac{i\cdot k-\lambda-1}{2}\right) n_i\right) - \\
& \left(\frac{k-\lambda-1}{k^2+k+\lambda+1}\right)\left(|X| + \sum_{i=1}^{\infty} i\cdot n_i\right) \\
=\; & |X| + \sum_{i=1}^{k}\left(\frac{(i^2-i)(k+2)-2i(k-\lambda-1)}{2(k^2+k+\lambda+1)}\right)\cdot n_i + \\
& \sum_{i=k+1}^{\infty}\left(\frac{(ik-\lambda-1)(k+2)-2i(k-\lambda-1)}{2(k^2+k+\lambda+1)}\right)\cdot n_i + \\
\overset{(7),(8)}{\leq}\; & |X| + \sum_{i=2}^{k}\left(\frac{i-1}{2}\right)\cdot n_i \;+\; \sum_{i=k+1}^{\infty}\left(\frac{i-1}{2}\right)\cdot n_i \\
=\; & |X| + \sum_{i=2}^{\infty}\left(\frac{i-1}{2}\right)\cdot n_i. \\
\leq\; & \alpha'(G).
\end{aligned}$$

This completes the proof of Theorem 9. □

5.2.2 λ Even and k Odd

In this section we prove the following result.

Theorem 10 *Let $\lambda \geq 2$ be an even integer and $k \geq 3$ an odd integer where $k > \lambda$. If $G \in \mathcal{G}_{k,\lambda}(n,m)$, then $\alpha'(G) \geq \frac{1}{2}(n-1)$ or $\alpha'(G) \geq d_{k,\lambda} \cdot m - e_{k,\lambda} \cdot n$.*

Proof. We adopt the notation in the proof of Theorem 9. In particular, X is a subset of vertices of G achieving equality in the Tutte-Berge formula in Theorem 3. If $X = \emptyset$, then $\alpha'(G) \geq (n-1)/2$ as desired. Hence we may assume that $X \neq \emptyset$. Recall that in this case,

$$d_{k,\lambda} = \frac{k+3}{k^2+2k+\lambda+1} \quad \text{and} \quad e_{k,\lambda} = \frac{k-\lambda-1}{k^2+2k+\lambda+1}.$$

It therefore suffices for us to show that the following inequality holds.

$$|X| + \sum_{i=2}^{\infty} \left\lceil \frac{i-1}{2} \right\rceil \geq d_{k,\lambda} \cdot m \; - \; e_{k,\lambda} \cdot n. \tag{9}$$

As in the proof of Theorem 9, Equation (4) holds here. Further, every odd component of $G - X$ of order $i \geq k+2$ has size at most $(i \cdot k - \lambda - 1)/2$. We point out that the inequality

$$\frac{i(i-1)(k+3) - 2i(k-\lambda-1)}{2(k^2+2k+\lambda+1)} \leq \frac{i-1}{2} \tag{10}$$

holds for all i where $1 \leq i \leq k$, since the above inequality is equivalent to

$$0 \leq (i-1)((k-i)(k+3) + (k-\lambda-1)) + 2(k-\lambda-1),$$

which clearly holds under the conditions of $1 \leq i \leq k$ and $\lambda + 1 \leq k$. Let $\omega_i \equiv i \pmod 2$. That is, ω_i is 1 if i is odd and 0 if i is even. Note that in this case $\lceil \frac{i-1}{2} \rceil = \frac{i-\omega_i}{2}$. We remark next that the inequality

$$\frac{(k+3)(ki-\lambda-\omega_i) - 2i(k-\lambda-1)}{2(k^2+2k+\lambda+1)} \leq \frac{i-\omega_i}{2} \tag{11}$$

holds for all i where $i \geq k+1$. The above inequality is equivalent to

$$\omega_i(k^2+k+\lambda-2) \leq i(k-\lambda-1) + \lambda(k+3).$$

When $i = k+1$, we note that i is even (as k is odd) and therefore $\omega_i = 0$. In this case the above inequality reduces to $0 \leq (k+1)(k-\lambda-1) + \lambda(k+3)$, or, equivalently, $0 \leq k^2 + 2\lambda - 1$, which is clearly true since here $\lambda \geq 2$ and $k \geq \lambda + 1$. Hence, we may assume that $i \geq k+2$, for otherwise Inequality (11) holds as desired. In this case, we note that

$$\begin{aligned} \omega_i(k^2+k+\lambda-2) &\leq k^2+k+\lambda-2 && \text{(since } \omega_i \leq 1\text{)} \\ &= (k+2)(k-\lambda-1) + \lambda(k+3) && \\ &\leq i(k-\lambda-1) + \lambda(k+3) && \text{(since } k+2 \leq i\text{),} \end{aligned}$$

implying once again that Inequality (11) holds. Therefore by Equation (4) and Inequality (5), and by definition of $e_{k,\lambda}$ and $d_{k,\lambda}$, the following holds.

$$
\begin{aligned}
& d_{k,\lambda} \cdot m \,-\, e_{k,\lambda} \cdot n \\
\leq \quad & \left(\frac{k+3}{k^2+2k+\lambda+1}\right)\left(k|X| + \sum_{i=1}^{k}\binom{i}{2}n_i + \sum_{i=k+1}^{\infty}\left(\frac{i\cdot k-\lambda-\omega_i}{2}\right)n_i\right) - \\
& \left(\frac{k-\lambda-1}{k^2+2k+\lambda+1}\right)\left(|X| + \sum_{i=1}^{\infty} i\cdot n_i\right) \\
= \quad & |X| + \sum_{i=1}^{k}\left(\frac{i(i-1)(k+3)-2i(k-\lambda-1)}{2(k^2+2k+\lambda+1)}\right)\cdot n_i + \\
& \sum_{i=k+1}^{\infty}\left(\frac{(k+3)(ki-\lambda-\omega_i)-2i(k-\lambda-1)}{2(k^2+2k+\lambda+1)}\right)\cdot n_i \\
\overset{(10),(11)}{\leq} \quad & |X| + \sum_{i=2}^{k}\left(\frac{i-1}{2}\right)\cdot n_i \,+\, \sum_{i=k+1}^{\infty}\left(\frac{i-\omega_i}{2}\right)\cdot n_i \\
\leq \quad & |X| + \sum_{i=2}^{\infty}\left\lceil\frac{i-1}{2}\right\rceil\cdot n_i = \alpha'(G).
\end{aligned}
$$

This completes the proof of Theorem 10. □

5.2.3 The case when G has a perfect matching

Recall that the constants $d_{k,\lambda}$ and $e_{k,\lambda}$ are related by Equation (2). Further, $e_{k,\lambda} \geq 0$, and equality holds if $k = \lambda + 1$. Since the graph G has maximum degree at most k, we note that $m \leq \frac{1}{2}kn$. Combining this inequality with $k \cdot d_{k,\lambda} - e_{k,\lambda} = 1$, we have

$$
\begin{aligned}
\frac{n}{2} - (d_{k,\lambda}\cdot m - e_{k,\lambda}\cdot n) \quad & \geq \quad \frac{n}{2} - d_{k,\lambda}\cdot\frac{1}{2}kn + e_{k,\lambda}\cdot n \\
& = \quad \frac{n}{2}\left(1 - d_{k,\lambda}\cdot k + 2e_{k,\lambda}\right) \\
& = \quad \frac{n}{2}\cdot e_{k,\lambda} \geq 0.
\end{aligned}
$$

The above inequality implies that in the statements of Theorem 9 and Theorem 10 if G has a perfect matching, then $\alpha'(G) \geq d_{k,\lambda}\cdot m - e_{k,\lambda}\cdot n$. We state this observation formally as follows.

Observation 1 *For $k > \lambda \geq 2$ integers where k and λ have different parities, if $G \in \mathcal{G}_{k,\lambda}(n,m)$ and $\alpha'(G) = \frac{1}{2}n$, then $\alpha'(G) \geq d_{k,\lambda}\cdot m - e_{k,\lambda}\cdot n$.*

5.3 Proof of Corollary 2

In this section, we prove Corollary 2. Recall its statement.

Corollary 2. *For integers $k > \lambda \geq 2$ where k and λ have different parities, if $G \in \mathcal{G}_{k,\lambda}(n,m)$, then $\alpha'(G) \geq d_{k,\lambda}\cdot m - e_{k,\lambda}\cdot n - f_{k,\lambda}$.*

Proof. By Theorem 9 and Theorem 10, if $\alpha'(G) < \frac{1}{2}(n-1)$, then $\alpha'(G) \geq d_{k,\lambda}\cdot m - e_{k,\lambda}\cdot n$. By Observation 1, if $\alpha'(G) = \frac{1}{2}n$, then $\alpha'(G) \geq d_{k,\lambda}\cdot m - e_{k,\lambda}\cdot n$. Hence, the only possible exception to the lower bound $\alpha'(G) \geq d_{k,\lambda} \cdot m - e_{k,\lambda} \cdot n$ is if G has an almost perfect matching; that is, if $\alpha'(G) = \frac{1}{2}(n-1)$. In this case, n is odd. We now consider the following two cases.

Case 1. **k is even.** If $n \leq k$, which is possible as k is an upper bound on the maximum degree in G and not necessarily the actual maximum degree, then define θ such that $n = k + 1 - \theta$. In this case, $m \leq \frac{n(n-1)}{2} = \frac{n}{2}(k - \theta)$. If $n > k$, then let $\theta = 0$ and note that $m \leq \frac{1}{2}kn = \frac{1}{2}kn - \frac{\theta n}{2}$. So in all cases, $n \geq k + 1 - \theta$ and $m \leq \frac{1}{2}kn - \frac{\theta n}{2}$. Recall that $d_{k,\lambda}$ and $e_{k,\lambda}$ are related by Equation (2). Thus, the following holds (as, by the definition of $e_{k,\lambda}$, we have $e_{k,\lambda} \geq 0$ and $ke_{k,\lambda} \leq 1 \leq n$).

$$
\begin{array}{lrcl}
 & \frac{1}{2}\left(1 - (k+1)e_{k,\lambda}\right) & = & f_{k,\lambda} \\
\Downarrow & & & \\
 & \frac{1}{2}\left(1 - e_{k,\lambda} \cdot (n + \theta)\right) & \leq & f_{k,\lambda} \\
\Updownarrow & & & \\
 & \left(\frac{n}{2} - \frac{\theta n}{2k}\right)(1 + e_{k,\lambda}) - e_{k,\lambda} \cdot n - \frac{n}{2} + \frac{\theta}{2k}\left(n + ne_{k,\lambda} - ke_{k,\lambda}\right) & \leq & f_{k,\lambda} - \frac{1}{2} \\
\Downarrow & & & \\
 & \left(\frac{n}{2} - \frac{\theta n}{2k}\right) \cdot k\, d_{k,\lambda} - e_{k,\lambda} \cdot n - f_{k,\lambda} & \leq & \frac{n-1}{2} \\
\Downarrow & & & \\
 & d_{k,\lambda} \cdot m - e_{k,\lambda} \cdot n - f_{k,\lambda} & \leq & \frac{n-1}{2} = \alpha'(G)
\end{array}
$$

Case 2. **k is odd.** Recall that n is odd. We first consider the case when $n \leq k$ and in this case define q such that $\lambda = k - 1 - q$. Note that $q \geq 0$ as $\lambda < n \leq k$. Define $h(q)$ as the following function.

$$h(q) = \frac{n(n-1)}{2} \cdot d_{k,\lambda} - n \cdot e_{k,\lambda} = \frac{n(n-1)(k+3)}{2(k^2+3k-q)} - \frac{nq}{k^2+3k-q}.$$

Differentiating $h(q)$ with respect to q gives us the following.

$$h'(q) = \frac{n(k+3)}{(k^2+3k-q)^2}\left(\frac{n-1}{2} - k\right).$$

As $n \leq k$ we observe that $h'(q) < 0$ for all $0 \leq q \leq k-1$. Therefore, as $m \leq \frac{1}{2}n(n-1)$ and $f_{k,\lambda} \geq 0$ the following holds.

$$d_{k,\lambda} \cdot m - e_{k,\lambda} \cdot n - f_{k,\lambda} \leq h(q) \leq h(0) = \frac{n(n-1)(k+3)}{2(k^2+3k)} - 0 = \frac{n(n-1)}{2k} \leq \frac{n-1}{2} = \alpha'(G).$$

This proves the case when $n \leq k$.

So now assume that $n > k$. As n and k are both odd this implies that $n \geq k+2$. Not all vertices in G can have degree k (as n and k are odd), and so

$$2m = \sum_{v \in V(G)} d(v) \leq nk - 1.$$

Recall that $d_{k,\lambda}$ and $e_{k,\lambda}$ are related by Equation (2). The following now holds (as $n \geq k+2$).

$$\begin{array}{rcl} \frac{1}{2}(1-d_{k,\lambda}-(k+2)e_{k,\lambda}) & = & f_{k,\lambda} \\ \Downarrow & & \\ \frac{1}{2}(1-e_{k,\lambda}\cdot n) & \le & f_{k,\lambda}+\frac{d_{k,\lambda}}{2} \\ \Updownarrow & & \\ \frac{n}{2}(1+e_{k,\lambda})-e_{k,\lambda}\cdot n-\frac{n}{2} & \le & f_{k,\lambda}-\frac{1}{2}+\frac{d_{k,\lambda}}{2} \\ \Updownarrow & & \\ \frac{n}{2}\cdot k\,d_{k,\lambda}-e_{k,\lambda}\cdot n-f_{k,\lambda} & \le & \frac{n-1}{2}+\frac{d_{k,\lambda}}{2} \\ \Updownarrow & & \\ \left(\frac{nk-1}{2}\right)d_{k,\lambda}-e_{k,\lambda}\cdot n-f_{k,\lambda} & \le & \frac{n-1}{2} \\ \Downarrow & & \\ d_{k,\lambda}\cdot m-e_{k,\lambda}\cdot n-f_{k,\lambda} & \le & \frac{n-1}{2}=\alpha'(G) \end{array}$$

This completes the proof of Case 2, and completes the proof of Corollary 2. □

5.4 Tightness of Bounds

In this section, we show that the lower bounds on the matching number presented in Theorem 6, Theorem 7, Theorem 9 and Theorem 10 are all tight, in the sense that for every $k \ge \lambda \ge 2$ there exist examples where the theorems obtain equality.

5.4.1 Tightness of Bound in Theorem 6

Let k and λ be integers where $k \ge \lambda \ge 2$. As remarked in the proof of Theorem 6, the only possible exception to the lower bound $\alpha'(G) \ge n - \frac{1}{\lambda}m$ on the matching number is if $\alpha'(G) = \frac{1}{2}(n-1)$. In this exceptional case, we consider the graph $G = K_{\lambda+1}$ where λ is even. We note that $G \in \mathcal{G}_{k,\lambda}(n,m)$, where $n = \lambda+1$ is odd, $m = \frac{1}{2}n\lambda$ and $\alpha'(G) = \frac{1}{2}(n-1)$. Therefore, $n - \frac{1}{\lambda}m - \frac{1}{2} = n - \frac{1}{\lambda}\cdot\frac{1}{2}n\lambda - \frac{1}{2} = \frac{1}{2}(n-1) = \alpha'(G)$, proving the tightness of Theorem 6 in the case when $\alpha'(G) = \frac{1}{2}(n-1)$.

We show next tightness of the bound in Theorem 6 when $\alpha'(G) = n - \frac{1}{\lambda}m$ (and $\alpha'(G) \ne \frac{1}{2}(n-1)$). Let $G \in \mathcal{G}_{k,\lambda}(n,m)$ be a λ-edge-connected bipartite graph with partite sets X and Y, where each vertex in X has degree k and each vertex in Y has degree λ (and where possibly $k=\lambda$). Thus, $|X| = c\cdot\lambda$ and $|Y| = c\cdot k$ for some $c \ge 1$. We remark that if $k = \lambda+1$ and $c = 1$, then $G = K_{\lambda,\lambda+1}$ which satisfies $\alpha'(G) = \frac{1}{2}(n-1)$. Hence if $k = \lambda+1$, then we take $c > 1$, while if $k = \lambda$ or $k \ge \lambda+2$, then we take $c \ge 1$ (where possibly $c = 1$). The graphs G so constructed satisfy $\alpha'(G) \ne \frac{1}{2}(n-1)$. Moreover,

$$m = \sum_{v\in X} d_G(v) = ck\lambda$$

and

$$n - \frac{1}{\lambda}m = (ck + c\lambda) - \frac{1}{\lambda}\cdot ck\lambda = c\lambda = |X| \ge \alpha'(G) \ge n - \frac{1}{\lambda}m, \tag{12}$$

where the inequality $\alpha'(G) \ge n - \frac{1}{\lambda}m$ on the far right hand side of the inequality chain (12) follows from Theorem 6 noting that by construction $\alpha'(G) \ne \frac{1}{2}(n-1)$. Consequently, we must have equality throughout the inequality chain (12). In particular, this implies that the graph $G \in \mathcal{G}_{k,\lambda}(n,m)$ satisfies $\alpha'(G) = n - \frac{1}{\lambda}m$.

5.4.2 Tightness of Bound in Theorem 7

The tightness of the lower bounds on the matching number presented in Theorem 7 follows from the tightness of the bounds in Theorem 9 and Theorem 10 given in the following two Sections 5.4.3 and 5.4.4, respectively. For this purpose, we shall need the following two constructions given in [6].

Definition 1 ([6]) Let $\lambda \ge 2$ be an arbitrary integer and let $k \ge 2$ be an even integer. Let $\mathcal{R}_{k,\lambda}$ be the family of graphs constructed as follows. If $k = \lambda \ge 2$, then let $\mathcal{R}_{k,\lambda}$ be the family of all λ-edge-connected k-regular graphs of even order. In particular, we note that $\mathcal{R}_{2,2}$ consist of all even cycles. For $k > \lambda \ge 2$, let G be a λ-edge-connected bipartite graph with partite sets X and Y where every vertex in X has degree λ and every vertex in Y has degree k. Let $|X| = \ell$ and let $X = \{x_1, x_2, \dots, x_\ell\}$, and let $|Y| = r$ and let $Y = \{y_1, y_2, \dots, y_r\}$. Let $X_1, X_2, \dots, X_\ell$ be a number of vertex disjoint graphs such that each X_i where $i \in [\ell]$ is either a single vertex or it is a K_{k+1} where the edges of a matching of size $\lceil \frac{\lambda}{2} \rceil$ have been deleted.

For $k > \lambda \ge 2$, let $R_{k,\lambda}$ be obtained from the disjoint union of the graphs $X_1, X_2, \dots, X_\ell$ by adding to it the vertices in Y and, furthermore, for every $i \in [\ell]$, adding λ edges between vertices in X_i and vertices in Y in such a way that if $x_i y_j$ is an edge of G for some j where $j \in [r]$, then y_j is joined in $R_{k,\lambda}$ to some vertex in X_i and no vertex in X_i has degree more than k. By construction, every vertex in Y has degree k in $R_{k,\lambda}$. If X_i has order $k+1$, then every vertex in X_i has degree k in $R_{k,\lambda}$, except in the case when λ is odd when exactly one vertex in X_i has degree $k-1$ and the remaining k vertices have degree k in $R_{k,\lambda}$. Further since G is λ-edge-connected, the graph $R_{k,\lambda}$ is λ-edge-connected. Let $\mathcal{R}_{k,\lambda}$ be the family of all such graph $R_{k,\lambda}$. We note that if X_i has order $k+1$ for all $i \in [\ell]$ and λ is even, then the graph $R_{k,\lambda}$ is k-regular.

Definition 2 ([6]) Let $\lambda \ge 2$ be an arbitrary integer and let $k \ge 3$ be an odd integer. Let $\mathcal{H}_{k,\lambda}$ be the family of graphs constructed as follows. If $k = \lambda \ge 3$, then let $\mathcal{H}_{k,\lambda}$ be the family of all k-edge-connected k-regular graphs.

For $k > \lambda \ge 2$ where λ is even, let F_{k+2} be the graph of (odd) order $k+2$ obtained from a complete graph K_{k+2} by removing the edges of a cycle on $\lambda+1$ vertices and from the remaining $k+1-\lambda$ vertices removing a perfect matching. Thus in this case, the complement of F_{k+2} is the graph $C_{\lambda+1} \cup \left(\frac{k+1-\lambda}{2}\right) K_2$. Further, $\lambda+1$ vertices have degree $k-1$ in F_{k+2} and the remaining $k+1-\lambda$ vertices have degree k in F_{k+2}. We note that if $k=3$ and $\lambda=2$, then F_{k+2} is the complete bipartite graph $K_{2,3}$.

For $k > \lambda \ge 2$ where λ is odd, let F_{k+2} be the graph of (odd) order $k+2$ obtained from a complete graph K_{k+2} by removing the edges of a cycle on λ vertices and from the remaining $k+2-\lambda$ vertices removing a perfect matching. Thus in this case, the complement of F_{k+2} is the graph $C_{\lambda} \cup \left(\frac{k+2-\lambda}{2}\right) K_2$. Further, λ vertices have degree $k-1$ in F_{k+2} and the remaining $k+2-\lambda$ vertices have degree k in F_{k+2}.

For $k > \lambda \ge 2$, let H be a λ-edge-connected bipartite graph with partite sets X and Y where every vertex in X has degree λ and every vertex in Y has degree k in H. Let $|X| = \ell$ and let $X = \{x_1, x_2, \dots, x_\ell\}$, and let $|Y| = r$ and let $Y = \{y_1, y_2, \dots, y_r\}$. Let $X_1, X_2, \dots, X_\ell$ be a number of vertex disjoint graphs such that each X_i where $i \in [\ell]$ is either a single vertex or it is a copy of F_{k+2}.

For $k > \lambda \ge 2$, let $H_{k,\lambda}$ be obtained from the disjoint union of the graphs $X_1, X_2, \dots, X_\ell$ by adding to it the vertices in Y and, furthermore, for every $i \in [\ell]$, adding λ edges between vertices in X_i and vertices in Y in such a way that if $x_i y_j$ is an edge of H for some j where $j \in [r]$, then y_j is joined in $H_{k,\lambda}$ to some vertex in X_i and no vertex in X_i has degree more than k. By construction, every vertex in Y has degree k in $H_{k,\lambda}$. If X_i has order $k+2$, then every vertex in X_i has degree k in $H_{k,\lambda}$, except in the case when λ is even when exactly one vertex in X_i has degree $k-1$ and the remaining $k+1$ vertices have degree k in $H_{k,\lambda}$. Further since H is λ-edge-connected, the graph $H_{k,\lambda}$ is λ-edge-connected. Let $\mathcal{H}_{k,\lambda}$ be the family of all such graph $H_{k,\lambda}$.

5.4.3 Tightness of Bound in Theorem 9

Let $\lambda \geq 3$ be an odd integer and $k \geq 4$ an even integer. Let $\mathcal{R}_{k,\lambda}$ be the family of graphs defined in Definition 1. We remark that the graphs G in $\mathcal{R}_{k,\lambda}$ have even order and hence do not satisfy $\alpha'(G) = \frac{1}{2}(n-1)$. Suppose that $k = \lambda + 1$. In this case, let $G = R_{k,\lambda}$ where X_i has order 1 for all $i \in [\ell]$. We note that $n = r + \ell$, $m = rk$ and $\alpha'(G) = r$. Further we note that $d_{k,\lambda} = \frac{1}{k}$ and $e_{k,\lambda} = 0$. Thus, $d_{k,\lambda} \cdot m - e_{k,\lambda} \cdot n = \frac{1}{k} \cdot rk = r = \alpha'(G)$. Suppose next that $k > \lambda + 1$. We first consider the graph $G = R_{k,\lambda}$ where X_i has order $k+1$ for all $i \in [\ell]$. Thus, each X_i is a copy of K_{k+1} where the edges of a matching of size $\frac{1}{2}(\lambda+1)$ have been deleted. We note that $G \in \mathcal{G}_{k,\lambda}(n,m)$, where $n = r + \ell(k+1)$, $m = rk + \frac{1}{2}\ell\,(k(k+1) - \lambda - 1)$ and $\alpha'(G) = r + \ell \cdot \frac{k}{2}$. Thus,

$$
\begin{aligned}
& d_{k,\lambda} \cdot m - e_{k,\lambda} \cdot n \\
&= \left(\frac{k+2}{k^2+k+\lambda+1}\right) \cdot m - \left(\frac{k-\lambda-1}{k^2+k+\lambda+1}\right) \cdot n \\
&= \left(\frac{k+2}{k^2+k+\lambda+1}\right) \cdot \left(rk + \frac{1}{2}\ell\left(k^2+k-\lambda-1\right)\right) - \left(\frac{k-\lambda-1}{k^2+k+\lambda+1}\right) \cdot (r + \ell(k+1)) \\
&= \left(\frac{k(k+2) - (k-\lambda-1)}{k^2+k+\lambda+1}\right) \cdot r + \frac{(k+2)(k^2+k-\lambda-1) - 2(k+1)(k-\lambda-1)}{k^2+k+\lambda+1} \cdot \frac{1}{2}\ell \\
&= 1 \cdot r + k \cdot \frac{1}{2}\ell = \alpha'(G).
\end{aligned}
$$

We next consider the graph $G = R_{k,\lambda+1}$ where X_i has order $k+1$ for all $i \in [\ell]$. Thus, each X_i is a copy of K_{k+1} where the edges of a matching of size $\frac{1}{2}(\lambda+1)$ have been deleted. We note that $G \in \mathcal{G}_{k,\lambda+1}(n,m) \subseteq \mathcal{G}_{k,\lambda}(n,m)$, where $n = r + \ell(k+1)$, $m = rk + \frac{1}{2}\ell\,(k(k+1) - \lambda - 1)$ and $\alpha'(G) = r + \ell \cdot \frac{k}{2}$, implying as before that $d_{k,\lambda} \cdot m - e_{k,\lambda} \cdot n = \alpha'(G)$. We note that in this case, the graph $G = R_{k,\lambda+1} \in \mathcal{G}_{k,\lambda}(n,m)$ is a k-regular graph.

5.4.4 Tightness of Bound in Theorem 10

Let $\lambda \geq 2$ be an even integer and $k \geq 3$ an odd integer. If $k = \lambda + 1$, then as in Section 5.4.3, we let $G = R_{k,\lambda}$ where X_i has order 1 for all $i \in [\ell]$. In this case, $d_{k,\lambda} \cdot m - e_{k,\lambda} \cdot n = \alpha'(G)$. Hence, we may assume here that $k > \lambda + 1$.

Let $G = H_{k,\lambda}$ where X_i has order $k+2$ for all $i \in [\ell]$. Thus, each X_i is the graph of (odd) order $k+2$ obtained from a complete graph K_{k+2} by removing the edges of a cycle on $\lambda+1$ vertices and from the remaining $k+1-\lambda$ vertices removing a perfect matching. We note that $G \in \mathcal{G}_{k,\lambda}(n,m)$, where $n = r + \ell(k+2)$, $m = rk + \frac{1}{2}\ell\left(k^2+2k-\lambda-1\right)$ and $\alpha'(G) = r + \frac{1}{2}\ell(k+1)$. Thus,

$$
\begin{aligned}
& d_{k,\lambda} \cdot m - e_{k,\lambda} \cdot n \\
&= \left(\frac{k+3}{k^2+2k+\lambda+1}\right) \cdot m - \left(\frac{k-\lambda-1}{k^2+2k+\lambda+1}\right) \cdot n \\
&= \left(\frac{k+3}{k^2+2k+\lambda+1}\right) \cdot \left(rk + \frac{1}{2}\ell\left(k^2+2k-\lambda-1\right)\right) - \left(\frac{k-\lambda-1}{k^2+2k+\lambda+1}\right) \cdot (r + \ell(k+2)) \\
&= \left(\frac{k(k+3) - (k-\lambda-1)}{k^2+2k+\lambda+1}\right) \cdot r + \frac{(k+3)(k^2+2k-\lambda-1) - 2(k+2)(k-\lambda-1)}{k^2+2k+\lambda+1} \cdot \frac{1}{2}\ell \\
&= 1 \cdot r + (k+1) \cdot \frac{1}{2}\ell \\
&= \alpha'(G).
\end{aligned}
$$

We next consider the graph $G = H_{k,\lambda+1}$ where X_i has order $k+2$ for all $i \in [\ell]$. We note that $G \in \mathcal{G}_{k,\lambda+1}(n,m) \subseteq \mathcal{G}_{k,\lambda}(n,m)$, where $n = r + \ell(k+1)$, $m = rk + \frac{1}{2}\ell\left(k(k+1) - \lambda - 1\right)$ and $\alpha'(G) = r + \ell \cdot \frac{k}{2}$, implying as before that $d_{k,\lambda} \cdot m - e_{k,\lambda} \cdot n = \alpha'(G)$. We note that in this case, the graph $G = H_{k,\lambda+1} \in \mathcal{G}_{k,\lambda}(n,m)$ is a k-regular graph.

5.5 The Families in the Statement of Theorem 8

The Family $\mathcal{F}^a_{k,\lambda}$. For k even and $k = \lambda \geq 2$, we define $\mathcal{F}^a_{k,\lambda}$ to be the family $\mathcal{R}_{k,\lambda}$ of all k-edge-connected k-regular graphs of even order, while for k odd and $k = \lambda \geq 3$, we define $\mathcal{F}^a_{k,\lambda}$ to be the family $\mathcal{H}_{k,\lambda}$ of all k-edge-connected k-regular graphs. For k and λ even where $k > \lambda \geq 2$, we define $\mathcal{F}^a_{k,\lambda}$ to be the family consisting of all graphs $R_{k,\lambda} \in \mathcal{R}_{k,\lambda}$ defined in Definition 1 where X_i has order $k+1$ for all $i \in [\ell]$. For k and λ odd where $k > \lambda \geq 3$, we define $\mathcal{F}^a_{k,\lambda}$ to be the family consisting of all graphs $H_{k,\lambda}$ defined in Definition 2 where X_i has order $k+2$ for all $i \in [\ell]$. In all the above cases, k and λ have the same parity and as shown in [6] every graph $G \in \mathcal{F}^a_{k,\lambda}$ is a λ-edge-connected k-regular graph of order n and size m satisfying $a_{k,\lambda} \cdot n + b_{k,\lambda} \cdot m = \alpha'(G)$. This proves Theorem 8(a).

The Family $\mathcal{F}^b_{k,\lambda}$. For $k \geq 4$ even and $\lambda \geq 3$ odd where $k > \lambda$, we define $\mathcal{F}^b_{k,\lambda}$ to consist of all graphs $R_{k,\lambda+1}$ defined in Definition 1 where X_i has order $k+1$ for all $i \in [\ell]$. For $k \geq 3$ odd and $\lambda \geq 2$ even where $k > \lambda$, we define $\mathcal{F}^b_{k,\lambda}$ to consist of all graphs $H_{k,\lambda+1}$ defined in Definition 2 where X_i has order $k+2$ for all $i \in [\ell]$. As shown in Sections 5.4.3 and 5.4.4, every graph $G \in \mathcal{F}^b_{k,\lambda}$ of order n and size m is a k-regular graph with edge-connectivity at least λ, such that $\alpha'(G) = -e_{k,\lambda} \cdot n + d_{k,\lambda} \cdot m$. This proves Theorem 8(b).

The Family $\mathcal{F}^c_{k,\lambda}$. For $k > \lambda \geq 3$ where k is even and λ is odd, we define $\mathcal{F}^c_{k,\lambda}$ to be the family of all graphs $R_{k,\lambda}$ defined in Definition 1 where X_i has order $k+1$ for all $i \in [\ell]$. For $k > \lambda \geq 2$ where k is odd and λ is even, we define $\mathcal{F}^c_{k,\lambda}$ to be the family of all graphs $H_{k,\lambda}$ defined in Definition 2 where X_i has order $k+2$ for all $i \in [\ell]$. As shown in Section 5.4.2, every graph $G \in \mathcal{F}^c_{k,\lambda}$ of order n and size m has edge-connectivity λ, such that $\alpha'(G) = -e_{k,\lambda} \cdot n + d_{k,\lambda} \cdot m$. Further as shown in [6], $\alpha'(G) = a_{k,\lambda} \cdot n + b_{k,\lambda} \cdot m$. This proves Theorem 8(c).

The Family $\mathcal{F}^d_{k,\lambda}$. For integers k and λ where $k \geq \lambda \geq 2$, we define $\mathcal{F}^d_{k,\lambda}$ to be the family of λ-edge-connected bipartite graph with partite sets X and Y, where each vertex in X has degree k and each vertex in Y has degree λ (and where possibly $k = \lambda$). As shown in Section 5.4.1, every graph $G \in \mathcal{F}^d_{k,\lambda}$ of order n and size m satisfies $\alpha'(G) = n - \frac{1}{\lambda}m$. Further as shown in [6], $\alpha'(G) = a_{k,\lambda} \cdot n + b_{k,\lambda} \cdot m$. This proves Theorem 8(d).

The Family $\mathcal{F}^e_{k,\lambda}$. For integers k and λ where $k \geq \lambda \geq 2$, we define $\mathcal{F}^d_{k,\lambda}$ to be the family of λ-edge-connected bipartite graphs where every vertex has degree λ. As shown in Section 5.4.1, every graph $G \in \mathcal{F}^e_{k,\lambda}$ of order n and size m satisfies $\alpha'(G) = n - \frac{1}{\lambda}m$. This proves Theorem 8(e).

5.5.1 Tightness of the Bound in Corollary 2

Let k and λ be integers of different parities where $k > \lambda \geq 2$. Suppose firstly that k is even, and so $\lambda \geq 3$ is odd. To show that the bound in Corollary 2 is tight in this case, we consider the graph $G = K_{k+1}$. We note that $G \in \mathcal{G}_{k,\lambda}(n,m)$, where $n = k+1$, $m = \frac{1}{2}k(k+1)$ and $\alpha'(G) = \frac{1}{2}(n-1) = \frac{1}{2}k$. Therefore,

$$\begin{aligned}
& d_{k,\lambda} \cdot m - e_{k,\lambda} \cdot n - f_{k,\lambda} \\
= \quad & d_{k,\lambda} \cdot \frac{1}{2}k(k+1) - e_{k,\lambda} \cdot (k+1) - \frac{1}{2}\left(1 - (k+1)e_{k,\lambda}\right) \\
= \quad & d_{k,\lambda} \cdot \frac{1}{2}k(k+1) - e_{k,\lambda} \cdot \frac{1}{2}(k+1) - \frac{1}{2} \\
= \quad & \frac{1}{2}(k+1)\left(d_{k,\lambda} \cdot k - e_{k,\lambda}\right) - \frac{1}{2} \\
\overset{(2)}{=} \quad & \frac{1}{2}(k+1) \cdot 1 - \frac{1}{2} \\
= \quad & \tfrac{1}{2}k = \alpha'(G).
\end{aligned}$$

Suppose, secondly, that k is odd, and so $\lambda \geq 2$ is even. To show that the bound in Corollary 2 is tight in this case, we consider the graph G of (odd) order $k+2$ obtained from a complete graph K_{k+2} by removing the edges of a path P_3 and from the remaining $k-1$ vertices removing a perfect matching. We note that $G \in \mathcal{G}_{k,\lambda}(n,m)$, where $n = k+2$, $m = \frac{1}{2}(k+2)(k+1) - \frac{1}{2}(k+3) = \frac{1}{2}(k^2+2k-1)$ and $\alpha'(G) = \frac{1}{2}(n-1) = \frac{1}{2}(k+1)$. Therefore,

$$\begin{aligned}
& d_{k,\lambda} \cdot m - e_{k,\lambda} \cdot n - f_{k,\lambda} \\
= \quad & d_{k,\lambda} \cdot \frac{1}{2}(k^2+2k-1) - e_{k,\lambda} \cdot (k+2) - \frac{1}{2}\left(1 - d_{k,\lambda} - (k+2)e_{k,\lambda}\right) \\
= \quad & d_{k,\lambda} \cdot \frac{1}{2}k(k+2) - e_{k,\lambda} \cdot \frac{1}{2}(k+2) - \frac{1}{2} \\
= \quad & \frac{1}{2}(k+2)\left(d_{k,\lambda} \cdot k - e_{k,\lambda}\right) - \frac{1}{2} \\
\overset{(2)}{=} \quad & \frac{1}{2}(k+2) \cdot 1 - \frac{1}{2} \\
= \quad & \tfrac{1}{2}(k+1) = \alpha'(G).
\end{aligned}$$

6 Proof of Theorem 1

We are now in a position to give a proof of our main result, namely Theorem 1. We show that the bounds given in Sections 4 and 5 are enough to give a complete description of the set $L_{k,\lambda}$ of pairs (γ,β). We adopt similar notation to that employed by the authors in [5]. For every pair (γ,β) of real numbers γ and β we define the concept of (k,λ)-good, (k,λ)-bad and (k,λ)-tight as follows.

- (γ,β) is called (k,λ)-*good* if there exists a constant $T_{\gamma,\beta}$ such that
$$\alpha'(G) \geq \gamma|V(G)| + \beta|E(G)| - T_{\gamma,\beta}$$
holds for every graph $G \in \mathcal{G}_{k,\lambda}$.
- (γ,β) is called (k,λ)-*bad* if it is not (k,λ)-good.
- (γ,β) is called (k,λ)-*tight* if it is (k,λ)-good and there exists a constant $S_{\gamma,\beta}$ such that
$$\alpha'(G) \leq \gamma|V(G)| + \beta|E(G)| - S_{\gamma,\beta}$$
holds for infinitely many graphs $G \in \mathcal{G}_{k,\lambda}$.

If we say that (γ, β) is (k, λ)-tight for a certain subset of $\mathcal{G}_{k,\lambda}$ (for example, the class of λ-regular or k-regular graphs), then we mean that there are infinitely many graphs from this class that satisfy $\alpha'(G) \le \gamma|V(G)| + \beta|E(G)| - S_{\gamma,\beta}$ for some constant $S_{\gamma,\beta}$.

If (γ, β) is (k, λ)-good and $\varepsilon \ge 0$, then there exists a constant $T_{\gamma,\beta}$ such that $\alpha'(G) \ge \gamma|V(G)| + \beta|E(G)| - T_{\gamma,\beta} \ge \gamma|V(G)| + (\beta - \varepsilon)|E(G)| - T_{\gamma,\beta}$, implying that $(\gamma, \beta - \varepsilon)$ is (k, λ)-good where $T_{\gamma,\beta-\varepsilon} = T_{\gamma,\beta}$. We state this formally as follows.

Observation 2 *If (γ, β) is (k, λ)-good and $\varepsilon \ge 0$, then $(\gamma, \beta - \varepsilon)$ is (k, λ)-good.*

The proof of the following lemma is analogous to the proof of Lemma 6 given by the authors in [5]. For completeness, we present the details of the proof in our case.

Lemma 11 *If (γ, β) is (k, λ)-good and $\varepsilon \ge 0$, then both $(\gamma + \varepsilon\lambda, \beta - 2\varepsilon)$ and $(\gamma - \varepsilon k, \beta + 2\varepsilon)$ are (k, λ)-good. Furthermore the following holds.*

(a) *If (γ, β) is (k, λ)-tight for λ-regular graphs, then $(\gamma + \varepsilon\lambda, \beta - 2\varepsilon)$ is (k, λ)-tight.*

(b) *If (γ, β) is (k, λ)-tight for k-regular graphs, then $(\gamma - \varepsilon k, \beta + 2\varepsilon)$ is (k, λ)-tight.*

Proof. Let $G \in \mathcal{G}_{k,\lambda}$ have order n and size m. Since $\delta(G) \ge \lambda(G) \ge \lambda$, we note that $2m \ge \lambda n$. Since (γ, β) is (k, λ)-good, this implies that there exists a constant $T_{\gamma,\beta}$ such that the following also holds for $\varepsilon \ge 0$.

$$\begin{aligned} \alpha'(G) &\ge \gamma \cdot n + \beta \cdot m - T_{\gamma,\beta} \\ &\ge \gamma \cdot n + \beta \cdot m - T_{\gamma,\beta} + \varepsilon(\lambda n - 2m) \\ &= (\gamma + \varepsilon\lambda)n + (\beta - 2\varepsilon)m - T_{\gamma,\beta}, \end{aligned}$$

implying that letting $T_{\gamma+\varepsilon\lambda,\beta-2\varepsilon} = T_{\gamma,\beta}$, the pair $(\gamma + \varepsilon\lambda, \beta - 2\varepsilon)$ is (k, λ)-good. If (γ, β) is (k, λ)-tight for λ-regular graphs, then there exists a constant $S_{\gamma,\beta}$ such that for infinitely many λ-regular graphs, G', in $\mathcal{G}_{k,\lambda}$ we have $\alpha'(G') \le \gamma|V(G')| + \beta|E(G')| - S_{\gamma,\beta}$. Let G' have order n' and size m'. Then, $2m' = \lambda n'$ and, analogously as before, the following holds.

$$\alpha'(G') \le \gamma \cdot n' + \beta \cdot m' - S_{\gamma,\beta} = (\gamma + \varepsilon\lambda)n' + (\beta - 2\varepsilon)m' - S_{\gamma,\beta}.$$

Thus letting $S_{\gamma+\varepsilon\lambda,\beta-2\varepsilon} = S_{\gamma,\beta}$, the pair $(\gamma+\varepsilon\lambda, \beta-2\varepsilon)$ is (k, λ)-tight in this case. This proves part (a). Since $\Delta(G) \le k$, we note that $2m \le kn$. Since (γ, β) is (k, λ)-good, the following holds for all $\varepsilon \ge 0$.

$$\begin{aligned} \alpha'(G) &\ge \gamma \cdot n + \beta \cdot m - T_{\gamma,\beta} \\ &\ge \gamma \cdot n + \beta \cdot m - T_{\gamma,\beta} + \varepsilon(2m - kn) \\ &= (\gamma - \varepsilon k)n + (\beta + 2\varepsilon)m - T_{\gamma,\beta}. \end{aligned}$$

Hence, letting $T_{\gamma-\varepsilon\cdot k,\beta+2\varepsilon} = T_{\gamma,\beta}$, we note that $(\gamma - \varepsilon k, \beta + 2\varepsilon)$ is (k, λ)-good. If (γ, β) is (k, λ)-tight for k-regular graphs, then there exists a constant $S_{\gamma,\beta}$ such that for infinitely many k-regular graphs, G', in $\mathcal{G}_{k,\lambda}$ we have $\alpha'(G') \le \gamma|V(G')| + \beta|E(G')| - S_{\gamma,\beta}$. Let G' have order n' and size m'. Then, $2m' = kn'$ and, analogously as before, the following holds.

$$\begin{aligned} \alpha'(G) &\le \gamma \cdot n' + \beta \cdot m' - S_{\gamma,\beta} \\ &= \gamma \cdot n' + \beta \cdot m' - S_{\gamma,\beta} + \varepsilon(2m' - kn') \\ &= (\gamma - \varepsilon k)n' + (\beta + 2\varepsilon)m' - S_{\gamma,\beta}. \end{aligned}$$

So letting $S_{\gamma-\varepsilon k,\beta+2\varepsilon} = S_{\gamma,\beta}$, the pair $(\gamma - \varepsilon k, \beta + 2\varepsilon)$ is k-tight in this case. This proves part (b). □

We omit the proofs of Lemma 12 and Lemma 13 below since they are identical to the proofs of Lemma 7 and Lemma 8 given by the authors in [5].

Lemma 12 *If (γ, β) is (k, λ)-tight, then both $(\gamma + \varepsilon, \beta)$ and $(\gamma, \beta + \varepsilon)$ are (k, λ)-bad for all $\varepsilon > 0$.*

Lemma 13 *The following holds.*

(a) *If (γ_1, β_1) and (γ_2, β_2) are both (k, λ)-good, then $(\varepsilon\gamma_1 + (1-\varepsilon)\gamma_2, \varepsilon\beta_1 + (1-\varepsilon)\beta_2)$ is also (k, λ)-good for all $0 \le \varepsilon \le 1$.*

(b) *Furthermore if (γ_1, β_1) and (γ_2, β_2) are both (k, λ)-tight for the same infinite class in $\mathcal{G}_{k,\lambda}$, then $(\varepsilon\gamma_1 + (1-\varepsilon)\gamma_2, \varepsilon\beta_1 + (1-\varepsilon)\beta_2)$ is also (k, λ)-tight.*

6.1 λ and k of the Same Parity

In this section, we consider the case when λ and k have the same parity.

Theorem 14 *Assume that $k \ge \lambda \ge 2$ where k and λ have the same parity. For any pair (γ, β), the following holds.*

(a) *If $\gamma \le a_{k,\lambda}$, then (γ, β) is (k, λ)-good if and only if $\beta \le b_{k,\lambda} + 2\frac{a_{k,\lambda}-\gamma}{k}$.*

(b) *If $a_{k,\lambda} \le \gamma \le 1$, then (γ, β) is (k, λ)-good if and only if $\beta \le \frac{\gamma - a_{k,\lambda}}{1-a_{k,\lambda}}(-\frac{1}{\lambda} - b_{k,\lambda}) + b_{k,\lambda}$.*

(c) *If $1 \le \gamma$, then (γ, β) is (k, λ)-good if and only if $\beta \le -\frac{1}{\lambda} - 2\frac{\gamma-1}{\lambda} = \frac{1-2\gamma}{\lambda}$.*

Note that if $\gamma = a_{k,\lambda}$ or $\gamma = 1$, then two of the above conditions hold, but we get the same bound on β in both cases.

Proof. The "only if" part of the theorem follows from Lemma 12. To prove the "if" part of the theorem, let $(\gamma_1, \beta_1) = (a_{k,\lambda}, b_{k,\lambda})$ and let $(\gamma_2, \beta_2) = (1, -\frac{1}{\lambda})$, and note that $\gamma_1 \le \gamma_2$. We first consider the case when $\gamma \le \gamma_1$. By Corollary 1 and Theorem 8(a), (γ_1, β_1) is (k, λ)-tight. By Lemma 11 we note that $(\gamma_1 - \varepsilon k, \beta_1 + 2\varepsilon)$ is (k, λ)-good and (k, λ)-tight for every $\varepsilon \ge 0$. Letting $\varepsilon = \frac{\gamma_1 - \gamma}{k}$ we note that $\gamma = \gamma_1 - \varepsilon k$, and therefore $(\gamma, \beta_1 + 2\varepsilon) = (\gamma, \beta_1 + 2\frac{\gamma_1-\gamma}{k})$ is both (k, λ)-good and (k, λ)-tight. Inserting the values for β_1, γ_1 we obtain the proof for part (a).

We next consider the case when $\gamma_1 \le \gamma \le \gamma_2$. By Theorem 8(d), $\mathcal{F}^d_{k,\lambda}$ is a family of graphs showing that both (γ_1, β_1) and (γ_2, β_2) are (k, λ)-tight. Lemma 13 now implies that the points

$$(\varepsilon\gamma_1 + (1-\varepsilon)\gamma_2, \varepsilon\beta_1 + (1-\varepsilon)\beta_2)$$

are all (k, λ)-good and (k, λ)-tight, for $0 \le \varepsilon \le 1$. Letting $\varepsilon = \frac{\gamma_2 - \gamma}{\gamma_2 - \gamma_1}$ we note that $0 \le \varepsilon \le 1$ and that the points

$$\left(\frac{\gamma_2 - \gamma}{\gamma_2 - \gamma_1}\gamma_1 + \frac{\gamma - \gamma_1}{\gamma_2 - \gamma_1}\gamma_2, \frac{\gamma_2 - \gamma}{\gamma_2 - \gamma_1}\beta_1 + \frac{\gamma - \gamma_1}{\gamma_2 - \gamma_1}\beta_2\right)$$
$$= \left(\gamma, \frac{(\gamma_2 - \gamma)\beta_1 + (\gamma - \gamma_1)\beta_2}{\gamma_2 - \gamma_1}\right) = \left(\gamma, \frac{\gamma - \gamma_1}{\gamma_2 - \gamma_1}(\beta_2 - \beta_1) + \beta_1\right)$$

are all (k, λ)-good and (k, λ)-tight, as $1 - \varepsilon = \frac{\gamma - \gamma_1}{\gamma_2 - \gamma_1}$. Inserting the values for $\beta_1, \beta_2, \gamma_1, \gamma_2$ we obtain the proof for part (b).

Now consider the case when $\gamma \ge \gamma_2$. By Theorem 8(e), $\mathcal{F}^e_{k,\lambda}$ is a family of λ-regular graphs showing that (γ_2, β_2) is (k, λ)-tight. By Lemma 11 we note that $(\gamma_2 + \varepsilon\lambda, \beta_2 - 2\varepsilon)$ is (k, λ)-good and (k, λ)-tight for every $\varepsilon \ge 0$. Letting $\varepsilon = \frac{\gamma - \gamma_2}{\lambda}$ we note that $\gamma = \gamma_2 + \varepsilon\lambda$, and therefore $(\gamma, \beta_2 - 2\varepsilon) = (\gamma, \beta_2 - 2\frac{\gamma - \gamma_2}{\lambda})$ is both (k, λ)-good and (k, λ)-tight. Inserting the values for β_2, γ_2 we obtain the proof for part (c). □

6.2 λ and k of Different Parity

In this section, we consider the case when λ and k have different parity.

Theorem 15 *Assume that $k \geq \lambda \geq 2$ where k and λ have different parity. For any pair (γ, β), the following holds.*

(a) *If $\gamma \leq -e_{k,\lambda}$, then (γ, β) is (k, λ)-good if and only if $\beta \leq d_{k,\lambda} - \frac{2}{k}(e_{k,\lambda} + \gamma)$.*

(b) *If $-e_{k,\lambda} \leq \gamma \leq a_{k,\lambda}$, then (γ, β) is (k, λ)-good if and only if $\beta \leq \frac{\gamma + e_{k,\lambda}}{a_{k,\lambda} + e_{k,\lambda}}(b_{k,\lambda} - d_{k,\lambda}) + d_{k,\lambda}$.*

(c) *If $a_{k,\lambda} \leq \gamma \leq 1$, then (γ, β) is (k, λ)-good if and only if $\beta \leq \frac{\gamma - a_{k,\lambda}}{1 - a_{k,\lambda}}(-\frac{1}{\lambda} - b_{k,\lambda}) + b_{k,\lambda}$.*

(d) *If $1 \leq \gamma$, then (γ, β) is (k, λ)-good if and only if $\beta \leq -\frac{1}{\lambda} - 2\frac{\gamma - 1}{\lambda} = \frac{1 - 2\gamma}{\lambda}$.*

Note that if $\gamma \in \{-e_{k,\lambda}, a_{k,\lambda}, 1\}$, then two of the above conditions hold, but we get the same bound on β in both cases.

Proof. Let $(\gamma_1, \beta_1) = (-e_{k,\lambda}, d_{k,\lambda})$ and $(\gamma_2, \beta_2) = (a_{k,\lambda}, b_{k,\lambda})$ and let $(\gamma_3, \beta_3) = (1, -\frac{1}{\lambda})$ and note that $\gamma_1 \leq 0 \leq \gamma_2 \leq \gamma_3$. First consider the case when $\gamma \leq \gamma_1$. By Theorem 8(b), $\mathcal{F}^b_{k,\lambda}$ is a family of k-regular graphs showing that (γ_1, β_1) is (k, λ)-tight. By Lemma 11 we note that $(\gamma_1 - \varepsilon k, \beta_1 + 2\varepsilon)$ is (k, λ)-good and (k, λ)-tight for every $\varepsilon \geq 0$. Letting $\varepsilon = \frac{\gamma_1 - \gamma}{k}$ we note that $\gamma = \gamma_1 - \varepsilon k$, and therefore $(\gamma, \beta_1 + 2\varepsilon) = (\gamma, \beta_1 + 2\frac{\gamma_1 - \gamma}{k})$ is both (k, λ)-good and (k, λ)-tight. Inserting the values for β_1, γ_1 we obtain the proof for part (a).

Now next consider the case when $\gamma_1 \leq \gamma \leq \gamma_2$. By Theorem 8(c), we note that $\mathcal{F}^c_{k,\lambda}$ is a family of graphs showing that both (γ_1, β_1) and (γ_2, β_2) are (k, λ)-tight. Lemma 13 now implies that the points

$$(\varepsilon\gamma_1 + (1 - \varepsilon)\gamma_2, \varepsilon\beta_1 + (1 - \varepsilon)\beta_2)$$

are all (k, λ)-good and (k, λ)-tight, for $0 \leq \varepsilon \leq 1$. Letting $\varepsilon = \frac{\gamma_2 - \gamma}{\gamma_2 - \gamma_1}$ we note that $0 \leq \varepsilon \leq 1$ and that the points

$$\left(\frac{\gamma_2 - \gamma}{\gamma_2 - \gamma_1}\gamma_1 + \frac{\gamma - \gamma_1}{\gamma_2 - \gamma_1}\gamma_2, \frac{\gamma_2 - \gamma}{\gamma_2 - \gamma_1}\beta_1 + \frac{\gamma - \gamma_1}{\gamma_2 - \gamma_1}\beta_2\right)$$
$$= \left(\gamma, \frac{(\gamma_2 - \gamma)\beta_1 + (\gamma - \gamma_1)\beta_2}{\gamma_2 - \gamma_1}\right) = \left(\gamma, \frac{\gamma - \gamma_1}{\gamma_2 - \gamma_1}(\beta_2 - \beta_1) + \beta_1\right)$$

are all (k, λ)-good and (k, λ)-tight, as $1 - \varepsilon = \frac{\gamma - \gamma_1}{\gamma_2 - \gamma_1}$. Inserting the values for $\beta_1, \beta_2, \gamma_1, \gamma_2$ we obtain the proof for part (b).

Now consider the case when $\gamma_2 \leq \gamma \leq \gamma_3$. Considering the family $\mathcal{F}^d_{k,\lambda}$ given in Theorem 8 (d), we analogously to above observe that the points

$$\left(\gamma, \frac{\gamma - \gamma_2}{\gamma_3 - \gamma_2}(\beta_3 - \beta_2) + \beta_2\right)$$

are all (k, λ)-good and (k, λ)-tight. Inserting the values for $\beta_2, \beta_3, \gamma_2, \gamma_3$ we obtain the proof for part (c).

Now finally consider the case when $\gamma \geq \gamma_3$. By Theorem 8(e), $\mathcal{F}^e_{k,\lambda}$ is a family of λ-regular graphs showing that (γ_3, β_3) is (k, λ)-tight. By Lemma 11 we note that $(\gamma_3 + \varepsilon\lambda, \beta_3 - 2\varepsilon)$ is (k, λ)-good and (k, λ)-tight for every $\varepsilon \geq 0$. Letting $\varepsilon = \frac{\gamma - \gamma_3}{\lambda}$ we note that $\gamma = \gamma_3 + \varepsilon\lambda$, and therefore $(\gamma, \beta_3 - 2\varepsilon) = (\gamma, \beta_3 - 2\frac{\gamma - \gamma_3}{\lambda})$ is both (k, λ)-good and (k, λ)-tight. Inserting the values for β_3, γ_3 we obtain the proof for part (d). □

Theorem 1 now follows from Theorems 14 and 15.

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