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Revisiting Ellsberg and Machina’s Paradoxes: A Two-Stage Evaluation Model under Ambiguity

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Abstract

In this paper, a two-stage evaluation (TSE) model for decision making under ambiguity is proposed. Events in state space are classified into risky and ambiguous events, which correspond to different types of uncertainty generated by different sources. In this TSE model, uncertainty of two different types are evaluated by DM in different stages. In the first stage, DM evaluates more uncertain consequences of an act locally by applying local subjective expected utility (SEU) models, which are then embedded into the second stage evaluation based on SEU defined globally over all events. To axiomatize such a model, [Kopylov \(2007\)](#)’s “small” domain SEU over risky acts is extended to both risky and non-risky(ambiguous) acts. When evaluating a risky act, TSE model reduces to Savage’s SEU with one stage. When evaluating an ambiguous act, local SEU with a different uncertainty aversion defined on ambiguous events gives TSE model some flexibility in describing preferences. It can be shown that TSE model can accommodate Ellsberg’s paradoxes and Machina’s paradoxes in the literature ([Ellsberg 1961](#), [Machina 2009](#), [2014](#)). When applied to portfolio selection problem, TSE model enjoys some nice properties other models do not have.

Keywords: Ambiguity, Ellsberg Paradox, Machina Paradox, Small domains, Two-stage, Source dependent, Myopic utility, Portfolio selection

1 Introduction

In the literature of decision theory, a rational decision maker (DM) is usually assumed to be an expected utility (EU) maximizer. The EU theory was axiomatized by [von Neumann & Morgenstern \(1944\)](#) for decision making under risk, where objective probabilities are available. However, there are situations where probabilities may not objectively exist. Building on the subjective belief theories developed by [Ramsey \(1964\)](#) and [de Finetti \(1937\)](#), [Savage \(1954\)](#) extended EU to subjective expected utility (SEU) for decision making under uncertainty, where subjective probabilities are derived from preferences. A key axiom used in EU to obtain the “expected utility” form is “independence axiom”; and the corresponding one in SEU is “sure-thing principle”. However, just a few years after the SEU theory was proposed by [Savage \(1954\)](#), [Ellsberg \(1961\)](#) constructed two thought experiments which conjectured appealing preferences that are inconsistent with SEU

to challenge its validity. These thought experiments are referred to as “Ellsberg’s paradoxes” in the literature and these conjectured preferences have been confirmed by many empirically studies (Camerer & Weber 1992, Machina & Siniscalchi 2014).

Specifically, Ellsberg (1961) showed that prevailing preferences would violate Savage’s “sure-thing principle” in a choice problem involving both objective uncertainty and subjective uncertainty. Extensive empirical studies (Camerer & Weber 1992, Machina & Siniscalchi 2014) have documented that most people would prefer betting on acts involving objective uncertainty rather than subjective uncertainty, which is known as “ambiguity aversion”. Motivated by Ellsberg’s paradoxes, many models have been proposed to generalize SEU to accommodate preferences in Ellsberg’s paradox, including the Two-Stage model in Segal (1987), Choquet Expected Utility (CEU) in Schmeidler (1989), Maxmin Expected Utility (MEU) in Gilboa & Schmeidler (1989), Bayesian approach based model in Dobbs (1991), Cumulative Prospect Theory in Tversky & Kahneman (1992), Variational Preferences model (VP) in Maccheroni et al. (2006), α -Maxmin Expected Utility model (α MEU) in Ghirardato et al. (2004), the smooth model of ambiguity aversion in Klibanoff et al. (2005) (KMM), Vector Expected Utility (VEU) in Siniscalchi (2009), Expected Uncertainty Utility (EUU) in Gul & Pesendorfer (2014), among others.

While new models were proposed to accommodate ambiguity averse preferences in Ellsberg’s paradoxes, Machina (2009) constructed two thought experiments, i.e., the 50:51 example and the reflection example, which falsify the CEU in Schmeidler (1989). The preferences in these examples have partially been confirmed in recent empirical studies (L’Haridon & Placido 2010, Aerts et al. 2018). Baillon et al. (2011) referred to these two examples as Machina’s paradoxes and further showed that the preferences in these two paradoxes cannot be accommodated by the other four popular ambiguity models, i.e., MEU, VP, α MEU, and KMM. Therefore, they concluded that Machina’s paradoxes call for effort going beyond the family of models they reviewed. Following these two paradoxes, Machina (2014) further challenged the major existing ambiguity models by showing that these models are also subject to one or more challenges from his conjectured examples involving ambiguity aversion with three or more outcomes.

In a more recent work, Dominiak & Lefort (2020) cast some doubts on the challenge made by Machina’s thought experiments to the above mentioned models. Contrary to the claims by Baillon et al. (2011), it showed that if DM chooses a different way to incorporate exogenous probabilistic information into preferences represented by MEU, VP, α MEU, and KMM, they can indeed accommodate the preferences in Machina’s reflection example. They argued that there are essentially two ways to incorporate exogenous probabilistic information into preferences in ambiguity models. The argument in Baillon et al. (2011) were based on the assumption that preferences over known-probability acts, i.e. acts associated with pure exogenous probabilistic information which are measurable with respect to known-probability events, represented by MEU, VP, α MEU, and KMM are consistent with expected utility theory. However, these models were built with more

general subjective probabilities which do not specify how DM incorporates exogenous probabilistic information into decisions. They showed that if exogenous probabilistic information is incorporated into these models such that preferences over known-probability acts are *probabilistically sophisticated* (PS) (Machina & Schmeidler 1992), the preferences in the reflection example can be indeed accommodated by them.

In this paper, a two-stage evaluation (TSE) model is proposed as a model for decision making under ambiguity, which accommodates both Ellsberg’s and Machina’s paradoxes (Ellsberg 1961, Machina 2009, 2014) without assuming more general PS preferences over known-probability acts. It can be shown that this model can accommodate some preferences that cannot be represented by some major models in the extant literature and enjoys some nice modeling properties compared with these models. In particular, when compared with the KMM model, it is shown that the TSE model enjoys the myopic property in inter-temporal setting discussed in Baucells & Sarin (2019), which makes solving a dynamic programming problem with ambiguity averse DM equivalent to solving a myopic one-period optimization problem. For the KMM model, this property only holds when DM is ambiguity neutral (Baucells & Sarin 2019). In an application to a static portfolio selection problem, it can be shown that in the context considered in the paper TSE allows a larger set of optimal portfolios than KMM, which is essentially due to the fact that TSE can represent preferences that cannot be accommodated by KMM. Actually, it is shown that TSE and KMM overlap in representing preferences, but each also represents preferences that cannot be accommodated by the other. Besides KMM, TSE is also compared with the EUU model which also belongs to the family of two-stage models. The scopes of preferences represented by TSE, KMM, and EUU are summarized in Figure 2 based on discussions in section 4.

In the TSE model, uncertainty of two different types are evaluated by DM in different stages. In the first stage, DM evaluates more uncertain consequences of an act locally by applying local SEU models, which are then embedded into the second stage evaluation based on a SEU defined over less uncertain events. To axiomatize such a model, Kopylov (2007)’s “small” domain SEU over risky acts is extended to both risky and non-risky(ambiguous) acts. Events in state space are classified into risky and ambiguous events, which correspond to different types of uncertainty generated by different sources respectively. An act is risky if its preimage contains only risky events; otherwise it is ambiguous. When evaluating a risky act, TSE model reduces to Savage’s SEU with one stage. When evaluating an ambiguous act, the local SEU with a different uncertainty aversion defined on ambiguous events gives TSE model some flexibility of representing preferences. The two types of uncertainty can be either objective uncertainty vs subjective uncertainty as in Ellsberg’s paradoxes or ambiguities from different sources as in Machina (2014)’s Slightly-Bent Coin problem.

This modeling approach captures a more general phenomenon called source preferences, which is coined in Fox & Tversky (1995) and used to interpret ambiguity aversion. There has been some behavioral and neuroimaging experiments showing that DM’s attitude towards uncertainty depends

on its source. Interested readers are referred to [Chew et al. \(2008\)](#) for a review. These experiments have confirmed that different regions of brain are used by DM when handling objective uncertainty and subjective uncertainty. The model in this paper is consistent with this finding. To capture source-dependent, a roulette-horse lottery is adopted in this paper, which is different from the commonly used [Anscombe & Aumann \(1963\)](#)'s horse-roulette lottery in the extant literature. Such an approach is first adopted by [Jaffray \(1989\)](#) to model ambiguity, which is extended by [Gul & Pesendorfer \(2014\)](#) recently. [Wakker \(2011\)](#) discussed some advantages of this approach and argued that it might be more reasonable to use roulette-horse lottery rather than horse-roulette lottery in modeling ambiguity. The TSE model contributes to this line of the literature. Regarding the issue of incorporating exogenous probabilistic information into preferences discussed in [Dominiak & Lefort \(2020\)](#), TSE is consistent with EU for known-probability acts. However, the idea of two-stage evaluation can be extended by embedding ([Machina & Schmeidler 1992](#))'s probabilistically sophisticated preferences in a "small domain" probabilistic sophistication theory ([Kopylov 2007](#)), which is beyond the scope of this paper. The main text of the paper focuses on discussing the preferences represented by the TSE model and its potential applications. The axiomatic foundation of the model is presented in the online appendix of the paper.

2 Motivating TSE by Revisiting Ellsberg and Machina's Paradoxes

2.1 Ellsberg's Three Color Urn Paradox

The Three-Color [Ellsberg \(1961\)](#)'s paradox is a thought experiment involving an urn containing 90 balls. It is known that there are exactly 30 red balls and each of the other 60 balls is either black or yellow. But the exact number of black or yellow balls is unknown. A ball is drawn from the urn, and a DM is presented with two pairs of bets, whose payoffs are contingent on the color of the ball drawn from the urn. Specifically, the two pairs of bets offered (f_1 vs g_1 and f_2 vs g_2) are listed in [Table 1](#). [Ellsberg \(1961\)](#) conjectured that most people would prefer f_1 to g_1 and g_2 to f_2 , which was also confirmed later by experiments.¹ However, according to Savage's sure-thing principle (STP), the preference should be independent of the common consequence on the state of yellow ball when comparing f_1 and g_1 as well as f_2 and g_2 . Under the STP, $f_1 \succeq g_1$ if and only if $f_2 \succeq g_2$. Therefore, when f_1 and g_1 become f_2 and g_2 after replacing \$0 by \$10 paid on yellow, the preference should not be reversed.

Different explanations have been proposed in the literature to address Ellsberg's paradox as reviewed in [section 1](#). This paper provides an alternative explanation to the preference pattern in the Three-Color-Ellsberg's paradox based on TSE process. In the first stage, for any

¹See review of experimental study on preferences in Ellsberg's paradox in [Machina & Siniscalchi \(2014\)](#)

Table 1: Three Color Ellsberg Paradox

Acts/states		30		60	
		Red(R)	Black(B)	Black(B)	Yellow(Y)
f_1	\succ	\$ 10	\$ 0	\$ 0	\$ 0
g_1		\$ 0	\$ 10	\$ 0	\$ 0
f_2	\succ	\$ 10	\$ 0	\$ 10	\$ 10
g_2		\$ 0	\$ 10	\$ 10	\$ 10

act f , DM evaluates the sub-act restricted to event $\{B, Y\}$ by a local SEU v conditional on this event. This stage gives a certainty equivalent (CE) of the act on the event $\{B, Y\}$, i.e. $CE = v^{-1}(p(B)v(f(B)) + p(Y)v(f(y)))$. In the second stage, the consequences of the act on event $\{B, Y\}$ is replaced with this CE , and f is evaluated by using a global utility u as follows

$$\begin{aligned} u(f) &= P(R)u(f(R)) + P(B \cup Y)u(CE) \\ &= P(R)u(f(R)) + P(B \cup Y)u\left(v^{-1}\left(p(B)v(f(B)) + p(Y)v(f(y))\right)\right) \end{aligned}$$

To apply this model, suppose that the sub-acts g_1 and f_2 restricted to $\{B, Y\}$ are evaluated to be indifferent due to DM's symmetric beliefs on the proportion of B and Y , whose CE s are denoted by δ . Using the model above, the acts are evaluated as:

$$\begin{aligned} U(f_1) &= \frac{1}{3}u(10) + \frac{2}{3}u(0), \quad U(g_1) = \frac{1}{3}u(0) + \frac{2}{3}u(\delta) \\ U(f_2) &= \frac{1}{3}u(10) + \frac{2}{3}u(\delta), \quad U(g_2) = \frac{1}{3}u(0) + \frac{2}{3}u(10) \end{aligned}$$

Clearly, $U(f_1) \geq U(g_1) \iff U(g_2) \geq U(f_2)$, which represents the preferences exhibited in Table 1. This TSE model captures the idea that DM treats different uncertainties, i.e. uncertainties from different sources, separately by applying different SEU models.

2.2 Definition of TSE

Following [Kopylov \(2007\)](#), consider an infinite state space S and a set of deterministic consequences $X \subset \mathbb{R}$. The set of events Σ is a collection of subsets of S , which is assumed to be an *algebra*. A Savage's simple act f is a function $f : S \rightarrow X$ which is Σ -measurable with finite range. Any consequence $x \in X$ is identified as a constant act $f(s) = x, \forall s \in S$. By abusing the notation, $x \in X$ is also used to denote a constant act that pays x in all states. The set of all simple acts is denoted by \mathcal{F} . The DM's preference over all simple acts is a binary relation \succeq , which is assumed to be a continuous weak order.

Assume that there exists a collection of risky events, denoted by $\mathcal{R} \subset 2^S$. Denote the collection of ambiguous events by $\Sigma \setminus \mathcal{R}$. An act $f \in \mathcal{F}$ is risky if it is \mathcal{R} -measurable, namely $f^{-1}(x) \in \mathcal{R}$ for all $x \in X$; otherwise, call it ambiguous act. Let $\mathcal{G} \subset \mathcal{F}$ denote the set of all risky acts. Following

the convention, for any $A \in \Sigma$, the notation fAg is used to denote an act, which has consequences given by $f(s)$ for $s \in A$ and $g(s)$ for $s \in A^c$. The axiomatic theory for the TSE model is presented in detail in the online appendix of the paper, where it is shown that there exists an endogenous partition of S denoted by $\{H_i\}_{i \in I}^f$ with $H_i \in \mathcal{R}$ associated with any ambiguous act f .² DM's preferences are separable on all H_i . Due to the separability, it is assumed that in the **first stage** DM evaluates the certainty equivalent of f restricted to each H_i based on a local SEU with utility index v_i without considering the consequences of f on the complement of H_i .

$$c_f^{H_i} = v_i^{-1} \left(\sum_{x \in X} v_i(x) p_i(f(x)^{-1} \cap H_i) \right) \quad (1)$$

where p_i and v_i are local probability measure and utility index in [Savage \(1954\)](#)'s SEU theory conditional on H_i respectively. Using these certainty equivalents, denoted by $c_f^{H_i}$ on each H_i , DM converts an ambiguous act f to an act \hat{f} which pays a constant consequence $c_f^{H_i}$ on each H_i . Such an \hat{f} is indifferent to f for the DM by construction.³ In the **second stage**, a global SEU with utility index u is used to evaluate the act f through evaluating \hat{f} by aggregating utilities of certainty equivalents on each H_i 's. This two-stage evaluation process results in a functional $U : \mathcal{F} \rightarrow \mathbb{R}$ representing preference \succeq as follows.

$$U(f) = \sum_{H_i \in \{H_i\}_{i \in I}^f} u(c_f^{H_i}) P(H_i) \quad (2)$$

where probability measure P is defined over risky events based on [Kopylov \(2007\)](#)'s "small" domain SEU theory. Embedding the first stage evaluation (1) into the second stage evaluation (2) delivers the TSE model below.

$$U(f) = \sum_{H_i \in \{H_i\}_{i \in I}^f} \phi_i \left(\sum_{x \in X} v_i(x) p_i(f(x)^{-1} \cap H_i) \right) P(H_i) \quad (3)$$

where $\phi_i := u \circ v_i^{-1}$. The utility u is a global utility representing DM's attitude towards less uncertain (risky) events, and each v_i represents the attitude towards the more uncertain (ambiguous) events which is hedge dependent. Therefore, ambiguity aversion is captured by v_i 's that are more concave than u on each hedge, which implies ϕ_i 's are convex under ambiguity aversion. Loosely speaking, hedges are events on which DM has enough information to assign probabilities, whereas the sub-events contained in hedge are events on which DM is unable to assign probabilities.⁴ The intuition is that separability (independence) of preferences only applies to the same class of events, either risky or ambiguous, but not across events from two classes. This modeling approach also allows DM's ambiguity aversion varies across different H_i . The descriptive flexibility and the hedge dependent ambiguity aversion are further discussed in sections 3 and 4.

²When the hedge partition associated with all acts under consideration are the same, the superscript f is dropped for simplicity.

³See online appendix for the definition and proof of existence of \hat{f} .

⁴The rigorous definitions of these two types of events are discussed in the online appendix.

2.3 Machina's Paradoxes

In this subsection, Machina's paradoxes (Machina 2009, 2014) are revisited to show that TSE is capable of accommodating the preferences in these paradoxes. To accommodate these preferences, it is enough to assume v_i and hence ϕ_i are hedge independent and the conditional probabilities on each hedge are uniform. Such a TSE model is a special case of the general TSE model (3) under assumptions discussed in the online appendix.

Machina (2009) conjectured some examples which falsify CEU in Schmeidler (1989). Baillon et al. (2011) further found that the other four major models, i.e., MEU (Gilboa & Schmeidler 1989), VP (Maccheroni et al. 2006), α M (Ghirardato et al. 2004), and KMM (Klibanoff et al. 2005) also fail to accommodate the preferences in Machina's 50:51 example shown in Table 2.

Table 2: Machina's 50:51 Paradox

Acts/Events	50 balls		51 balls	
	E_1	E_2	E_3	E_4
f_1	\$ 8k	\$ 8k	\$ 4k	\$ 4k
f_2 \succ	\$ 8k	\$ 4k	\$ 8k	\$ 4k
f_3	\$ 12k	\$ 8k	\$ 4k	\$ 0k
f_4 \succ	\$ 12k	\$ 4k	\$ 8k	\$ 0k

It is easy to verify that all acts in the table share the same hedge partition $\{H_i\}_{i \in \{1,2\}}$ with $H_1 = \{E_1, E_2\}$ and $H_2 = \{E_3, E_4\}$. Using the model (3), the following evaluations are obtained

$$\begin{aligned}
 U(f_1) &= \frac{50}{101} \phi \left(\frac{1}{2} v(8) + \frac{1}{2} v(8) \right) + \frac{51}{101} \phi \left(\frac{1}{2} v(4) + \frac{1}{2} v(4) \right) = \frac{50}{101} u(8) + \frac{51}{101} u(4) \\
 U(f_2) &= \frac{50}{101} \phi \left(\frac{1}{2} v(8) + \frac{1}{2} v(4) \right) + \frac{51}{101} \phi \left(\frac{1}{2} v(8) + \frac{1}{2} v(4) \right) = \phi \left(\frac{1}{2} v(8) + \frac{1}{2} v(4) \right) \\
 U(f_3) &= \frac{50}{101} \phi \left(\frac{1}{2} v(12) + \frac{1}{2} v(8) \right) + \frac{51}{101} \phi \left(\frac{1}{2} v(4) + \frac{1}{2} v(0) \right) \\
 U(f_4) &= \frac{50}{101} \phi \left(\frac{1}{2} v(12) + \frac{1}{2} v(4) \right) + \frac{51}{101} \phi \left(\frac{1}{2} v(8) + \frac{1}{2} v(0) \right)
 \end{aligned}$$

The evaluation of f_1 is totally determined by the risk aversion captured by utility function u . In contrast, the evaluation of f_2 is determined by the ambiguity aversion captured by v . Suppose the DM's certainty equivalent for the ambiguous sub-lottery $\{8,4\}$ is 5, namely $\phi \left(\frac{1}{2} v(8) + \frac{1}{2} v(4) \right) = \phi(v(5)) = u(5)$. Notice that f_1 has an expectation close to that of f_2 , i.e. $\mathbb{E}f_1 = \frac{604}{101}$ and $\mathbb{E}f_2 = \frac{600}{100}$. Suppose the DM is ambiguity averse, i.e., v is more concave than u . It is possible that her certainty equivalent for the risky lottery equals to 5.5, namely $\frac{50}{101} u(8) + \frac{51}{101} u(4) = u(5.5)$. Thus, the DM exhibits the preference $f_1 \succeq f_2$.

For f_3 and f_4 , it is intuitive to assume that the sub-act $\{E_1, 12; E_2, 8\}$ should have a higher certainty equivalent than $\{E_1, 12; E_2, 4\}$; and $\{E_3, 4; E_4, 0\}$ should have a lower certainty equivalent than $\{E_3, 8; E_4, 0\}$. For instance, suppose the DM evaluates the certainty equivalents for these

ambiguous sub-acts on each hedge in the following way

$$\{E_1, 12; E_2, 8\} \sim \{E_1, 9; E_2, 9\}; \{E_3, 4; E_4, 0\} \sim \{E_3, 1; E_4, 1\};$$

$$\{E_1, 12; E_2, 4\} \sim \{E_1, 7; E_2, 7\}; \{E_3, 8; E_4, 0\} \sim \{E_3, 3; E_4, 3\}$$

Then, acts f_3 and f_4 are evaluated as indifferent to the two respective lotteries below

$$f_3 \sim \left\{ \frac{50}{101}, 9; \frac{51}{101}, 1 \right\}; \quad f_4 \sim \left\{ \frac{50}{101}, 7; \frac{51}{101}, 3 \right\}.$$

For a risk averse DM, it is possible that she prefers f_4 to f_3 , which requires u to satisfy

$$U(f_3) = \frac{50}{101}u(9) + \frac{51}{101}u(1) < \frac{50}{101}u(7) + \frac{51}{101}u(3) = U(f_4)$$

Therefore, the preference $f_3 \preceq f_4$ is likely to be produced by DM's risk aversion who is trying to avoid the larger variance in the lottery induced by f_3 ; while the preference $f_1 \succeq f_2$ is due to her ambiguity aversion.

Following the similar reasoning, it can be easily shown that [Machina \(2009\)](#)'s "reflection paradox" can also be accommodated by the TSE model. In a more recent paper, [Machina \(2014\)](#) argued that it is reasonable that a DM may exhibit different attitudes towards ambiguity at different levels of outcomes. He showed that none of the four popular models of decision making under ambiguity, namely MEU ([Gilboa & Schmeidler 1989](#)), CEU ([Schmeidler 1989](#)), KMM ([Klibanoff et al. 2005](#)), and VP ([Maccheroni et al. 2006](#)), allows such outcome level dependent ambiguity aversion. In the following discussion, it is shown that the TSE model allows this dependence, which is illustrated by using [Machina \(2014\)](#)'s "ambiguity at low versus at high outcomes problem" shown in [Table 3](#).

Table 3: Ambiguity at low versus at high outcomes problem

	1 ball	1 ball	1 ball		
b_0	\$0	\$c	\$100		
Urn I			Urn II		
2 balls		1 ball	vs	1 ball	2 balls
Black	White	Red		Red	Black White
\$0	\$c	\$100		\$0	\$c \$100

Note: \$c = your certainty equivalent of the objective bet (\$0, 1/2; \$100, 1/2)

This problem consists of a choice between two bets on Urn I vs Urn II, each can be constructed from the pure objective bet $b_0 = \{\frac{1}{3}, \$0; \frac{1}{3}, \$c; \frac{1}{3}, \$100\}$, where \$c is the certainty equivalent of an objective bet $\{\frac{1}{2}, \$0; \frac{1}{2}, \$100\}$. Although, the bet on Urn I and II are both obtained by adding the same ambiguity to the original non-ambiguous bet b_0 , the ambiguity is added at different outcome levels. [Machina \(2014\)](#) argued that DM may prefer to bet on Urn II rather than to bet on Urn I, as risk averter may prefer bearing the same risk at higher outcome levels (decreasing risk aversion).

In this problem, the natural state space is $S_{\mathcal{N}} = \{B_I, W_I, R_I\} \times \{B_{II}, W_{II}, R_{II}\}$. It is also easy to verify that both bets (acts) share the same hedge partition $\{H_i\}_{i \in I} = \{H_1, H_2, H_3, H_4\}$ where

$$\begin{aligned} H_1 &= \{B_I B_{II}, B_I W_{II}, W_I B_{II}, W_I W_{II}\} & H_2 &= \{R_I B_{II}, R_I W_{II}\} \\ H_3 &= \{B_I R_{II}, W_I R_{II}\} & H_4 &= \{R_I R_{II}\} \end{aligned}$$

The objective probabilities are $P(H_1) = \frac{4}{9}, P(H_2) = \frac{2}{9}, P(H_3) = \frac{2}{9}, P(H_4) = \frac{1}{9}$. For simplicity, assume that the DM exhibits risk neutral attitude towards lotteries involving objective probabilities, which implies that the certainty equivalent $c = 50$. The two acts are evaluated by the TSE model (3) as

$$\begin{aligned} U(\text{Urn I}) &= \frac{2}{3}\phi\left(\frac{1}{2}v(0) + \frac{1}{2}v(50)\right) + \frac{1}{3}\phi(v(100)) \\ U(\text{Urn II}) &= \frac{2}{3}\phi\left(\frac{1}{2}v(50) + \frac{1}{2}v(100)\right) + \frac{1}{3}\phi(v(0)) \end{aligned}$$

Let $c_l = v^{-1}(\frac{1}{2}v(0) + \frac{1}{2}v(50))$ and $c_h = v^{-1}(\frac{1}{2}v(50) + \frac{1}{2}v(100))$. Clearly, a decreasing absolute risk aversion (DARA) utility function v can model the decreasing ambiguity aversion conjectured by Machina (2014).

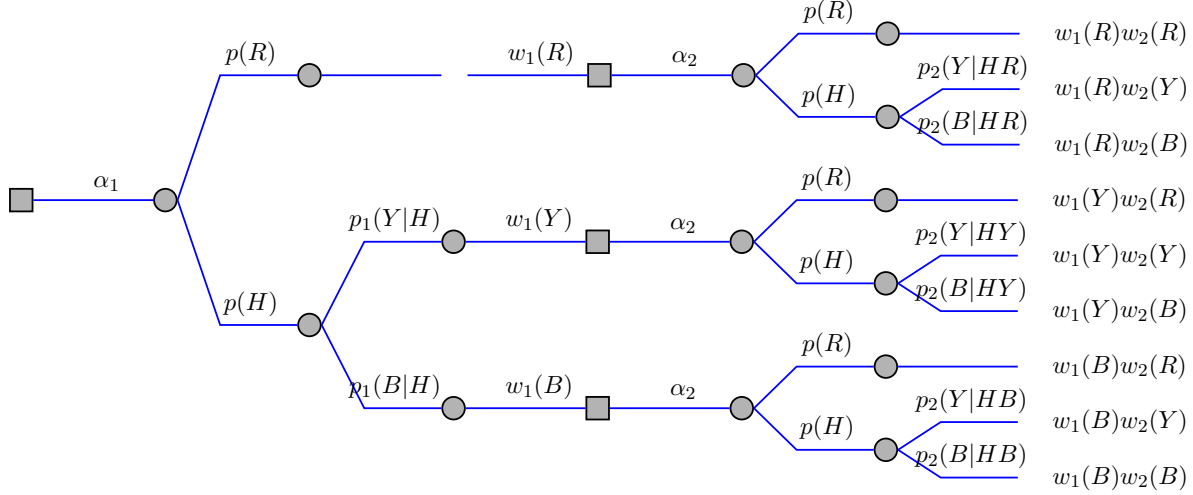
Finally, it can also be shown that the preference in the ‘‘Slightly-Bent Coin Problem’’ conjectured by Machina (2014) involving two different sources of ambiguity can also be accommodated by the TSE model by following similar arguments above when an appropriate $S_{\mathcal{N}}$ is chosen.

3 Applications in Portfolio Selection

3.1 Myopic Property of TSE

Consider betting on Ellsberg’s urn twice. Suppose that DM’s initial wealth level is C , she chooses α_1 percent of initial wealth to bet on the color of a ball drawn from an Ellsberg’s urn in her first bet. If a Black (B) ball is drawn, she receives $\theta_B > 1$ times her bet; if a Yellow (Y) ball is drawn, she receives $\theta_Y > \theta_B > 1$ times her bet; if Red (R) is drawn, she loses her bet. Therefore, after the first bet, her wealth becomes $w_1(R, \alpha_1) = C(1 - \alpha_1)$ if R is drawn, $w_1(Y, \alpha_1) = C(1 - \alpha_1 + \alpha_1\theta_Y)$ if Y is drawn, and $w_1(B, \alpha_1) = C(1 - \alpha_1 + \alpha_1\theta_B)$ if B is drawn. After the uncertainty of the first bet is resolved, the DM makes a second bet α_2 as a percentage of her wealth after receiving prize from the first bet. Both the composition of the urn and the payment rule are unchanged. Define $w_2(R, \alpha_2) = (1 - \alpha_2)$, $w_2(Y, \alpha_2) = (1 - \alpha_2 + \alpha_2\theta_Y)$, and $w_2(B, \alpha_2) = (1 - \alpha_2 + \alpha_2\theta_B)$. After the second bet, her wealth becomes $w_1(s_1, \alpha_1)w_2(s_2, \alpha_2)$ depending on the event path (s_1, s_2) where $s_1, s_2 \in \{R, Y, B\}$. The DM may update her subjective belief on the composition of the urn (the proportion of the Yellow and Black ball which has 60 in total) after the uncertainty in the first bet is resolved. The structure of this decision making problem is illustrated in Figure 1, where the arguments α_1 and α_2 are suppressed in w_1 and w_2 for simplicity.

Figure 1: Bet on Ellsberg's Urn in Two Periods



Note: $p(R) = \frac{1}{3}, p(H) = \frac{2}{3}$; $p_1(Y|H) = p, p_1(B|H) = 1 - p$; $p_2(Y|HR) = p; p_2(B|HR) = 1 - p$; $p_2(Y|HY) = q; p_2(B|HY) = 1 - q$; $p_2(Y|HB) = r; p_2(B|HB) = 1 - r$; and $q > p > r$. $\theta_Y > \theta_B > 1$. $w_1(R) = C(1 - \alpha_1)$, $w_1(Y) = C(1 - \alpha_1 + \alpha_1\theta_Y)$, $w_1(B) = C(1 - \alpha_1 + \alpha_1\theta_B)$, $w_2(R) = (1 - \alpha_2)$, $w_2(Y) = (1 - \alpha_2 + \alpha_2\theta_Y)$, $w_2(B) = (1 - \alpha_2 + \alpha_2\theta_B)$.

As TSE is not a Bayesian model, the DM may not apply the Bayesian rule to update beliefs. In general, the updating is modeled as correlation of subjective probabilities in the following way. Denote by p the subjective probability that a Yellow ball is drawn conditional on the hedge event $H = \{B, Y\}$ in the first bet, i.e. $p = p_1(Y|H)$. Denote by q (r) the probability that a Yellow (Black) ball is drawn conditional on the hedge event $H = \{B, Y\}$ in the second bet and a Yellow (Black) ball in the first bet, i.e. $q = p_2(Y|HY)$ ($r = p_2(Y|HB)$). When a Red ball is drawn in the first bet, DM has no evidence to update her information on the composition of the urn. Therefore, her subjective probabilities about the composition will not be updated, i.e. $p_2(Y|HR) = p$. It is assumed that $q > p > r$, which implies that if the DM observes a Yellow ball in the first bet, she increases the subjective probability of drawing a Yellow ball from p to q in the second bet; if she observes a Black ball in the first bet, she decreases her subjective probability of drawing a Yellow ball from p to r in the second bet.

The intertemporal TSE model is dynamically consistent (He 2020), which allows the use of backward induction (rolling back decision tree) approach. The *global optimal solution* of the problem can be obtained by solving the two-period betting problem backward. In the second period, the DM solves the following problem after she chooses α_1 and observes the state $s_1 \in \{R, Y, B\}$ resolved from the first bet.

$$\max_{\alpha_2} U_2(s_1, \alpha_1, \alpha_2) = \frac{2}{3} \phi \left(p_2(Y|Hs_1)v(w_1(s_1, \alpha_1)w_2(Y, \alpha_2)) + p_2(B|Hs_1)v(w_1(s_1, \alpha_1)w_2(B, \alpha_2)) \right) + \frac{1}{3} \phi \left(v(w_1(s_1, \alpha_1)w_2(R, \alpha_2)) \right)$$

Denote by $\alpha_2^*(s_1, \alpha_1)$ the optimal bet and $U_2^*(s_1, \alpha_1) := U_2(s_1, \alpha_1, \alpha_2^*(s_1, \alpha_1))$ the optimal utility solved from the problem above. Denote by $CE(s_1, \alpha_1) := u^{-1}(U_2^*(s_1, \alpha_1))$ the certainty equivalent from the optimal second bet. In the first period, the DM solves the following problem.

$$\max_{\alpha_1} U_1(\alpha_1) = \frac{2}{3}\phi\left(p_1(Y|H)v(CE(Y, \alpha_1)) + p_1(B|H)v(CE(B, \alpha_1))\right) + \frac{1}{3}u(CE(R, \alpha_1)) \quad (4)$$

It can be shown that when $v(x) = \log_b(x)$ with $b > 0$ and $\phi(x) = \exp(x)$, the myopic optimal solution of the first period is also global optimal. Specifically, under these utility functions,

$$\begin{aligned} & U_2(s_1, \alpha_1, \alpha_2) \\ &= \frac{2}{3}\exp\left(p_2(Y|Hs_1)\log_b Cw_1w_2(Y, \alpha_2) + p_2(B|Hs_1)\log_b Cw_1w_2(B, \alpha_2)\right) + \frac{1}{3}\exp\left(\log_b Cw_1w_2(R, \alpha_2)\right) \\ &= \frac{2}{3}\exp\left(\log_b Cw_1 + p_2(Y|Hs_1)\log_b w_2(Y, \alpha_2) + p_2(B|Hs_1)\log_b w_2(B, \alpha_2)\right) + \frac{1}{3}\exp\left(\log_b Cw_1 + \log_b w_2(R, \alpha_2)\right) \\ &= \exp(\log_b Cw_1) \left(\frac{2}{3}\exp\left(p_2(Y|Hs_1)\log_b w_2(Y, \alpha_2) + p_2(B|Hs_1)\log_b w_2(B, \alpha_2)\right) + \frac{1}{3}\exp\left(\log_b w_2(R, \alpha_2)\right) \right) \\ &= \exp(\log_b Cw_1) U_2^0(s_1, \alpha_2) \end{aligned}$$

where $U_2^0(s_1, \alpha_2) := \frac{2}{3}\exp(p_2(Y|Hs_1)\log_b w_2(Y, \alpha_2) + p_2(B|Hs_1)\log_b w_2(B, \alpha_2)) + \frac{1}{3}\exp(\log_b w_2(R, \alpha_2))$, $w_1 = w_1(s_1, \alpha_1)$. In general, α_2^* depends on α_1 . However, in this case, maximizing U_2 is equivalent to maximizing U_2^0 , which makes α_2 only depend on s_1 , i.e. $\alpha_2^*(s_1)$. Consequently, $U_2^* = \exp(\log_b Cw_1) U_2^0(s_1, \alpha_2^*(s_1))$.

Since $u = \phi \circ v$, $u^{-1} = v^{-1} \circ \phi^{-1}$. It follows that

$$CE(s_1, \alpha_1) = u^{-1}(U_2^*) = b^{\ln(\exp(\log_b Cw_1) U_2^0(s_1, \alpha_2^*))} = Cw_1(s_1, \alpha_1) b^{\ln(U_2^0(s_1, \alpha_2^*))}$$

In the first bet, the DM solves the problem below

$$\begin{aligned} & \max_{\alpha_1} U_1(\alpha_1) \\ &= \frac{2}{3}\phi\left(p_1(Y|H)v\left(Cw_1(Y, \alpha_1)b^{\ln(U_2^0(Y, \alpha_2^*))}\right) + p_1(B|H)v\left(Cw_1(B, \alpha_1)b^{\ln(U_2^0(B, \alpha_2^*))}\right)\right) \\ &+ \frac{1}{3}\phi\left(v\left(Cw_1(R, \alpha_1)b^{\ln(U_2^0(R, \alpha_2^*))}\right)\right) \\ &= \frac{2}{3}\exp\left(p_1(Y|H)\log_b Cw_1(Y, \alpha_1) + p_1(B|H)\log_b Cw_1(B, \alpha_1) + \ln(U_2^0(Y, \alpha_2^*))^{p_1(Y|H)} (U_2^0(B, \alpha_2^*))^{p_1(B|H)}\right) \\ &+ \frac{1}{3}\exp\left(\log_b(Cw_1(R, \alpha_1)) + \ln(U_2^0(R, \alpha_2^*))\right) \\ &= \frac{2}{3}\exp\left(p_1(Y|H)\log_b Cw_1(Y, \alpha_1) + p_1(B|H)\log_b Cw_1(B, \alpha_1)\right) (U_2^0(Y, \alpha_2^*))^{p_1(Y|H)} (U_2^0(B, \alpha_2^*))^{p_1(B|H)} \\ &+ \frac{1}{3}\exp\left(\log_b(Cw_1(R, \alpha_1))\right) (U_2^0(R, \alpha_2^*)) \end{aligned}$$

It is easy to observe that when $(U_2^0(Y, \alpha_2^*))^{p_1(Y|H)} (U_2^0(B, \alpha_2^*))^{p_1(B|H)} = (U_2^0(R, \alpha_2^*))$, the problem above reduces to

$$\begin{aligned} & \max_{\alpha_1} U_1(\alpha_1) \\ &= \left(\frac{2}{3}\exp\left(p_1(Y|H)\log_b Cw_1(Y, \alpha_1) + p_1(B|H)\log_b Cw_1(B, \alpha_2)\right) + \frac{1}{3}\exp\left(\log_b(Cw_1(R, \alpha_1))\right) \right) U_2^0(R, \alpha_2^*) \end{aligned}$$

which is equivalent to solving a *myopic problem* in the first bet below

$$\begin{aligned} & \max_{\alpha_1} U^M(\alpha_1) \\ & = \frac{2}{3} \exp\left(p_1(Y|H) \log_b C w_1(Y, \alpha_1) + p_1(B|H) \log_b C w_1(B, \alpha_2)\right) + \frac{1}{3} \exp\left(\log_b(C w_1(R, \alpha_1))\right) \end{aligned}$$

The condition $(U_2^0(Y, \alpha_2^*))^{p_1(Y|H)} (U_2^0(B, \alpha_2^*))^{p_1(B|H)} = (U_2^0(R, \alpha_2^*))$ defines a class of subjective probabilities $p = p_1(Y|H) = p_2(Y|HR)$, $q = p_2(Y|HY)$, $r = p_2(Y|HB)$ in the two bets such that this myopic property holds. Recall that α_2^* is a function of s_1 , which depends on s_1 through the conditional probabilities $p_2(Y|Hs_1)$ and $p_2(B|Hs_1) = 1 - p_2(Y|Hs_1)$ for $s_1 \in \{R, Y, B\}$. Therefore, α_2^* can be written as functions of p, q, r , namely $\alpha_2^*(R) = \alpha_2^*(p)$, $\alpha_2^*(Y) = \alpha_2^*(q)$ and $\alpha_2^*(B) = \alpha_2^*(r)$. The utility $U_2^0(s_1, \alpha_2)$ can be written as a function of these conditional probabilities p, q, r . Consequently, the condition reduces to

$$\left(U_2^0(q, \alpha_2^*(q))\right)^p \left(U_2^0(r, \alpha_2^*(r))\right)^{(1-p)} = U_2^0(p, \alpha_2^*(p)) \quad (5)$$

where $U_2^0(p, \alpha_2^*(p)) \equiv \max_{\alpha_2} U_2^0(p, \alpha_2)$; and for any $p \in [0, 1]$

$$U_2^0(p, \alpha_2) = \frac{2}{3} \exp\left(p \cdot \log_b w_2(Y, \alpha_2) + (1-p) \cdot \log_b w_2(B, \alpha_2)\right) + \frac{1}{3} \exp\left(\log_b w_2(R, \alpha_2)\right)$$

Thus, $U_2^0(p, \alpha_2^*(p))$ is the value function obtained from solving a parameterized family of optimization problems. Using envelope theorem, it is easy to verify the value function is continuous and the derivative $\frac{\partial U_2^0(p, \alpha_2^*(p))}{\partial p} > 0$ for any $p \in [0, 1]$.

It is easy to see that when $p = q = r$, the condition holds trivially. Therefore, when there is no correlation between probabilities from bets in different periods, the myopic property holds. When correlation exists, the condition can be used to find the class of probabilities so that the myopic property holds. For any given p , pick $r < p$. The existence of a unique q is implied by the condition (5) based on the intermediate value theorem.⁵ Therefore, TSE holds the “*myopic property*” under correlated beliefs in two periods even when DM is not ambiguity neutral (ϕ is not linear). In contrast, KMM does not have this myopic property when beliefs are correlated unless DM is ambiguity neutral, which makes KMM reduce to SEU (see theorem 4 in [Baucells & Sarin \(2019\)](#)). This myopic property is very nice for modeling decision analysis problems. It simplifies solving a dynamic programming problem to solving a static one period problem. Following the reasoning above, the statement below can be obtained.

Proposition 1. (Myopic Property of TSE in Two-Period Betting on Ellsberg’s Urn)
*In a two period betting on Ellsberg’s Urn, when $\phi(x) = \exp(x)$ and $v(x) = \log_b(x)$ for $b > 0$, the *myopic proerty* holds when the conditional probabilities p, q , and r satisfy the condition (5).*

Proof of Proposition 1 The proof follows the analysis above for $v(x) = \log_b(x)$ and $\phi(x) = \exp(x)$ when beliefs are correlated. ■

⁵Thus, q can be solved numerically.

3.2 Static portfolio selection under TSE

This subsection shows how to apply TSE to a static portfolio selection problem. A simple numerical example shows that the optimal portfolios selected based on TSE contains the portfolios selected based on KMM as a subset in the special case considered in the paper. The more detailed comparison between TSE and KMM in term of the scope of preferences is further discussed in subsection 4.1.

Imagine that an investor who was making investing decisions on stock market in the beginning of Jan 2020. One of the biggest relevant uncertainty would be whether a trade war deal could be signed by US and China. The investor believed that if a deal was signed, the market would go up; otherwise, it would go down. Given each scenario, she might also have some beliefs about how market could go up and down respectively. If there is no deal, it is hard to predict how trade war would go further, it is unclear how higher and wider the further tariff could be imposed by each side. Therefore, the investor might feel that the no deal scenario would make the market become more ambiguous than the scenario with a deal. She also probably had different ambiguity aversions towards the scenario with deal and the one without deal. She might also feel that more information would be available when there is a deal. In most models for decision making under ambiguity in the extant literature, source-dependent ambiguity attitudes are not explicitly captured, which do not allow a comparative static analysis on how source-dependent ambiguity aversion influences the optimal portfolios. In contrast, TSE explicitly models source-dependent ambiguity aversion by allowing v_i 's differ across different hedges, which allows such an analysis.

Consider a set of state contingent assets f_k indexed by k from a finite set K . For any $k \in K$, $f_k : S \rightarrow X$ where X is a payoff space. Denote by α_k the money allocated to asset f_k and by w the total wealth. The payoff of the portfolio is given by $f_p = \sum_{k \in K} \alpha_k f_k$. A portfolio selection problem under TSE can be formulated as follows.

$$\begin{aligned} \max_{\alpha_k} U(f_p) &= \sum_{H_i \in \{H_i\}_{i \in I}^f} u \left(v_i^{-1} \left(\sum_{x \in X} v_i(x) p_i(f_p(x)^{-1} \cap H_i) \right) \right) P(H_i) \\ s.t. \sum_{k \in K} \alpha_k &= w \end{aligned} \tag{6}$$

The portfolio selection problem above allows the optimal allocation α_k depend on the source-dependent ambiguity aversion captured by v_i . In the example considered above, one can select the “no deal” vs “deal” as a hedge partition of state space with $v_{\text{no deal}}$ and v_{deal} modeling different ambiguity aversions respectively. It is possible to conduct comparative static on how optimal allocation α_k depends on the ambiguity aversion captured by v_i on each hedge H_i .

TSE may also allow optimal portfolios that will not be selected by other models as it represents preferences that cannot be represented by others. In the subsequent subsections, how TSE preferences differ from the EUU preferences and KMM preferences are discussed in detail with

conjectured choice problems. The example below shows that in some situations the set of optimal portfolios under TSE contains the set of optimal portfolios under KMM, which will be revisited in section 4.1 when comparing the scope of preferences represented by TSE and KMM.

Example 1. Consider two ambiguous acts f and g whose consequences depend on the color of ball drawn from an urn containing 10,000 balls as presented in the left panel of Table 4. Assume the consequences are presented on utility u scale.

Table 4: Global vs Local Ambiguity Aversion on State Space

Acts/Composition	1000 balls		9000 balls		Acts/Sub-acts	$P(H_1) = 0.1$	$P(H_2) = 0.9$
	Black	White	Red	Yellow		sub-act on H_1	sub-act on H_2
f	50	50	10	0	f	50	$u_2 \sim U(0, 10)$
g	100	0	5	5	g	$u_1 \sim U(0, 100)$	5

There are two hedges $H_1 = \{B, W\}$ and $H_2 = \{R, Y\}$ in the problem with objective probabilities 0.1 and 0.9 of obtaining each respectively. Given the symmetric information regarding the proportion of different colors, it is reasonable to assume uniform subjective distributions on each hedge. This converts the two acts to these presented in the right panel of Table 4, where the consequences on two hedges are uniformly distributed according to $u_1 \sim U(0, 100)$ and $u_2 \sim U(0, 10)$ respectively.⁶

To simplify the discussion, it is assumed that there are two state contingent assets available to an investor, which are represented by f and g in Table 4. Assume that both assets are traded at the same price and the investor has decided to allocated the money enough to buy two units of assets in her portfolio. Then, all possible investment portfolios are: 1 share f plus 1 share g ($f \oplus g$), 2 shares of f ($2f$), and 2 shares of g ($2g$), which are listed in Table 5.

Table 5: Portfolio Consisting Two Shares based on f and g

Portfolio f_p	$p(H_1) = 0.1$		$p(H_2) = 0.9$		$\mathbb{E}(f_p)$	$Var(f_p)$
	Black	White	Red	Yellow		
$2f$	100	100	20	0	19	$\frac{324}{12}$
$2g$	200	0	10	10	19	$\frac{400}{12}$
$f \oplus g$	150	50	15	5	19	$\frac{181}{12}$

Consider applying KMM in this example. Under the uniform distribution assumption on each hedge, it is easy to verify that these three portfolios share the same first order expected utility with different variance, which are shown in the last two columns in the Table 5. Thus, under the KMM model, one can conclude that $U_{KMM}(f \oplus g) > U_{KMM}(2f) > U_{KMM}(2g)$, namely the optimal

⁶A continuous uniform distribution is used to approximate discrete uniform distribution here.

portfolio is $f \oplus g$ based on the KMM model. Detailed calculations of utility under KMM of acts f and g are presented in subsection 4.1.2.

In contrast, under TSE, due to the extra flexibility from different ambiguity aversions on different hedges, all of the three portfolios in Table 5 could be selected as the optimal one depending on investor's ambiguity aversions on H_1 and H_2 . Consider the three cases regarding the investor's ambiguity aversions on H_1 and H_2 in Table 6. In Case 1, the investor is assumed to be extremely uncertainty averse on H_1 but uncertainty neutral on H_2 . On H_1 , when utilities are uncertain, her utility of CE on the hedge takes the minimum possible utility. On H_2 , uncertainty neutral implies that the investor assumes a uniform distribution of the utility and takes the mean as the utility of the CE . This results in the corresponding utilities on each hedge for each portfolio in Case 1. In Case 2, the utilities on each hedge are obtained by assuming that the investor is extremely uncertainty averse on H_2 but uncertainty neutral on H_1 . In Case 3, the investor is assumed to be uncertainty averse when utilities on a hedge are uncertain. In particular, the bigger the uncertain utility interval is, the lower the utility obtained on the hedge. For instance, on H_2 , $2f$ delivers an uncertain utility interval $[0, 20]$ which is bigger than the interval $[5, 15]$ delivered by $f \oplus g$, thus the utility obtained on $[0, 20]$ is assumed to be 9 which is lower than 9.5 obtained on $[5, 15]$, both of which are lower than 10 when DM is uncertainty neutral. On H_1 , 95 and 93 are assumed to be obtained from $f \oplus g$ and $2g$ respectively by the same reason. Since uncertainty aversions on H_1 and H_2 are independent, these assumed attitudes are plausible. Finally, the third column in each case holds the expected aggregated utility in the second stage evaluation for each portfolio. Clearly, each portfolio could be selected as the optimal one depending on which case occurs.

Table 6: Portfolio Selection under TSE (Three Cases)

	Case 1			Case 2			Case 3		
	$p = 0.1$	$p = 0.9$	$U(f_p)$	$p = 0.1$	$p = 0.9$	$U(f_p)$	$p = 0.1$	$p = 0.9$	$U(f_p)$
f_p	U_{H_1}	U_{H_2}		U_{H_1}	U_{H_2}		U_{H_1}	U_{H_2}	
$2f$	100	10	19*	100	0	10	100	9	18.1
$2g$	0	10	9	100	10	19*	93	10	18.3
$f \oplus g$	50	10	14	100	5	14.5	95	9.5	18.5*

Note: The starred row indicates the optimal portfolio in each case under TSE for the problem.

4 Comparing TSE with Some Existing Models

The TSE model represents preferences that have not been studied in the extant literature. There are some special characteristics of TSE which make it become a nice candidate as a modeling tool for problems in decision analysis, finance, as well as other areas.

Besides the applications discussed in section 3, TSE also differs from existing models by representing preferences that cannot be represented by them. In the extant literature, there are two types of *two-stage* models for decision making under ambiguity. The first one includes the EUU model axiomatized in Gul & Pesendorfer (2014) which can be roughly termed as a subjective version of the model based on belief function in Dempster-Shafer’s evidence theory (Dempster 1967, Shafer 1976) proposed by Jaffray (1989). Wakker (2011) term this approach as “roulette-horse” lottery where less uncertain events are resolved first, then the more uncertain events are resolved. The advantages of this approach are discussed in Jaffray & Wakker (1993) and Wakker (2011). The second type, which is more common, is represented by the smooth ambiguity model (KMM) (Klibanoff et al. 2005),⁷ which includes other similar models in Nau (2006), Segal (1987), Seo (2009), Grant et al. (2009), and Neilson (2010). All these models feature a two-stage lottery where more uncertain events are resolved first, then the less uncertain events are resolved. The TSE belongs to the first type, which contains the EUU for simple acts as a special case. This relationship is discussed in detail in subsection 4.2. In subsection 4.1, the TSE is compared with KMM in detail, which reveals that TSE overlaps with KMM in presenting preferences; but both model can represent preferences that cannot be represented by the other one. The relationships between TSE, EUU, and KMM can be illustrated in Figure 2, which also shows the examples that will be discussed in detail in the sequel. Finally, as TSE also captures source dependent uncertainty, the model is also compared with a source-dependent preference model recently proposed by Cappelli et al. (2020) in subsection 4.3.

4.1 Comparison with KMM

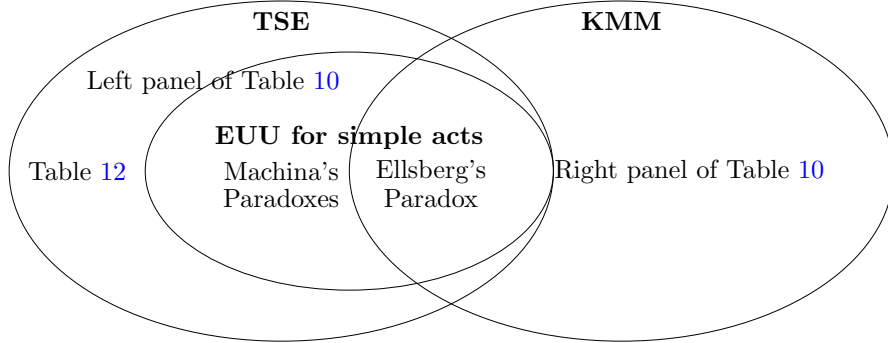
4.1.1 Connection between KMM and TSE

KMM is axiomatized in a setting that is different from the one used in this paper. In simple words, TSE is based on “roulette-horse” lottery, whereas KMM is based on “horse-roulette” lottery. Before giving a formally argument on how TSE is connected to KMM, Figure 3 helps to illustrate the intuition of their relationship. In short, KMM and TSE can be interpreted as “projections” of a three-stage model to different two-stage “planes”.

KMM models preference under ambiguity by considering DM’s uncertainty of possible probability measures over states. In particular, DM does not have a unique probability measure π over state space S . Instead, she has a set of π contained in Δ she is uncertain about, which can be interpreted as model uncertainty (Marinacci 2015). In contrast, TSE captures ambiguity as uncertainties orig-

⁷The smooth ambiguity model in Klibanoff et al. (2005) is not axiomatized in AA’s setting, which is axiomatized by embedding SEU over objective lotteries into another SEU that represents preference over second order acts (see a review in section 4.1). But, such a recursive model shares similar representation as the one axiomatized in a simple AA’s setting in Neilson (2010), both of which are usually classified into the same family in the literature. Such a tradition is followed in this paper.

Figure 2: Preferences Described by TSE, EEU, and KMM



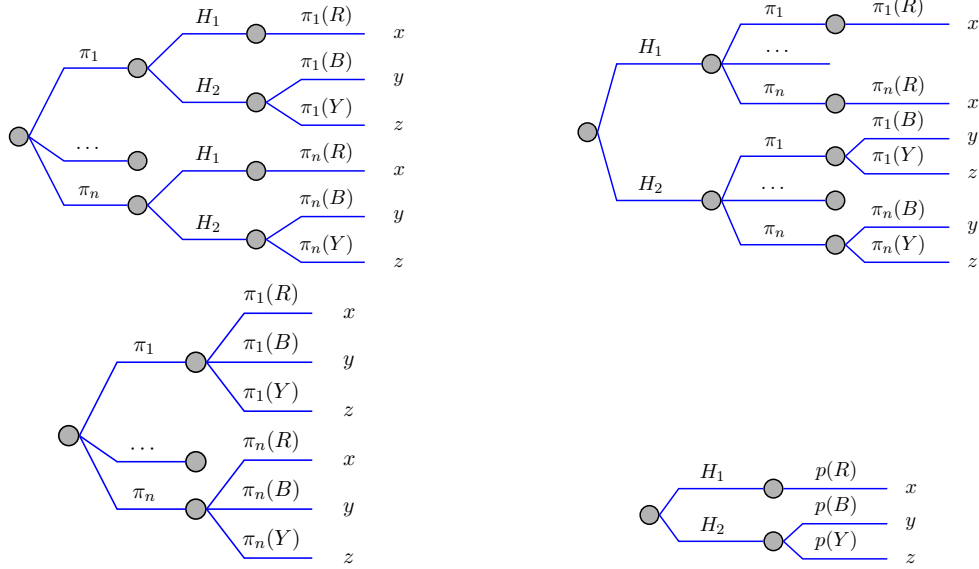
Note: The left panel of Table 10 can be used to create preference that is consistent with TSE but not with KMM; and the right panel of Table 10 can be used to create preference that is consistent with KMM but not with TSE. According to Dominiak & Lefort (2020), the Machina’s reflection example can be accommodated by KMM if exogenous probabilistic information is included in KMM such that preferences are *probabilistically sophisticated* over known-probability acts.

inating from different sources represented by two classes of events. In a more general setting where both model uncertainty and source dependent uncertainty are present, KMM and TSE are special cases obtained by applying the Reduction of Compound Lottery (ROCL) to different stages of a three-stage compound lottery. To illustrate this point, consider the Three Color Ellsberg paradox, where a hedge partition is $\{H_i\}_{i \in \{1,2\}} := \{H_1, H_2\}$ with $H_1 = \{R\}, H_2 = \{B, Y\}$. A generic act (choice object) is illustrated in Figure 3.

Both trees in the first row in Figure 3 have three stages. They differ in whether “model uncertainty” chance node proceeds the “hedge uncertainty” chance node. When hedge uncertainty is independent of model uncertainty, i.e. $\pi_i(H_k) = \pi_j(H_k)$ for any $\pi_i \neq \pi_j \in \Delta$ and $k \in \{1, 2\}$, one can flip the order of these two types of nodes. A more general decision model can be obtained by considering choice objects described by three-stage lotteries. Both KMM and TSE are special cases of such a more general three stage model obtained by applying ROCL to different stages. In the second row of Figure 3, the left tree is the choice object in KMM, which can be obtained from the left tree in the first row by applying ROCL to merge hedge uncertainty chance nodes with state uncertainty chance nodes; and the right tree is the choice object in TSE, which can be obtained from the right tree in the first row by applying ROCL to merge model uncertainty chance nodes with state uncertainty chance nodes. Therefore, both KMM and TSE can be interpreted as “projections” of a three-stage model to different two-stage “planes”.

The Proposition 2 in the sequel shows that there exists a subset of preferences represented by KMM which can also be represented by TSE. Although such a claim can be easily obtained by noticing that the preference in the Three-Color Ellsberg paradox can be represented by both KMM

Figure 3: Converting Ellsberg’s Choice Objects in KMM to TSE



Note: The trees (compound lotteries) in the upper left and right panel are equivalent when H_i 's are independent of π_i 's. The lower left tree is the choice object of KMM; the lower right tree is the choice object of TSE.

and TSE, the proposition gives a formal argument as well as conditions on preference under which this situation occurs.

Formally, besides the state space S , KMM also assumes that there exists a set of possible probability measures defined on S as primitives, i.e. a set Δ where $\pi \in \Delta$ is a probability measure $\pi : S \rightarrow [0, 1]$ that is considered to be a possible measure on S by the DM. Fix any $f \in \mathcal{F}$, where $f : S \rightarrow X$, for each π , there is an induced lottery, i.e. a probability measure defined on the consequence set X , defined as $\pi_f(B) = \pi(f^{-1}(B))$ for any $B \subset X$. It also assumes that there are some acts, denoted by $l_f(\pi)$, which are essentially lotteries associated with objective distributions on consequences. These acts are referred to as “lottery acts”.⁸ Using these lottery acts to “replicate” the distribution π_f , the certainty equivalent $c_f(\pi)$ of a lottery act $l_f(\pi)$ can be assigned to π_f that shares the same distribution with $l_f(\pi)$. Therefore, given a specific f , $c_f(\pi)$ maps each $\pi \in \Delta$ to a consequence given by $c_f(\pi)$, which is referred to as second order act in KMM. It is assumed that there exists a second order preference \succeq^2 over the set of second order acts. To illustrate the primitives in KMM, suppose there are n finite states in S and m finite possible probabilities π (possible models of uncertainty), a generic f can be illustrated in Table 7.

In KMM, the first order act models the choice object faced by the DM, which is observable.

⁸In KMM, state space is assumed to be of product form $S = \Omega \times (0, 1]$ where $(0, 1]$ is associated with a Lebesgue measure, if an act only depends on events in $(0, 1]$, the Lebesgue measure induces an objective probability measure from the act, which is independent of π . This makes this type acts becomes lotteries involving only objective uncertainty.

Table 7: Choice Object f in KMM

f							
probabilities/states	s_1	s_2	s_3	...	s_{n-1}	s_n	second order act
π_1	$f(s_1)$	$f(s_2)$	$f(s_3)$...	$f(s_{n-1})$	$f(s_n)$	$c_f(\pi_1)$
π_2	$f(s_1)$	$f(s_2)$	$f(s_3)$...	$f(s_{n-1})$	$f(s_n)$	$c_f(\pi_2)$
...
π_m	$f(s_1)$	$f(s_2)$	$f(s_3)$...	$f(s_{n-1})$	$f(s_n)$	$c_f(\pi_m)$
first order act	$f(s_1)$	$f(s_2)$	$f(s_3)$...	$f(s_{n-1})$	$f(s_n)$	

This is illustrated in the last row in Table 7. The second order act associated with f corresponds to the last column, which maps different possible probability measures (π_i) over states to different consequences. KMM is derived from three assumptions: (i) the first order preference can be represented by a SEU model; (ii) the second order preference can be represented by a SEU model; (iii) for any $f, g \in \mathcal{F}$, $f \succeq g$ iff $c_f \succeq^2 c_g$, i.e. acts are ranked in the same way by both first order and second order preferences. Under these assumptions, the preference \succeq over acts can be represented by the KMM model below.

$$U_{KMM}(f) = \int_{\Delta} \psi \left(\int_S u(f(s)) d\pi \right) d\mu \quad (7)$$

where μ is the second order probability over all probability measures in Δ implied by \succeq^2 .

To obtain the TSE model in the KMM's setting, three assumptions are needed.

Assumption 1. *There exists a partition $\{H_i\}_{i=1}^m$ of S such that for any i and any $\pi, \pi' \in \Delta$ with $\pi \neq \pi'$, $\pi(H_i) = \pi'(H_i)$.*

Under this assumption, these H_i 's can be interpreted as hedge events in TSE. Besides this assumption, it also needs to presuppose that there exists a preference $\succeq_{|H_i}$ conditional on each H_i over acts restricted to H_i , denoted by f_{H_i} for any f . This preference can be interpreted as a counterfactual preference in an intermediate time point when uncertainty of S is partially resolved and DM is notified that H_i is obtained in a way similar to the explanation in Ghirardato (2002). The preference $\succeq_{|H_i}$ is different from the conditional preference \succeq_{H_i} in the axiomatic theory of TSE in the online appendix. In the axiomatic theory of TSE, due to separability, the conditional preference \succeq_{H_i} is an induced preference of \succeq . In contrast, in the current KMM's setting, separability of preference on H_i is not satisfied and $\succeq_{|H_i}$ is a primitive.

Assumption 2. *For any H_i , $\succeq_{|H_i}$ can be represented by a continuous SEU model.*

Under this assumption, $\succeq_{|H_i}$ can be represented by SEU with v_{H_i} depending on H_i . It also implies that there exists a certainty equivalent for f_{H_i} defined as $c_f^{H_i} \sim_{|H_i} f_{H_i}$, where $c_f^{H_i}$ is a constant act defined on H_i . Then, for any $f \in \mathcal{F}$, there exists an act $f_c := c_f^{H_1} H_1 c_f^{H_2} H_2 \dots c_f^{H_{m-1}} H_{m-1} c_f^{H_m} H_m$. The last assumption says that DM ranks f in the same way as she ranks f_c .

Assumption 3. For any $f, g \in \mathcal{F}$, $f \succeq g$ if and only if $f_c \succeq g_c$.

Proposition 2. (KMM preferences represented by TSE) If a preference \succeq over \mathcal{F} can be represented by KMM, then under Assumptions 1, 2, and 3, the preference can also be represented by TSE model.

Proof of Proposition 2

Under Assumption 3, $f \succeq g$ if and only if $f_c \succeq g_c$. Using KMM Model to evaluate f_c and g_c , one obtains

$$f \succeq g \iff \sum_{\pi \in \Delta} P(\pi) \psi \left(\sum_{H_i \in \mathcal{H}} u(c_f^{H_i}) \pi(H_i) \right) \geq \sum_{\pi \in \Delta} P(\pi) \psi \left(\sum_{H_i \in \mathcal{H}} u(c_g^{H_i}) \pi(H_i) \right)$$

where P is the second order probability measure defined on Δ ; and ψ and u are utility functions in the KMM. Under Assumption 1, for any H_i , $\pi(H_i) = \pi'(H_i)$ for any $\pi, \pi' \in \Delta$. It follows that

$$f \succeq g \iff \sum_{H_i \in \mathcal{H}} u(c_f^{H_i}) \pi(H_i) \geq \sum_{H_i \in \mathcal{H}} u(c_g^{H_i}) \pi(H_i) \quad \forall \pi \in \Delta$$

Assumption 2 implies that for any $f \in \mathcal{F}$, $c_f^{H_i} = v_i^{-1}(\sum_{x \in X} v_i(x) \mu_i(f^{-1}(x) \cap H_i))$, where v_i and μ_i are utility function and probability measure in the SEU representing $\succeq_{|H_i}$. Substituting this into the KMM model above,

$$\begin{aligned} f \succeq g &\iff \\ &\sum_{H_i \in \mathcal{H}} u \left(v_i^{-1} \left(\sum_{x \in X} v_i(x) \mu_i(f^{-1}(x) \cap H_i) \right) \right) \pi(H_i) \geq \sum_{H_i \in \mathcal{H}} u \left(v_i^{-1} \left(\sum_{x \in X} v_i(x) \mu_i(g^{-1}(x) \cap H_i) \right) \right) \pi(H_i) \\ &\sum_{H_i \in \mathcal{H}} \phi_i \left(\sum_{x \in X} v_i(x) \mu_i(f^{-1}(x) \cap H_i) \right) \pi(H_i) \geq \sum_{H_i \in \mathcal{H}} \phi_i \left(\sum_{x \in X} v_i(x) \mu_i(g^{-1}(x) \cap H_i) \right) \pi(H_i) \end{aligned}$$

where $\phi_i \equiv u \circ v_i^{-1}$. Both sides of the last inequality above are evaluated according to a general TSE model ■.

4.1.2 Different preferences represented by KMM and TSE

This subsection focuses on discussing how TSE and KMM differ from each other in term of the preferences represented by them. The examples presented in this subsection, together with the discussion in subsection 4.1.1, imply the relationship between KMM and TSE described in Figure 2.

Recall the Example 1 presented in section 3. In KMM, the first order expected utility of f and g are $u(f) = 0.1 \times 50 + 0.9 \times u_2$ and $u(g) = 0.1 \times u_1 + 0.9 \times 5$ respectively. Thus, the two acts are evaluated as

$$\begin{aligned} U_{KMM}(f) &= \int_0^{10} \psi(5 + 0.9 \times u_2) \frac{1}{10} du_2 \\ U_{KMM}(g) &= \int_0^{100} \psi(0.1 \times u_1 + 4.5) \frac{1}{100} du_1 \end{aligned}$$

It is easy to observe that $\mathbb{E}(5 + 0.9 \times u_2) = 9.5 = \mathbb{E}(0.1 \times u_1 + 4.5)$, but $Var(5 + 0.9 \times u_2) = \frac{81}{12} < \frac{100}{12} = Var(0.1 \times u_1 + 4.5)$. Therefore, a DM with smooth ambiguity averse preference represented by KMM with a concave ψ will prefer f to g . However, it may also be possible that an ambiguity averse DM may actually prefer g to f , since the ambiguity in g happens with a smaller chance than the ambiguity in f . Such a preference cannot be accommodated by KMM. To further elaborate on this issue, the preference in the Example 1 is extended to a general pattern in Table 8.

Table 8: Global vs Local Ambiguity Aversion on State Space: General Pattern

	$P(H_1) = p$	$P(H_2) = 1 - p$
Acts/Sub-acts	sub-act on H_1	sub-act on H_2
f	x	$Y \sim U(y - \xi, y + \xi)$
g	$X \sim U(x - \delta, x + \delta)$	y

In the Table 8, f and g are two acts with ambiguous sub-act on hedge H_1 and H_2 respectively, where x and y are certain consequences on utility scale. Y and X are two random variables on utility scale, which have uniform distributions on domain $[y - \xi, y + \xi]$ and $[x - \delta, x + \delta]$ respectively. In KMM, these two acts are evaluated as

$$U_{KMM}(f) = \mathbb{E}\psi(px + (1 - p)Y) = \int_{y-\xi}^{y+\xi} \psi(px + (1 - p)t) \frac{1}{2\xi} dt$$

$$U_{KMM}(g) = \mathbb{E}\psi(pX + (1 - p)y) = \int_{x-\delta}^{x+\delta} \psi(pt + (1 - p)y) \frac{1}{2\delta} dt$$

Under assumption of ambiguity aversion with concave ψ , Taylor expansion can be used to approximate the CE of these acts

$$CE_\psi(f) \approx \mathbb{E}(\bar{f}) - r_\psi(\bar{f})Var(px + (1 - p)Y) = \mathbb{E}(\bar{f}) - r_\psi(\bar{f})(1 - p)^2 \frac{(2\xi)^2}{12}$$

$$CE_\psi(g) \approx \mathbb{E}(\bar{g}) - r_\psi(\bar{g})Var(pX + (1 - p)y) = \mathbb{E}(\bar{g}) - r_\psi(\bar{g})p^2 \frac{(2\delta)^2}{12}$$

where $r_\psi = -\frac{\psi''}{\psi'}$ is the risk aversion measure and $\bar{f} = \bar{g} = px + (1 - p)y$ is the expected utility when DM is ambiguity neutral. Thus, KMM implies that

$$f \succeq g \iff r_\psi \left(p^2 \frac{(2\delta)^2}{12} - (1 - p)^2 \frac{(2\xi)^2}{12} \right) \geq 0 \iff p^2 \frac{(2\delta)^2}{12} - (1 - p)^2 \frac{(2\xi)^2}{12} \geq 0 \quad (8)$$

Clearly, the preference between f and g is independent of the degree of ambiguity aversion captured by r_ψ .

In contrast, the preference between f and g in TSE depends on ambiguity aversion measure, which allows the model to have an extra “degree of freedom” when describing this preference. To see this, again it is enough to adopt a utility with hedge independent ambiguity aversion and uniform conditional probabilities on each hedge as in section 2.3. Denote the local SEU on each

hedge by v and the global SEU by $u = \phi \circ v$, which is the same utility to the u in KMM. Assume that the underlying lotteries on consequence scale for Y and X are T and W respectively, namely $Y = u(T)$ and $X = u(W)$.⁹ Given $u = \phi \circ v$, $\phi^{-1} = v \circ u^{-1}$ can be obtained. In the TSE model, these two acts are evaluated as

$$\begin{aligned} U(f) &= px + (1-p)\phi(\mathbb{E}v(T)) = px + (1-p)\phi(\mathbb{E}v(u^{-1}(Y))) = px + (1-p)\phi(\mathbb{E}\phi^{-1}(Y)) \\ U(g) &= p\phi(\mathbb{E}v(W)) + (1-p)y = p\phi(\mathbb{E}v(u^{-1}(X))) + (1-p)y = p\phi(\mathbb{E}\phi^{-1}(X)) + (1-p)y \end{aligned}$$

Under ambiguity aversion assumption, ϕ is a convex transformation from v to u . Thus, ϕ^{-1} is concave. Then, $\phi(\mathbb{E}\phi^{-1}(Y))$ gives CE of lottery Y under a concave utility ϕ^{-1} . Based on Taylor expansion, the utilities above can be approximated by

$$\begin{aligned} U(f) &= px + (1-p)\phi(\mathbb{E}\phi^{-1}(Y)) = px + (1-p)CE_{\phi^{-1}}(Y) \approx px + (1-p) [\mathbb{E}(Y) - r_{\phi^{-1}}(\mathbb{E}(Y)) \cdot Var(Y)] \\ &= px + (1-p)y - (1-p)r_{\phi^{-1}}(y) \frac{(2\xi)^2}{12} \\ U(g) &= p\phi(\mathbb{E}\phi^{-1}(X)) + (1-p)y = pCE_{\phi^{-1}}(X) + (1-p)y \approx p [\mathbb{E}(X) - r_{\phi^{-1}}(\mathbb{E}(X)) \cdot Var(X)] + (1-p)y \\ &= px + (1-p)y - pr_{\phi^{-1}}(x) \frac{(2\delta)^2}{12} \end{aligned}$$

where $r_{\phi^{-1}}(y)$ and $r_{\phi^{-1}}(x)$ are risk aversion measures of concave function ϕ^{-1} evaluated at y and x respectively, which are different. Therefore, in the TSE model

$$f \succeq g \iff pr_{\phi^{-1}}(x) \frac{(2\delta)^2}{12} - (1-p)r_{\phi^{-1}}(y) \frac{(2\xi)^2}{12} \geq 0 \quad (9)$$

A comparison between (9) and (8) reveals that TSE allows the ambiguity aversion captured by ϕ to influence the preference between f and g , while KMM does not allow ψ to take effect in this preference. This is due to the fact that the ambiguity aversion captured by ψ is applied “globally” to “all states”, which does not allow the DM to have different degree of ambiguity aversion on different subsets of states. Suppose that $x = 50, y = 5, p = 0.1, \xi = 5, \delta = 50$, then (8) implies $f \succeq g$ since $p = 0.1 > \frac{1}{11}$. In contrast, the ambiguity aversion captured by ϕ in TSE is applied to different subsets, i.e. hedges, which allows risk aversion of ϕ at different levels to be used when evaluating acts involving ambiguity from different hedges. In this example, (9) implies that $f \succeq (\preceq)g$ iff $\frac{r_{\phi^{-1}}(50)}{r_{\phi^{-1}}(5)} \geq (\leq) \frac{9}{100}$. Thus, if the risk aversion measure is decreasing, when it decreases fast enough such that $\frac{r_{\phi^{-1}}(50)}{r_{\phi^{-1}}(5)} \leq \frac{9}{100}$, the TSE model can also represent preference $f \preceq g$.

The argument above shows that there exist situations where TSE is more flexible than KMM. Actually, if the roles of hedge and second order states (possible models $\pi \in \Delta$) are flipped, it follows that there are also situations where KMM is more flexible than TSE. To see this, the above example in Table 8 is modified to include second order states to represent possible probability models over the first order states shown in Table 9. For simplicity, a state space with six states is considered. Suppose that the state space S can be partitioned into two hedges $H_1 = \{s_1, s_2, s_3\}$

⁹ $Y = u(T)$ is used to denote a transformation from random variable T to random variable Y .

Table 9: Example in Table 8 with Second Order States

Acts/Hedges	$\pi(H_1) = p$			$\pi(H_2) = 1 - p$			
	H_1			H_2			
	s_1	s_2	s_3	s_4	s_5	s_6	
f	π_1	x	x	x	$y - \xi$	y	$y + \xi$
	π_2	x	x	x	$y - \xi$	y	$y + \xi$
	π_3	x	x	x	$y - \xi$	y	$y + \xi$
g	π_1	$x - \delta$	x	$x + \delta$	y	y	y
	π_2	$x - \delta$	x	$x + \delta$	y	y	y
	π_3	$x - \delta$	x	$x + \delta$	y	y	y

and $H_2 = \{s_4, s_5, s_6\}$; and DM is uncertain about the probabilities assigned to states in each hedge. This uncertainty is modeled by three second order states $\{\pi_1, \pi_2, \pi_3\}$. For each $i = \{1, 2, 3\}$, $\pi_i(s_i|H_1) = \pi_i(s_{i+3}|H_2) = 1$ and all other $\pi_i(s_k|H_m) = 0$ for any $k \neq \{i, i + 3\}$, $m \in \{1, 2\}$. All these π 's agree on assigning probability to H_1 , i.e. $\pi_i(H_1) = p$ for all $i \in \{1, 2, 3\}$. Suppose the second order states are equal likely under a second order probability P , i.e. $P(\pi_i) = 1/3$ for $i = \{1, 2, 3\}$. Such a state space and P make f and g in Table 8 become these exhibited in Table 9, which are also equivalent to these in the left panel of Table 10.

Table 10: TSE vs KMM

Acts/Hedges	$\pi(H_1) = p$ $\pi(H_2) = 1 - p$		Acts/Probabilities	$P(\pi_1) = p$ $P(\pi_2) = 1 - p$			
	H_1	H_2		π_1	π_2		
f	π_1	x	$y - \xi$	f'	H_1	x	$y - \xi$
	π_2	x	y		H_2	x	y
	π_3	x	$y + \xi$		H_3	x	$y + \xi$
g	π_1	$x - \delta$	y	g'	H_1	$x - \delta$	y
	π_2	x	y		H_2	x	y
	π_3	$x + \delta$	y		H_3	$x + \delta$	y

Note: The left one shows a case where TSE is more flexible than KMM; the right one shows a case where KMM is more flexible than TSE.

Table 11: Definition of f' and g' in Table 10

	H_1		H_2		H_3	
	s_1	s_2	s_3	s_4	s_5	s_6
f'	x	$y - \xi$	x	y	x	$y + \xi$
g'	$x - \delta$	y	x	y	$x + \delta$	y

Based on the argument above, when ranking f and g in the left panel of Table 10, TSE model is more flexible than KMM. To see the situation where KMM is more flexible than TSE, one only

needs to flip the roles of H_i and π_i in the left panel of Table 10, which gives the right panel of Table 10. The acts f' and g' in Table 10 are defined in Table 11 with two possible probability measures π_1 and π_2 such that $\pi_i(H_j) = 1/3$ for $i \in \{1, 2\}$ and $j \in \{1, 2, 3\}$, $\pi_1(s_1|H_1) = \pi_1(s_3|H_2) = \pi_1(s_5|H_3) = 1$, and $\pi_2(s_2|H_1) = \pi_2(s_4|H_2) = \pi_2(s_6|H_3) = 1$. The second order probabilities are $P(\pi_1) = p, P(\pi_2) = 1 - p$. Suppose the consequences in the right panel in Table 10 are on the utility v scale now, and both $Y \sim U(y - \xi, y + \xi)$ and $X \sim U(x - \delta, x + \delta)$ are uniform distributions. Since $u = \phi \circ v$, consequences x and y , after converted to u scale, become $\phi(x)$ and $\phi(y)$. Then, KMM evaluates f' and g' as follows

$$\begin{aligned} U_{KMM}(f') &= p\psi(\phi(x)) + (1 - p)\psi(\mathbb{E}\phi(Y)) \\ U_{KMM}(g') &= p\psi(\mathbb{E}\phi(X)) + (1 - p)\psi(\phi(y)) \end{aligned}$$

The TSE evaluates these acts as

$$\begin{aligned} U(f') &= \frac{1}{3}\phi(px + (1 - p)(y - \xi)) + \frac{1}{3}\phi(px + (1 - p)y) + \frac{1}{3}\phi(px + (1 - p)(y + \xi)) \\ &= \mathbb{E}\phi(px + (1 - p)Y) \\ U(g') &= \frac{1}{3}\phi(p(x - \delta) + (1 - p)y) + \frac{1}{3}\phi(px + (1 - p)y) + \frac{1}{3}\phi(p(x + \delta) + (1 - p)y) \\ &= \mathbb{E}\phi(pX + (1 - p)y) \end{aligned}$$

Using the similar reasoning applied to the left panel of Table 10, it is easy to show that the curvature of ψ at different levels are used in the KMM's evaluations. In contrast, the curvature of ϕ at the same level is used in the TSE's evaluations. In this case, KMM becomes more flexible than TSE when ranking acts f' and g' . Loosely speaking, when consequences vary over hedges and the variation occurs under different second order probability models, KMM is more flexible than TSE. In contrast, when consequences vary over different second order probability models and the variation occurs on different hedges, TSE is more flexible than KMM.

4.2 Comparison with Expected Uncertain Utility

In EUU (Gul & Pesendorfer 2014), the probabilities are define over all "ideal" events. Let \mathcal{E} denote the collection of all ideal events, which forms a σ -algebra in EUU. The preference over all acts $f : S \rightarrow [l, m]$ is considered. Let $I = \{[x, y] | l \leq x \leq y \leq m\}$ be the set of intervals. Let $\mathcal{F}_{\mathcal{E}}$ be the set of all \mathcal{E} -measurable functions $\mathbf{f} : S \rightarrow I$, which are called interval acts. \mathbf{f}_1 and \mathbf{f}_2 denote the minimum and maximum elements of the interval. For any act f , an interval act \mathbf{f} is the envelope of f if (i) $f(s) \in \mathbf{f}(s)$ for any $s \in S$ and (ii) for any interval act $\mathbf{g} \neq \mathbf{f}$, $f \in \mathbf{g}$ for all s implies $\mathbf{f}(S) \subset \mathbf{g}(S)$. The envelope of f is denoted by $[f] = ([f]_1, [f]_2)$. It can be shown that for any act f there exists an interval act $\mathbf{f} \in \mathcal{F}_{\mathcal{E}}$ which is f 's envelope. EUU evaluates an act f as

$$U_{EUU}(f) = \int u[\mathbf{f}]d\mu \tag{10}$$

where u is an interval utility $u : I \rightarrow \mathbb{R}$ and μ is the subjective probability measure derived over \mathcal{E} .

For simple acts, (10) above reduces to

$$U_{EUU}(f) = \sum_I u(x, y) \mu\{s|[f](s) = [x, y]\} \quad (11)$$

Denote by \mathcal{H} the set of all hedge events in TSE. In the online appendix, these hedge events are defined in the same way as “ideal” events in EUU. In contrast to EUU, the probabilities in TSE are derived based on [Kopylov \(2007\)](#)’s “small” domain SEU over risky events in a mosaic \mathcal{R} , which is a concept weaker than σ -algebra used in EUU. The hedge events (ideal events) in this paper are not used to define probabilities, which are only used to obtain \mathcal{H} measurable acts for any ambiguous acts. The collection of hedge events \mathcal{H} can be either finite or infinite, depending on DM’s preference. In EUU, when some consequences of an act is not measurable with respect to \mathcal{E} , the envelop of the act is used to evaluate it. This approach assumes that the DM ignores other consequences contained in the interval $[x, y] = [[f]_1, [f]_2]$ and evaluates the act by only considering the maximum $[f]_2$ and the minimum $[f]_1$ of the interval. This interval based approach fails to satisfy strict monotonicity as discussed in [Wakker \(2011\)](#) and [Gul & Pesendorfer \(2014\)](#). Here, it is further argued that this approach may also results in evaluations that reverse the ranking of acts. To see this, consider acts in Table 12, where H_1, H_2, H_3 are \mathcal{H} events in TSE which forms a hedge partition.

Table 12: Reversing Preference by EUU

Acts/Events	$P(H_1) = 0.45$			$P(H_2) = 0.1$	$P(H_3) = 0.45$		
	s_1	s_2	s_3	s_4	s_5	s_6	s_7
f	\$100	\$99	\$0	\$1	\$50	\$49	\$0
g	\$100	\$1	\$0	\$2	\$50	\$1	\$0

Under EUU, one obtains $U_{EUU}(f) < U_{EUU}(g)$ based on the evaluations below.

$$U_{EUU}(f) = 0.45u(0, 100) + 0.1u(1, 1) + 0.45u(0, 50) <$$

$$U_{EUU}(g) = 0.45u(0, 100) + 0.1u(2, 2) + 0.45u(0, 50)$$

However, a DM may notice that on both hedges H_1 and H_3 , f dominates g ; and g only dominates f on H_2 with a very small probability 0.1. Therefore, even though DM has a hard time to quantify probabilities of s_2 and s_6 , it is possible she prefers f to g as long as she believes that the chances of these two states are not too small. This kind of preference cannot be accommodated by EUU. But it is easy to show $f \succeq g$ can be accommodated by TSE.

In general, it can be shown that TSE contains EUU as a special case for simple acts.

Proposition 3. (EUU preferences represented by TSE) *If a preference \succeq over simple acts \mathcal{F} can be represented by an EUU model, it can also be represented by a TSE model.*

Proof of Proposition 3

It is enough to show that the EEU model (11) can be written as a TSE model. For a simple act f , there are finite intervals in the range of $[f]$. Denote these intervals by $\{I_1, I_2, \dots, I_n\} \subset I$. Then, $\{[f]^{-1}(I_i)\}_{i=1}^n$ forms a \mathcal{H} partition of S , where $[f]^{-1}(I_i) \in \mathcal{H}$ for all i . Let $[f]^{-1}(I_i) = H_i$, then $\mu\{s|[f](s) = I_i\} = \mu(H_i)$. Let $v_u(x) := u(x, x)$ as defined in Gul & Pesendorfer (2014). Let σ_u^{xy} be the unique $\sigma \in [0, 1]$ that satisfies

$$u(x, y) = v_u(\sigma x + (1 - \sigma)y)$$

Then, let $CE_{xy} = \sigma_u^{xy}x + (1 - \sigma_u^{xy})y$.¹⁰ Clearly, for any utility index ρ , there exists probability λ such that $CE_{xy} = \rho^{-1}(\rho(x)\lambda(x) + \rho(y)\lambda(y))$. Let $\phi = v_u \circ \rho^{-1}$,

$$u(x, y) = v_u(CE_{xy}) = v_u\left(\rho^{-1}\left(\sum_{\{x,y\}} \rho(x)\lambda(x)\right)\right) = \phi\left(\sum_{\{x,y\}} \rho(x)\lambda(x)\right)$$

Thus, for any I_i , $u(I_i) = \phi_i\left(\sum_{x \in \{\min(I_i), \max(I_i)\}} \rho_i(x)\lambda_i(x)\right)$. Substituting this expression of $u(I_i)$ and $\mu\{s|[f](s) = I_i\} = \mu(H_i)$ to (11) results in a special case of TSE model where λ_i is a hedge dependent probability measure that only assigns positive probability to maximum and minimum of $f(H_i)$. ■

4.3 Comparison with Source-Dependent Prospect Profile Model

In a recent work (Cappelli et al. 2020), a general source-dependent prospect profile (SDPP hereafter for simplicity) model is proposed to aggregate a profile of prospects that depends on different sources. In this model, a collection of independent source of uncertainty denoted by $\{S_i\}_{i \in I}$ is considered. For each $i \in I$, a prospect dependent on source i is a function $f_i : S_i \rightarrow X$, where X is the set of consequences. For any source, $\mathcal{F}_i := X^{S_i}$ denotes the set of all prospects dependent on source i . DM's preference is defined over a set of prospect profiles \mathbf{f} in $\mathcal{F} := \times_i \mathcal{F}_i$, where a prospect profile is a vector $\mathbf{f} = (f_1, \dots, f_n)$ with $n = |I|$. The paper shows that under some standard axioms, a prospect profile is evaluated by a two-step procedure: (i) for any i , the certainty equivalent of f_i is evaluated as $c_i(f_i)$; (ii) each certainty equivalent $c_i(f_i)$ is converted to the unit of a numeraire $o \in I$ by a tradeoff rate δ_{oi} , then aggregated by an aggregator W_0 to obtain the overall evaluation of the profile as:

$$V_o(\mathbf{f}) = W_0\left(\delta_{o1}(c_1(f_1)), \dots, \delta_{on}(c_n(f_n))\right)$$

In this model, c_i represents the *intra-source* preference \succeq_i over \mathcal{F}_i ; the rate δ_{ij} captures the *inter-source* trade-off between prospects dependent on different sources under the overall preference \succeq .

The SDPP model is a very general source-dependent model in the sense that (i) it considers a multiple source of uncertainty indexed by I ; (ii) c_i can be delivered by any model that admits

¹⁰This is the certainty equivalent of interval $[x, y]$ defined in Gul & Pesendorfer (2014).

a certainty equivalent, such as EU, Maxmin utility, cautious expected utility etc (see discussion section 3.1 in Cappelli et al. (2020)). In contrast, the TSE model is a source-dependent preference model where only two sources are considered.¹¹ In this regard, the SDPP model in Cappelli et al. (2020) is more general than the TSE. In a special case when only two sources are considered, a preference represented by the SDPP can also be represented by TSE.

For instance, the Two-Urn Ellsberg’s paradox and Machina (2014)’s “Slightly-Bent Coin Problem” are of this type. Cappelli et al. (2020) discussed how the SDPP model accommodates the Two-Urn Ellsberg’s paradox. To see how the preference in this paradox can be accommodated by the TSE model, consider $S_1 = \{R_1, B_1\}, S_2 = \{R_2, B_2\}$ where R_i (B_i) represents a red ball (black ball) is drawn from Urn $i \in \{1, 2\}$. In this problem, it is only known that Urn 1 contains 50 red and 50 black balls, but the composition of Urn 2 is unknown. DM considers her preference over bets f_i and g_i for $i \in \{1, 2\}$, where $f_i(g_i)$ denotes bet that pays \$100 if the ball drawn from Urn i is red (black). The typical preference is $f_1 \sim g_1 \succ g_2 \sim f_2$. To apply TSE model, set $S_N = S_1 \times S_2$ and choose events in S_1 as risky events and events S_2 as ambiguous such that $S_N = H_1 \cup H_2, H_1 = \{(R_1, R_2), (R_1, B_2)\}, H_2 = \{(B_1, R_2), (B_1, B_2)\}$. It is easy to verify that applying the TSE gives

$$CE(f_1) = CE(g_1) = u^{-1}(0.5u(100) + 0.5u(0)) > v^{-1}(0.5v(100) + 0.5v(0)) = CE(g_2) = CE(f_2)$$

as v is more concave than u in the TSE model.

Finally, it is worth remarking that although TSE only allows two sources of uncertainty, these two sources need not to be independent as required by the SDPP model. In the setting of DSPP, sources are independent to each other.

5 Conclusion

The paper proposes a TSE model for decision making under ambiguity. In the first stage, DM evaluates CE s of consequences on more uncertain events based on local SEU models. In the second stage, consequences on more uncertain events are replaced by those CE s and evaluated based on Kopylov (2007)’s “small” domain SEU. The TSE model features using roulette-horse lotteries rather horse-roulette lotteries in modeling ambiguity, which is consistent with the idea in Jaffray (1989) and Gul & Pesendorfer (2014) and differs from the other two-stage models in the literature (Segal 1987, Klibanoff et al. 2005, Nau 2006, Seo 2009, Grant et al. 2009, Neilson 2010).

The paper also discusses how TSE model accommodates the paradoxes in the extant literature that challenge most major models for decision making under ambiguity. To achieve this flexibility,

¹¹The spirit of TSE can be extended to multiple sources by considering decision trees with multiple tiers and modeling different sources of uncertainty by a different collection of events. But, the parsimony of the model needs to be compromised.

the parsimony of the model does not need to be sacrificed. The TSE model only requires different v and u to accommodate the paradoxes under uniform subjective probabilities. It is shown that the TSE enjoys some nice properties in the applications to finance problems which are not held by KMM. The model is also compared with some popular models that are closely relevant in the literature. It can be shown that when evaluating simple acts, the EUU model is a special case of the TSE model. The KMM can be interpreted as a two-stage evaluation in Bayesian paradigm. In contrast, the TSE is not Bayesian. It is in the source-dependent family, where uncertainties from different sources are treated separately. In term of the preferences represented by KMM and TSE, it can be shown that TSE model shares an overlap with the KMM model, but there are also preferences that can only be accommodated by each model respectively.

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Online Appendix for “Revisiting Ellsberg and Machina’s Paradoxes: A Two-Stage Evaluation Model under Ambiguity”

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In this online appendix, [Kopylov \(2007\)](#)’s “small” domain SEU is extended from risky acts to all acts. Then, the TSE model is axiomatized by assuming extra structure on preference over ambiguous acts in the “small” domain SEU framework. Following [Kopylov \(2007\)](#), call $S = \cup_{i=1}^m S_i$ a grand partition of state space S if the events S_1, \dots, S_m are disjoint and risky. [Kopylov \(2007\)](#)’s “small” domain SEU is obtained by restricting the original Savage’s axioms to smaller sets. The collection of risky events $\mathcal{R} \subset \Sigma$ is a *mosaic*, which satisfies the following conditions

1. $S \in \mathcal{R}$;
2. $A \in \mathcal{R} \Rightarrow A^c \in \mathcal{R}$ and;
3. $S = \cup_{i=1}^m S_i$ is a grand partition $\Rightarrow S_i \cup S_j \in \mathcal{R}$ for all $i, j = 1, \dots, m$.

In [Savage \(1954\)](#)’s SEU theory, six axioms (P1-P6) ([Fishburn 1970](#)) are used to derive an expected utility representation with subjective finite-additive probabilities defined on a σ -algebra of events in S . [Kopylov \(2007\)](#) relaxed [Savage \(1954\)](#)’s SEU by deriving probabilities on a “small” domain of events which has a mosaic structure. To achieve this goal, the sure-thing principle (P2 in [Fishburn \(1970\)](#)) is replaced by assuming separability of preference over a smaller set of acts conditioned on a smaller set of events. Specifically, for any risky event $A \in \mathcal{R}$, [Kopylov \(2007\)](#) defined a set $\mathcal{G} \cap \mathcal{E}$ which contains all risky acts that is also \mathcal{R} -measurable when restricted to event in \mathcal{E} . The relaxed sure-thing principle (STP) is stated as follows.

Axiom 1. (P2(\mathcal{R})) For any risky event $A \in \mathcal{R}$, and acts $f, g \in \mathcal{G} \cap \{A\}$ and outcomes $x, y \in X$,

$$fAx \succeq gAx \Rightarrow fAy \succeq gAy$$

If a DM does not distinguish risky events from ambiguous events, then \mathcal{R} becomes Σ and $\mathcal{G} \cap A$ becomes \mathcal{F} . The P2(\mathcal{R}) reduces to P2 in Savage’s SEU. Besides this axiom, [Kopylov \(2007\)](#) also relaxed the other Savage’s axioms (P1-P6) ([Fishburn 1970](#)) by rewriting these conditions such that they only involve risky acts and risky events to develop an SEU on a “small” domain.

Axiom 2. (P1(\mathcal{R})) \succeq is complete and transitive on \mathcal{G}

Axiom 3. (P3(\mathcal{R})) For each risky event $A \in \mathcal{R}$, at least one of the following statements holds simultaneously for all outcomes $x, y \in X$ and for all acts $h \in \mathcal{G} \cap A$,

1. $x \succeq y \iff xAh \succeq yAh$;

2. $xAh \sim yAh$.

Axiom 4. (P4(\mathcal{R})) For all risky events $A, B \in \mathcal{R}$, for all outcomes $x \succ x'$ and $z \succ z'$, if $xAx' \succeq xBx'$, then $zAz' \succeq zBz'$.

Axiom 5. (P5(\mathcal{R})) There exist outcomes x and x' such that $x \succ x'$.

Axiom 6. (P6(\mathcal{R})) For any outcome x , for any finite collection of risky events $\mathcal{E} \subseteq \mathcal{R}$, and for any \mathcal{E} -measurable acts $f \succ g$, there exists a grand partition $\{S_1, \dots, S_m\} \subseteq \mathcal{R} \cap \mathcal{E}$ such that for all $i = 1, \dots, m$, $xS_i f \succ g$ and $f \succ xS_i g$.

When $\mathcal{R} = \Sigma$, Axiom 1, 3, 4, 5, and 6 above become equivalent to the corresponding axioms used in Savage's SEU theory. Readers are referred to [Kopylov \(2007\)](#) for detailed discussions on these axioms. Using these modified axioms, [Kopylov \(2007\)](#) showed that

Theorem 1. ([Kopylov \(2007\)](#)'s "small" domain SEU) *Let Σ be an algebra, and let $\mathcal{R} \subseteq \Sigma$ be a mosaic. Then the following two statements are equivalent:*

1. \succeq satisfies Axiom 1-6.

2. \succeq is represented on the set \mathcal{G} by expected utility

$$U(f) = \sum_{x \in X} u(x) \cdot P(f^{-1}(x)) \text{ for } f \in \mathcal{G} \tag{1}$$

where $u : X \rightarrow \mathbb{R}$ is a non-constant utility index, and $P : \mathcal{R} \rightarrow [0, 1]$ is a finely ranged probability measure.¹

In this representation, the index u is unique up to a positive linear transformation, and the probability measure P is unique.

As emphasized by [Kopylov \(2007\)](#), this "small" domain SEU assumes no structure on the preferences over ambiguous acts, which allows a great amount of freedom to model DM's choice among ambiguous acts.

¹See [Kopylov \(2007\)](#) for the definition of finely ranged probability measure.

1 Extending “small” domain SEU to ambiguous acts

Consider the preference \succeq over the set of simple acts \mathcal{F} . The first axiom for the axiomatic TSE model assumes that [Kopylov \(2007\)](#)’s “small” domain SEU holds.

Axiom 7. (“Small” domain SEU) \succeq satisfies the axioms required by [Kopylov \(2007\)](#)’s “small” domain SEU.

To extend the SEU to all acts, extra assumptions are required. First, assume the consequence set $X = [l, m]$ is a closed interval in \mathbb{R} and preference \succeq is defined over all simple acts \mathcal{F} .

Axiom 8. (Weak Order on \mathcal{F}) \succeq is a weak order (complete and transitive) on \mathcal{F} .

Another axiom added to the list of [Kopylov \(2007\)](#)’s axioms requires that if act f dominates g in each state, then $f \succeq g$. Although it is a natural assumption, it is not implied by Savage’s (also [Kopylov’s](#)) axioms.²

Axiom 9. (State-wise Dominance) For any $f, g \in \mathcal{F}$, if $\forall s \in S, f(s) > g(s)$, then $f \succ g$.

In addition, the following axiom is added to obtain continuity in the model, which is standard in the literature ([Mas-Colell et al. 1995](#), [Ghirardato et al. 2003](#), [Gul & Pesendorfer 2014](#)).

Axiom 10. (Continuity) Let $g \succeq f_n \succeq h$ for all n , then $f_n \in \mathcal{F}$ converges uniformly to f implies $g \succeq f \succeq h$.

Clearly, the collection of “risky events” \mathcal{R} is a key concept of [Kopylov \(2007\)](#)’s “small” domain SEU, which is a “mosaic”. In the literature, there are different ways to define risky event, which is also referred to as unambiguous event. The most straightforward way is to use known objective probabilities to define risky events exogenously in a problem. Alternatively, different endogenous definition of risky event can be adopted ([Nehring 1999](#), [Epstein & Zhang 2001](#), [Zhang 2002](#), [Ghirardato & Marinacci 2002](#), [Klibanoff et al. 2005](#)). [Kopylov \(2007\)](#) discussed the application of his theory to both [Zhang \(2002\)](#) and [Epstein & Zhang \(2001\)](#)’s definition. Since the mosaic \mathcal{R} can be defined in different ways ([Kopylov 2007](#)), i.e. either exogenously based on known probabilities or endogenously based on preferences, [Kopylov \(2007\)](#)’s “small” domain SEU can be applied to different collections of risky events. [Kopylov \(2007\)](#) showed that both the set of risky events \mathcal{R}_Z and \mathcal{R}_{EZ} based on the definition in [Zhang \(2002\)](#) and [Epstein & Zhang \(2001\)](#) respectively are “mosaic”, which can be used as \mathcal{R} in the “small” domain SEU (1). In the same spirit, the TSE model in this paper also applies to general definitions of \mathcal{R} as long as the “mosaic” structure can be verified.

To extend the “small” domain SEU to preferences over ambiguous acts, a key step is to make some structure assumptions on the preferences. For this purpose, it is assumed that there exists a

²See [Gul & Pesendorfer \(2014\)](#)’s footnote 2 and detailed discussion in [Wakker \(1993\)](#).

sub-collection of “risky events”, on which the separability required by SEU is satisfied by all acts. Under this assumption, for any event in this sub-collection, there exists a well-defined preference over all acts conditional on the event. In this way, the representation of [Kopylov \(2007\)](#)’s “small” domain SEU can be extended to preferences over ambiguous acts. Formally, this sub-collection of risky events is defined below.

Definition 1. (Hedge Event) *A risky event $H \in \mathcal{R}$ is called a hedge event if for any acts $f, g, h, h' \in \mathcal{F}$, (i) $fHh \succeq gHh$ implies that $fHh' \succeq gHh'$; (ii) statement in (i) also holds when H is replaced with H^C . The collection of hedge event is denoted by \mathcal{H} .*

By this definition, if H is a hedge event, so is H^C . Both H and H^C are separable in preference in [Gorman \(1968\)](#)’s sense, which is exactly the separability required by the STP. Actually, when $\mathcal{R} = \mathcal{R}_Z$ or $\mathcal{R} = \mathcal{R}_{EZ}$, the separability condition in the above definition implies that a hedge is indeed a risky event defined in [Zhang \(2002\)](#) and [Epstein & Zhang \(2001\)](#).³ According to [Gorman \(1968\)](#)’s theorem, the intersection of separable sets are also separable. Therefore, it does not make sense to assume the separability required by the STP on all risky events, which implies that the intersection of any two events is also risky. Such a conclusion immediately contradicts with the intuition of using “mosaic” system to model risky events, as a DM may not be able to assign subjective probability to an intersection of events even when she can do so for each event. This is why a weaker version of separability is used to define risky events in ([Zhang 2002](#), [Epstein & Zhang 2001](#)). However, a weaker version of separability cannot be used to define conditional preferences on risky events to model DM’s choice over ambiguous acts restricted to these events. Therefore, in this paper, a subclass of risky events called hedge events is defined based on the separability required by the STP so that conditional preferences on hedge events are well-defined. Clearly, $\mathcal{H} = \{\emptyset, S\}$ always exists as a trivial case. The TSE model assumes that there exists non-trivial \mathcal{H} .

Axiom 11. (Existence of Hedge Events) There exists a non-trivial \mathcal{H} .

The existence of hedge partition can be verified behaviorally as verifying separability required by STP. In Savage’s theory, STP assumes separability applies to all events and all acts. In contrast, the axiom above only requires the separability to be verified on a sub-collection of risky events. For instance, a simple case that satisfies the axiom is the existence of a very small non-trivial $\mathcal{H} = \{\emptyset, H, H^C, S\}$, which only requires the separability to be verified on two events. Under these axioms, the following result holds.

Lemma 1. *If \mathcal{H} is a finite set, it is an algebra; if it is infinite, it is a σ -algebra.*

Proof of Lemma 1

³Both \mathcal{R}_Z and \mathcal{R}_{EZ} are defined based on the idea of relaxing the separability required by the STP to only a subset of acts on a subset of events.

Notice that the hedge event is defined in the same way as “ideal” events in Gul & Pesendorfer (2014). When \mathcal{H} is an infinite set as in Gul & Pesendorfer (2014), the Lemma B2 in the Appendix in Gul & Pesendorfer (2014) shows that the collection of ideal events is a σ -algebra. To show this result, Lemma B0 and Lemma B1 in the Appendix in Gul & Pesendorfer (2014) are proved based on the definition of ideal events and Axiom 4 and 6. Similar results in these two lemmas also hold here when they are restricted to simple acts.⁴ Axiom 6 in Gul & Pesendorfer (2014) restricted to simple acts is the Axiom 10 assumed in this paper;⁵ Axiom 4 in Gul & Pesendorfer (2014) is applied to ideal events when proving Lemma B1 (v), which is implied by Kopylov (2007)’s Axiom 4 (P4(\mathcal{R})) since $\mathcal{H} \subset \mathcal{R}$. Therefore, results in Lemma B0 and Lemma B1 in Gul & Pesendorfer (2014) also hold in this paper, and the Lemma B2 in Gul & Pesendorfer (2014) can be obtained in the same way. This also implies that when \mathcal{H} is a finite set, \mathcal{H} is an algebra. ■

Under Axiom 11, it immediately follows that \succeq restricted to any $H \in \mathcal{H}$ is a well defined preference. For any hedge $H \in \mathcal{H}$, the *conditional preference* \succeq_H on the hedge over all acts in \mathcal{F} is defined as: $\forall f, g \in \mathcal{F}$,

$$f \succeq_H g \iff fHh \succeq gHh, \forall h \in \mathcal{F}$$

Using continuity and the definition of \succeq_H , it also follows that the certainty equivalent of any act restricted to any H exists. For any $f \in \mathcal{F}$ and any $H \in \mathcal{H}$, there exists a consequence $c_f^H \in X$ which is referred to as *the certainty equivalent of f on hedge H* such that $c_f^H \sim_H f$.⁶

Using Lemma 1 and c_f^H , it can be shown that for any ambiguous act $f \in \mathcal{F} \setminus \mathcal{G}$, there exists a \mathcal{H} -measurable act \hat{f} that is indifferent to f . In fact, such a \hat{f} is not unique.

Example 1. Consider $S := \{s_1, s_2, s_3, s_4, s_5\}$. Suppose $\mathcal{H} = \sigma(\{H_1, H_2, H_3, H_4\})$ with $H_1 = \{s_1\}, H_2 = \{s_2, s_3\}, H_3 = \{s_4\}, H_4 = \{s_5\}$.⁷ Consider an act $f = (2, 2, 4, 4, 6)$ defined as $f(s_1) = f(s_2) = 2, f(s_3) = f(s_4) = 4, f(s_5) = 6$. Clearly, the act is non- \mathcal{H} -measurable. There are two ways to construct \hat{f} : 1) find $x = c_f^{H_1 \cup H_2 \cup H_3}$, define $\hat{f} = (x, x, x, x, 6)$; 2) find $x = c_f^{H_2}$, define $\hat{f} = (2, x, x, 4, 6)$. Both ways make $\hat{f} \sim f$.

In the second way of the example, when \hat{f} is constructed by using c_f^H , the H is chosen as small as possible. This keeps \hat{f} as “similar” as possible to the original f . The program below defines \hat{f} by following this principle to construct the act. To include the preference over \mathcal{G} as a special case of the extended “small” domain SEU, the definition of act \hat{f} is also extended to $f \in \mathcal{G}$ as a special case in the program. Denote by $R(f)$ the range of function f . Let $f(S)$ denote the set of consequences of f . Since f has finite range, $\{f^{-1}(x)\}_{x \in R(f)}$ is a finite partition of S . For simplicity,

⁴Gul & Pesendorfer (2014) considers a set of all acts, this paper only considers simple acts.

⁵Since point-wise convergence is implied by uniform convergence, Axiom 10 also implies the same convergence property holds when replacing uniformly with point-wise.

⁶The result is proved as result (vi) of Lemma B1 in the Appendix in Gul & Pesendorfer (2014).

⁷ $\sigma(X)$ denote the σ -algebra generated by X , which is a family of subsets of S .

let $D_x := f^{-1}(x)$. For any set E , define its “hedge interior” as $I(E) := \cup_{H \subseteq E, H \in \mathcal{H}} H$ and its “hedge interior complement” as $L(E) := E \setminus I(E)$.

1. For any $x \in f(S)$, if $f^{-1}(x) \in \mathcal{R}$, set $\hat{f}(f^{-1}(x)) = x$. If $\hat{f}(s)$ has been defined for any $s \in S$, stop; otherwise, go to step 2.
2. Let $M := \{x | f^{-1}(x) \in \mathcal{H}, x \in f(S)\}$. Index $x \in f(S) \setminus M$ such that $f(S) \setminus M = \{x_1, x_2, \dots, x_n\}$. For any x_i , if $I(D_{x_i}) \neq \emptyset$, let $\hat{f}(I(D_{x_i})) = x_i$; otherwise do nothing.
3. For simplicity denote $L_i = L(D_{x_i})$. Let $O_0 = \{L_1, L_2, \dots, L_n\}$ and $I_0 := \{1, 2, \dots, n\}$. Set $j = 1$,
4. Find $E_j = \cup_{i \in N} L_i$ for some $N \subseteq I_{j-1}$ with the minimum $|f(E_j)|$ such that $I(E_j) \neq \emptyset$. When a tie happens, choose an arbitrary one. Let $A_j = I(E_j)$ and , set $\hat{f}(A_j) = c_f^{A_j}$, set $T = O_{j-1} \setminus \{L_i | i \in N\}$.
5. Let $L_k = L(E_j)$ for an arbitrary $k \in N$. Set $O_j = T \cup \{L_k\}$, $I_j = (I_{j-1} \setminus N) \cup \{k\}$. If $O_j = \emptyset$, stop, otherwise, go to step 6.
6. Set $j = j + 1$, go to step 4.

Lemma 2. (Existence of Hedge Equivalent Act) *For any $f \in \mathcal{F} \setminus \mathcal{G}$, there exists a finite valued act \hat{f} constructed above called **hedge equivalent act** of f such that $f \sim \hat{f}$ and \hat{f} is \mathcal{H} -measurable.*

Proof of Lemma 2

For any $f \in \mathcal{F} \setminus \mathcal{G}$, \hat{f} is defined by the program above. By definition, $I(E) \in \mathcal{H}$ due to Lemma 1. Therefore, $L(E) \in \mathcal{H}$ iff $E \in \mathcal{H}$ due to Lemma 1 again. Moreover, by definition, $I(L(E)) = \emptyset$.

According to the program it is easy to see that:

$$S = f^{-1}(M) \cup (\cup_{i=1}^n D_{x_i}) = f^{-1}(M) \cup (\cup_{i=1}^n I(D_{x_i})) \cup (\cup_{i=1}^n L(D_{x_i}))$$

In step 1, \hat{f} has been assigned with finite consequences on $f^{-1}(M)$; in step 2, \hat{f} has been assigned with consequences on finite subsets $I(D_{x_i})$ for some $x_i \in f(S) \setminus M$. On $\cup_{i=1}^n L(D_{x_i})$, the assignment process described in step 4, 5, and 6 stops in finite steps as well. In step 4, for any j , E_j is formed by the union of at least two L_i , because $\forall L_i \in O_j, I(L_j) = \emptyset$. Therefore $|N| \geq 2$ in step 4, which makes $|O_j|$ decrease by at least 2 in each iteration. In step 5, $|O_j|$ increases by 1 in each iteration. Therefore, O_j becomes empty eventually, which stops the program.

Suppose the program stops at $j = m$, then $\cup_{i=1}^n L(D_{x_i}) = \cup_{j=1}^m A_j$,

$$S = f^{-1}(M) \cup (\cup_{i=1}^n I(D_{x_i})) \cup (\cup_{i=1}^n L(D_{x_i})) = f^{-1}(M) \cup (\cup_{i=1}^n I(D_{x_i})) \cup (\cup_{j=1}^m A_j)$$

At the end of iteration $j = m$, consequences on all A_j are assigned. Since program assigns consequences uniquely, \hat{f} is well-defined.

Finally, by the program, $\hat{f}(s) = f(s)$ for any $s \in S \setminus (\cup_{j=1}^m A_j)$. On each event $A_j \in \mathcal{H}$, \hat{f} is assigned with $c_f^{A_j}$. By separability and the definition of c_f^H , it follows that $f \sim \hat{f}$. Clearly, for $f \in \mathcal{F} \setminus \mathcal{G}$, the \hat{f} constructed by the program is \mathcal{H} -measurable. ■

Definition 2. (Hedge Partition) For any $f \in \mathcal{F} \setminus \mathcal{G}$, $\hat{f}^{-1}(\hat{f}(S))$ forms a finite partition of S , the element of which are all \mathcal{H} -measurable sets, namely, $S = \cup_{x \in \hat{f}(S)} \hat{f}^{-1}(x)$ and $\hat{f}^{-1}(x) \in \mathcal{H}$ for all $x \in \hat{f}(S)$. Call this partition the **hedge partition** associated with f , denoted by $\{H_i\}_{i \in I}^f$ with $I = |\hat{f}(S)|$.

Clearly, if an act f is \mathcal{H} -measurable, it is also \mathcal{R} -measurable. Therefore, evaluating an ambiguous act f can be done by evaluating its \hat{f} using “small” domain SEU.

Proposition 1. (“Small” domain SEU over all acts) Axiom 1 to 5 hold if and only if there exists a continuous utility function u and a finely ranged probability measure $P : \mathcal{R} \rightarrow [0,1]$ such that the preference \succeq over \mathcal{F} is represented by the functional $U : \mathcal{F} \rightarrow \mathbb{R}$ given by:

$$U(f) = \sum_{x \in \hat{f}(S)} u(x)P(\hat{f}^{-1}(x)), \quad \forall f \in \mathcal{F} \quad (2)$$

In the representation (2), u is unique up to positive linear transformation and P is a finely ranged probability measure.

Proof of Proposition 1

If $f \in \mathcal{G}$, then $\hat{f} = f$. Model (2) reduces to [Kopylov \(2007\)](#)’s “small” domain SEU. Therefore, Axiom 1-6 are necessary conditions. The necessity of Axiom 7-10 follows directly from applying (2). Only sufficiency of Axiom 1-10 is shown here. Axiom 1-6 implies the model (1). Axiom 7-10 implies the well-defined $\hat{f} \sim f$ exists. For any $f, g \in \mathcal{F}$, $f \succeq g$ if and only if $\hat{f} \succeq \hat{g}$. Since \hat{f}, \hat{g} are risky measurable, (2) follows from applying the model (1). To show that u is continuous, apply Axiom 10 and follow the similar reasoning in Lemma B5 in the Appendix in [Gul & Pesendorfer \(2014\)](#). ■

By the definition of \hat{f} , it is easy to verify that $\forall f \in \mathcal{G}$, $\hat{f} = f$. Model (2) reduces to [Kopylov \(2007\)](#)’s “small” domain SEU for $f \in \mathcal{G}$. When $f \in \mathcal{F} \setminus \mathcal{G}$, (2) evaluates f by evaluating its hedge equivalent act \hat{f} . Therefore, [Kopylov \(2007\)](#)’s “small” domain SEU is embedded in the model (2).

2 General TSE model

The extended “small” domain SEU (2) models DM’s **second stage** evaluation in the TSE model. In the **first stage**, DM employs the conditional preference over acts to evaluate a certainty equivalent c_f^H of act f on each hedge $H \in \{H_i\}_{i \in I}^f$. It is assumed that the conditional preference on a hedge is represented by a local SEU on the hedge.

Axiom 12. (Conditional SEU on hedges) For any $H \in \mathcal{H}$, the conditional preference \succeq_H is represented by a continuous SEU.

Such a continuous SEU can be delivered by assuming more fundamental preference conditions as these assumed in the Theorem V.3.4. in [Wakker \(1989\)](#). When conditioning on each hedge, the entire state space is updated to the hedge. Accordingly, the distinction between risky and ambiguous events originated from different sources in the original state space S disappears. Under [Axiom 12](#), one immediately concludes that

$$f \succeq_H g \iff \sum_{x \in X} v_H(x) p_H(f^{-1}(x) \cap H) \geq \sum_{x \in X} v_H(x) p_H(g^{-1}(x) \cap H) \quad (3)$$

where v_H and p_H are utility and probability measure conditional on H . To make the “small” domain SEU become a special case of TSE, the notation of hedge partition associated with ambiguous act f in the [Definition 2](#) is abused to risky act. For any $f \in \mathcal{G}$, define $H_i := f^{-1}(x_i), x_i \in f(S)$ and $\{H_i\}_{i \in I}^f := f^{-1}(S)$ with $I = |f(S)|$. Embedding the local SEU to model [\(2\)](#) results in the TSE model.

Theorem 2. (A General TSE Model) Axiom 1 to 6 hold if and only if, for any $f \in \mathcal{F}$, there exist continuous increasing functions ϕ_i and v_i on each $H_i \in \{H_i\}_{i \in I}^f$ such that the preference \succeq over \mathcal{F} is represented by the functional $U : \mathcal{F} \rightarrow \mathbb{R}$ given by:

$$U(f) = \sum_{H_i \in \{H_i\}_{i \in I}^f} \phi_i \left(\sum_{x \in X} v_i(x) p_i(f^{-1}(x) \cap H_i) \right) P(H_i) \quad \forall f \in \mathcal{F} \quad (4)$$

where $\{H_i\}_{i \in I}^f$ is a hedge partition of f , $\phi_i = u \circ v_i^{-1}$ for all i , and u is the utility in [\(2\)](#) which is unique up to a positive linear transformation. On each hedge $H_i \in \mathcal{H}$, v_i is unique under a positive linear transformation $\tilde{v}_i = \alpha_i v_i + \beta_i$. Given this transformation, $\tilde{\phi}_i(\alpha_i v_i + \beta_i) = \phi_i(v_i)$. P is a probability measure defined in [\(2\)](#) and p_i is a probability measure conditional on hedge H_i .

Proof of Theorem 2

The necessity of the Axiom 1-11 follows the same reasoning in the proof of [Proposition 1](#). Only the sufficiency is shown here.

In the case of risky act $f \in \mathcal{G}$, $H_i \in \{H_i\}_{i \in I}^f$ is not necessarily in \mathcal{H} . If $H_i \in \mathcal{H}$, v_i and p_i are defined by the local conditional preference \succeq_{H_i} under [Axiom 12](#) as in the subsequent proof. If $H_i \notin \mathcal{H}$, define ϕ_i and v_i such that $u = \phi_i \circ v_i$; and define $p_i(H_j) = 0$ for $j \neq i$ and $p_i(H_i) = 1$. In both cases, the “small” domain SEU can be written as [\(4\)](#), since $u(x) = \phi_i \circ v_i(x)$ and $p_i(f^{-1}(x) \cap H_i) = p_i(H_i) = 1$ if $x \in f(H_i)$ and $p_i(f^{-1}(x) \cap H_i) = p_i(\emptyset) = 0$ if $x \notin H_i$.

Consider the case of ambiguous acts $f \in \mathcal{F} \setminus \mathcal{G}$. For any $f, g \in \mathcal{F} \setminus \mathcal{G}$, $f \succeq g \iff \hat{f} \succeq \hat{g}$, by

model (2),

$$\begin{aligned}
f \succeq g &\iff \sum_{x \in \hat{f}(S)} u(x)P(\hat{f}^{-1}(x)) \geq \sum_{x \in \hat{g}(S)} u(x)P(\hat{g}^{-1}(x)) \\
&\iff \sum_{H_i \in \{H_i\}_{i \in I}^f} u(\hat{f}^{-1}(H_i))P(H_i) \geq \sum_{H_j \in \{H_j\}_{j \in J}^g} u(\hat{g}^{-1}(H_j))P(H_j) \quad (5)
\end{aligned}$$

Under Axiom 12, apply SEU again to local preferences \succeq_{H_i} on both sides of (5), one can conclude that the preference \succeq_{H_i} can be represented by a local SEU conditional on $\forall H_i \in \mathcal{H}$. Therefore, for any $f \in \mathcal{F} \setminus \mathcal{G}$ and its associated $\{H_i\}_{i \in I}^f$, since $\hat{f} \sim_{H_i} f$, the $\hat{f}^{-1}(H_i)$ is given by

$$\hat{f}^{-1}(H_i) = v_i^{-1} \left(\sum_{x \in X} v_i(x) p_i(f^{-1}(x) \cap H_i) \right) \quad (6)$$

where v_i and p_i are hedge dependent local utility and probability measures respectively. The utility function v_i is unique up to a positive linear transformation and p_i is a unique probability measure for each i . Evaluate both sides of (6) by u ,

$$u(\hat{f}^{-1}(H_i)) = u \left(v_i^{-1} \left(\sum_{x \in X} v_i(x) p_i(f^{-1}(x) \cap H_i) \right) \right) = \phi_i \left(\sum_{x \in X} v_i(x) p_i(f^{-1}(x) \cap H_i) \right) \quad (7)$$

where $\phi_i = u \circ v_i^{-1}$ for each i and its continuity follows from the continuity of u and v_i . Substituting (7) into (5),

$$\begin{aligned}
f \succeq g &\iff \\
&\sum_{H_i \in \{H_i\}_{i \in I}^f} \phi_i \left(\sum_{x \in X} v_i(x) p_i(f^{-1}(x) \cap H_i) \right) P(H_i) \geq \sum_{H_j \in \{H_j\}_{j \in J}^g} \phi_j \left(\sum_{x \in X} v_j(x) p_j(g^{-1}(x) \cap H_j) \right) P(H_j)
\end{aligned}$$

Define $U(f) := \sum_{H_i \in \{H_i\}_{i \in I}^f} \phi_i \left(\sum_{x \in X} v_i(x) p_i(f^{-1}(x) \cap H_i) \right) P(H_i)$, the representation in the theorem can be obtained.

The uniqueness can be proved in the standard way by following the similar idea in the proof of the uniqueness of representation in [Klibanoff et al. \(2005\)](#). \blacksquare

When applied to $f \in \mathcal{G}$, the representation (4) reduces to “small” domain SEU by definitions of ϕ_i, v_i, p_i , which is a trivial case. The TSE occurs when the representation (4) is applied to $f \in \mathcal{F} \setminus \mathcal{G}$. In this case, the utility v_i and subjective probability measure p_i are both hedge H_i dependent, where p_i is a conditional subjective probability measure $p_i(s) = p(s|H_i), \forall s \in H_i$.

Remark 1. *In practice, a natural state space is usually finite as in the Ellsberg’s paradoxes and Machina’s paradoxes. However, [Kopylov \(2007\)](#)’s “small” domain SEU is based on an infinite space. To bridge the gap between the theory and the application, one can always assume that $S = S_K \times S_N$ where S_K is an infinite state space which satisfies the [Kopylov \(2007\)](#)’s theory, and S_N is a natural state space used to define consequences of choice objects.*

The “hedge” and the conditional SEU on hedge share some similarities with the “conditional small worlds” studied in [Chew & Sagi \(2008\)](#). However, [Chew & Sagi \(2008\)](#)’s preference model on a conditional small world depends on the consequences of act outside the world. In contrast, the separability of preference across hedges ($P2(\mathcal{F})$) is assumed in this paper. Moreover, hedges form a partition of the state space S , but [Chew & Sagi \(2008\)](#)’s “conditional small worlds” are not necessarily so.

In a more recent work, [Burkovskaya \(2020\)](#) proposed a “state aggregation” model, which also shares a similar representation with TSE. However, these two models are proposed based on totally different motivations. The stage aggregation model motivated by DM’s limited attention and lack of focus. The TSE in this paper is motivated by source dependent preferences. This results in different two-stage evaluations. The second stage evaluation in state aggregation model is based on a partition π which reflects how DM aggregate states and is unique. However, the second stage evaluation in TSE is not unique which is act dependent.

The TSE model also shares a similar “two-stage” idea adopted in [Gul & Pesendorfer \(2014\)](#)’s Expected Uncertainty Utility (EUU) model. In the first stage in the EUU, all non- \mathcal{H} -measurable act is converted to \mathcal{H} -measurable act. Then, in the second stage, a \mathcal{H} -measurable act is evaluated by a SEU model. There are two major differences between EUU and TSE, first, in the first stage when evaluating acts restricted to a hedge, EUU only evaluates the act based on the maximum and minimum consequences of the act on the hedge. In contrast, in this stage, the TSE model evaluates the sub-act based on all consequences on the hedge. Second, the subjective probabilities are derived in EUU based on stronger assumptions on state space similar to [Savage \(1954\)](#)’s SEU, namely a σ -algebra of “ideal” (hedge) events. In contrast, the subjective probabilities are delivered by a weaker version of SEU based on “small” domain axiomatized by [Kopylov \(2007\)](#).

In [Klibanoff et al. \(2005\)](#)’s KMM, an ambiguity aversion is modeled by a concave ϕ . In contrast, ambiguity aversion is hedge dependent in TSE. Moreover, KMM is a Bayesian model ([Marinacci 2015](#)), whereas TSE is not which is in the family of source-dependent preferences. In [Dobbs \(1991\)](#)’s Bayesian approach based model, [Klibanoff et al. \(2005\)](#)’s smooth ambiguity model, [Segal \(1987\)](#)’s recursive model, and [Ergin & Gul \(2009\)](#)’s two-stage model, a second order probability distribution over the first order probabilities is employed to model DM’s imprecise information on first order probabilities. In contrast, the TSE model only involves first order probabilities, the probabilities in the two evaluation stages represent two types of probabilities assigned to events with uncertainties from different sources.

3 Hedge independent attitude and uniform local probabilities

In this section, the TSE model (4) is restricted to two cases: (1) the conditional local utility is hedge independent; (2) local probabilities are uniform over finite consequences.

Intuitively, if DM's preferences over lotteries on each hedge are the same, the local utility v_i in model (4) becomes hedge independent. Actually, a weaker condition that requires preferences over all even chance binary lotteries on all hedges are the same can deliver a unique v over all hedges. For any $H \in \mathcal{H}$, consider all acts with $|f(H)| = 2$, namely acts with two consequences on H . Denote the two consequences of f by x and y , i.e. $f(H) = \{x, y\}$. For some $B \subset H$, if $(xB y)Hf \sim (yBx)Hf$, call such act f an even binary act on H . Denote the set of all such acts on $H \in \mathcal{H}$ by \mathcal{F}_H^e which is defined as:

$$\mathcal{F}_H^e := \{f | f \in \mathcal{F}, f(H) = \{x, y\} \text{ for some } x \neq y \in X, B = H \cap f^{-1}(x), (xB y)Hf \sim (yBx)Hf\}$$

Axiom 13. For any $H_i, H_j \in \mathcal{H}$, if $f, g \in \mathcal{F}_{H_i}^e$ and $f', g' \in \mathcal{F}_{H_j}^e$ with $f(H_i) = f'(H_j)$ and $g(H_i) = g'(H_j)$, then $f \succeq_{H_i} g$ if and only if $f' \succeq_{H_j} g'$.

Theorem 3. (Hedge Independent Attitude Towards Ambiguity) Under Axiom 1 to 6, Axiom 13 holds if and only if v_i 's are all equal to v and ϕ_i 's are all equal to ϕ in (4).

Proof of theorem 3

It is easy to verify that given model (4), Axiom 13 is implied by the hedge independent v and ϕ , only the sufficiency is shown. For $f \in \mathcal{G}$, the result is trivial.

For $f \in \mathcal{F} \setminus \mathcal{G}$, consider any $f \in \mathcal{F}_{H_i}^e$ with $f(H_i) = \{x, y\}$ and $B = H_i \cap f^{-1}(x)$. The lottery induced by f under p_i , denoted by f_{p_i} , is an even chance binary lottery. To see this, use the definition of $\mathcal{F}_{H_i}^e$ and the local SEU on H_i , it follows that

$$p_i(B)v_i(x) + (1 - p_i(B))v_i(y) = p_i(B)v_i(y) + (1 - p_i(B))v_i(x)$$

Since $x \neq y$, the equation above implies that $p_i(B) = 0.5$. Axiom 13 implies that the induced preferences over even chance lotteries of the form $\{0.5, x; 0.5, y\}$ on both H_i and H_j are the same.

The induced preference over even chance binary lottery is represented by $0.5v_i(x) + 0.5v_i(y)$ on H_i , which can be interpreted as an additive conjoint measurement in [Krantz et al. \(1971\)](#). Such an additive representation is unique under affine transformation. Therefore, for any two hedges H_i and H_j , Axiom 13 implies that $y = v_j(x) = \alpha v_i(x) + \beta$ for some $\alpha > 0$ and β . Thus, $x = v_j^{-1}(y) = v_i^{-1}\left(\frac{y-\beta}{\alpha}\right)$. On hedge j ,

$$\begin{aligned} v_j^{-1}\left(\sum_{x \in X} v_j(x) p_j(f^{-1}(x) \cap H_j)\right) &= v_j^{-1}\left(\sum_{x \in X} \alpha v_i(x) p_j(f^{-1}(x) \cap H_j) + \beta\right) \\ &= v_i^{-1}\left(\frac{\sum_{x \in X} \alpha v_i(x) p_j(f^{-1}(x) \cap H_j) + \beta - \beta}{\alpha}\right) \\ &= v_i^{-1}\left(\sum_{x \in X} v_i(x) p_j(f^{-1}(x) \cap H_j)\right) \end{aligned}$$

Substituting the above result to the proof of Theorem 2, the local utility functions on hedge i and j become the same. Since i and j are arbitrary, the hedge indexes of local utility functions can be dropped to obtain the desired result. ■

In some situations where DM has no information to believe that some states are more likely to occur than the others, she may intend to form uniform subjective probabilities conditional on each hedge. The Ellsberg's paradoxes and Machina's paradoxes where ambiguity is generated by lacking the information on proportions of balls in the urn are of this type. The following axiom is used to reduce the model from Theorem 3 to such a case where ambiguity aversion is hedge independent and each conditional subjective probability distribution on a hedge is a uniform distribution.

Axiom 14. For any $H \in \mathcal{H}$ and $f, g \in \mathcal{F}$, $f(H) = g(H)$ implies $f \sim_H g$.

When $f(H) = g(H)$, both acts f and g share the same image set in the consequence space. The axiom says that if their image set on H are the same, the DM is indifferent between the two acts.

Theorem 4. (Uniform Local Probabilities) Under Axiom 1 to 6, Axiom 13 and 14 hold if and only if v_i 's are all equal to v , ϕ_i 's are all equal to ϕ , and p_i on each H_i becomes uniform such that $p_i(x) = \frac{1}{|f(H_i)|}$, $\forall x \in f^{-1}(H_i)$ in (4).

Proof of theorem 4

It easy to verify these axioms are necessary; only the sufficiency is shown. Given theorem 3, it is sufficient to argue Axiom 14 implies uniform distribution for local SEU on each hedge.

Consider any H_i , suppose that p_i is not uniform in the sense that for any f , there exists $x, y \in f(H_i)$ such that $p_i(f^{-1}(x)) > p_i(f^{-1}(y))$. Then, let $g = f$ on $H_i \setminus \{f^{-1}(x), f^{-1}(y)\}$, $g(f^{-1}(x)) = y$, and $g(f^{-1}(y)) = x$. Clearly, if $x > y$, using local SEU on H_i implies that $f \succ_{H_i} g$ even though they share the same image set on H_i , which contradicts the Axiom 14 ■

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