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Missov, Trifon I.

*Published in:*  
Demographic Research

*Publication date:*  
2021

*Document version:*  
Final published version

*Document license:*  
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*Citation for pulished version (APA):*

Missov, T. I. (2021). Response Letter to "Gamma-Gompertz life expectancy at birth", Demographic Research, 12 February 2013. *Demographic Research*. <https://www.demographic-research.org/volumes/vol28/9/letter.htm>

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# Response Letter to “Gamma-Gompertz life expectancy at birth”, *Demographic Research*, 12 February 2013

Trifon I. Missov

Interdisciplinary Centre on Population Dynamics, University of Southern Denmark

June 8, 2021

## **An error and a typo in the original manuscript**

The original article contains an incomplete expression, equation (10): to be equal to remaining life expectancy  $e_x$ , the expression on the right-hand-side of (10) needs to be divided by the survival function  $S(x)$  of the gamma-Gompertz model. In this sense, equation (10) in the original manuscript is incorrect, and readers should not use its right-hand side to calculate  $e_x$ . Here, I provide the correct version for  $e_x$  and present an alternative derivation method that is based on the same substitution  $t = 1 - e^{-bx}$  (see the transition from (5) to (6) in the original article) as the one I use to derive expression (2) for life expectancy  $e_0$ . The latter is also used by Castellares et al. (2020). In this Letter, I also correct a typo on p. 266, third (unnumbered) equation from the bottom, in which the interval endpoints for calculating the primitive on the right-hand side are missing. Throughout this Letter, I will use the Roman format for equation numbering to make a clear distinction from the equations in the original article.

## **A correct expression for $e_x$ in a gamma-Gompertz setting**

For a population in which a survival function  $S(x)$  characterizes individual lifetimes, remaining life expectancy at age  $x$  is given by

$$e_x = \frac{1}{S(x)} \int_x^{\infty} S(y) dy. \quad (\text{i})$$

The original article presents an equation for  $\int_x^{\infty} S(y) dy$  only, i.e., equation (10) should be re-written with its correct left-hand side:

$$\int_x^{\infty} S(y) dy = \frac{1}{bk} \left( \frac{b\lambda}{a} e^{-bx} \right)^k {}_2F_1 \left( k, k; k+1; \left( 1 - \frac{b\lambda}{a} \right) e^{-bx} \right). \quad (\text{ii})$$

Applying the linear transformation formula for hypergeometric functions (Abramowitz and Stegun, 1965, p.559, 15.3.4 or 15.3.5), namely,

$${}_2F_1(a, b; c; z) = (1-z)^{-a} {}_2F_1 \left( a, c-b; c; \frac{z}{z-1} \right).$$

or

$${}_2F_1(a, b; c; z) = (1-z)^{-b} {}_2F_1 \left( b, c-a; c; \frac{z}{z-1} \right).$$

for  $a = k, b = k, c = k+1$ , and  $z = \left( 1 - \frac{b\lambda}{a} \right) e^{-bx}$ , yields

$$\int_x^{\infty} S(y) dy = \frac{1}{bk} \left( \frac{1 - \left( 1 - \frac{b\lambda}{a} \right) e^{-bx}}{\frac{b\lambda}{a} e^{-bx}} \right)^{-k} {}_2F_1 \left( k, 1; k+1; \frac{1 - \frac{b\lambda}{a}}{1 - \frac{b\lambda}{a} + \frac{b\lambda}{a} e^{-bx}} \right). \quad (\text{iii})$$

Note that the expression

$$\left( \frac{1 - \left( 1 - \frac{b\lambda}{a} \right) e^{-bx}}{\frac{b\lambda}{a} e^{-bx}} \right)^{-k} = \left( 1 + \frac{a}{b\lambda} (e^{bx} - 1) \right)^{-k}.$$

The right-hand side of the latter represents the gamma-Gompertz survival function. As a result, remaining life expectancy in a gamma-Gompertz setting equals

$$e_x = \frac{1}{bk} {}_2F_1 \left( k, 1; k + 1; \frac{1 - \frac{b\lambda}{a}}{1 - \frac{b\lambda}{a} + \frac{b\lambda}{a} e^{-bx}} \right). \quad (\text{iv})$$

The same expression is reported in Castellares et al. (2020).

## Expressing $e_x$ by applying the derivation procedure for $e_0$

The original article misses the opportunity to derive an expression for  $e_x$  by applying the same substitution as the one used for  $e_0$ , namely  $t = 1 - e^{-bx}$ , under the integral. In fact, it takes a simple substitution  $u = y - x$  to reduce the initial integral to a form that resembles expression (5) in the original article

$$\int_x^\infty S(y) dy = \int_x^\infty \left( 1 + \frac{a}{b\lambda} (e^{by} - 1) \right)^{-k} dy = \int_0^\infty \left( 1 + \frac{a}{b\lambda} (e^{bu} e^{bx} - 1) \right)^{-k} du. \quad (\text{v})$$

Then the aforementioned substitution  $t = 1 - e^{-bu}$  leads to the integral representation of the Gaussian hypergeometric function:

$$\begin{aligned} \int_x^\infty S(y) dy &= \int_0^\infty \left( 1 + \frac{a}{b\lambda} (e^{bu} e^{bx} - 1) \right)^{-k} du = \frac{1}{b} \int_0^\infty \left( 1 + \frac{a}{b\lambda} \left( \frac{e^{bx}}{1-t} - 1 \right) \right)^{-k} \frac{1}{1-t} dt = \\ &= \frac{1}{b} \int_0^\infty (1-t)^{k-1} \left( 1 + \frac{a}{b\lambda} (e^{bx} - 1) - \left( 1 - \frac{a}{b\lambda} \right) t \right)^{-k} dt = \\ &= \frac{1}{b} \left( 1 + \frac{a}{b\lambda} (e^{bx} - 1) \right)^{-k} \int_0^\infty (1-t)^{k-1} \left( 1 - \left( 1 - \frac{a}{b\lambda} \right) t \right)^{-k} dt. \end{aligned}$$

From here, it takes only making the appropriate correspondence with (Abramowitz and Stegun, 1965, p.558, 13.3.1)

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tz)^{-a} dt, \quad \Re(c) > \Re(b) > 0 \quad (\text{vi})$$

to get (iv). Note that the gamma-Gompertz parameters are real positive numbers, and the condition in (vi) reduces to  $c > b > 0$ , which is fulfilled.

## A note on deriving (10) in the original article

The original article contains typos that were unfortunately overlooked in the proofs. In particular, the third equation from the bottom on p. 266 contains such a typo (the interval endpoints for calculating the primitive are missing) that makes the derivation of (10) look somewhat irrational. I also started deriving from the indefinite integral which is unnecessary. Here, I present the derivation once again more neatly. It is less elegant than the one in the previous section, but still holds.

When deriving (10), the original article uses a substitution  $t = e^{-y}$ . It leads to

$$\int_x^\infty S(y) dy = \int_x^\infty \left(1 + \frac{a}{b\lambda} (e^{by} - 1)\right)^{-k} dy = \frac{1}{b} \left(\frac{b\lambda}{a}\right)^k \int_0^{e^{-bx}} t^{k-1} \left(1 - \left(1 - \frac{b\lambda}{a}\right) t\right)^{-k} dt \quad (\text{vii})$$

The interchangeability of the first two arguments of the hypergeometric function, i.e.,  ${}_2F_1(a, b; c; z) = {}_2F_1(b, a; c; z)$  and the relationship (Lebedev, 1965, p. 258)

$$\left(1 - \left(1 - \frac{b\lambda}{a}\right) t\right)^{-k} = {}_2F_1\left(k, C; C; \left(1 - \frac{b\lambda}{a}\right) t\right) \quad \forall C \equiv \text{const} \quad (\text{viii})$$

yield

$$\int_x^\infty S(y) dy = \frac{1}{b} \left(\frac{b\lambda}{a}\right)^k \int_0^{e^{-bx}} t^{k-1} {}_2F_1\left(k, k; k; \left(1 - \frac{b\lambda}{a}\right) t\right) dt. \quad (\text{ix})$$

In addition, the relationship (see <https://functions.wolfram.com/07.23.21.0006.01>)

$$\int z^{c-1} {}_2F_1(a, b; c; z) dz = z^c \Gamma(c) {}_2\tilde{F}_1(a, b; c+1; z) = \frac{z^c}{c} {}_2F_1(a, b; c+1; z), \quad (\text{x})$$

where  ${}_2\tilde{F}_1(a, b; c+1; z) = \frac{1}{\Gamma(c+1)} {}_2F_1(a, b; c+1; z)$  is the regularized hypergeometric function, implies

$$\int_x^\infty S(y) dy = \frac{1}{bk} \left( \frac{b\lambda}{a} \right)^k t^k {}_2F_1 \left( k, k; k+1; \left( 1 - \frac{b\lambda}{a} \right) t \right) \Big|_0^{e^{-bx}}. \quad (\text{xi})$$

By the definition of the Gaussian hypergeometric function, i.e.,

$${}_2F_1(a, b; c; z) = \sum_{j=0}^{\infty} \frac{a(a+1) \dots (a-j+1) b(b+1) \dots (b-j+1)}{c(c+1) \dots (c-j+1) j!} z^j, \quad (\text{xii})$$

we have

$${}_2F_1 \left( k, k; k+1; \left( 1 - \frac{b\lambda}{a} \right) t \right) = \sum_{j=0}^{\infty} C_j t^j, \quad (\text{xiii})$$

where  $C_j = \frac{k \cdot k(k+1) \dots (k-j+1)}{(k-j+2) j!} \left( 1 - \frac{b\lambda}{a} \right)^j$ .

As  $k > 0$ ,

$$\lim_{t \rightarrow 0} t^k \cdot {}_2F_1 \left( k, k; k+1; \left( 1 - \frac{b\lambda}{a} \right) t \right) = \lim_{t \rightarrow 0} \sum_{j=0}^{\infty} C_j t^{j+k} = 0, \quad (\text{xiv})$$

and (xi) reduces to

$$\int_x^\infty S(y) dy = \frac{1}{bk} \left( \frac{b\lambda}{a} e^{-bx} \right)^k {}_2F_1 \left( k, k; k+1; \left( 1 - \frac{b\lambda}{a} \right) e^{-bx} \right). \quad (\text{xv})$$

The expression in the right-hand side of (xv) is equivalent to (10) in the original manuscript.

## Acknowledgements

I express my gratitude to Jesús-Adrián Alvarez for drawing my attention to the research by Castellares et al. (2020), as well as to James W. Vaupel for his comments and suggestions.

## References

- Abramowitz, M. and I. Stegun. 1965. *Handbook of Mathematical Functions*. Washington, DC: US Government Printing Office.
- Castellares, F., Silvio C. Patricio, and Artur J. Lemonte. 2020. "On gamma-Gompertz life expectancy." *Statistics and Probability Letters* 165:108832.
- Lebedev, N.N. 1965. *Special Functions and Their Applications*. Englewood Cliffs, N.J.: Prentice-Hall. Translated from the revised Russian edition (Moscow, 1963) by Richard A. Silverman.