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Published in:
Journal of Mathematical Economics

DOI:
[10.1016/j.jmateco.2019.10.002](https://doi.org/10.1016/j.jmateco.2019.10.002)

Publication date:
2019

Document version:
Accepted manuscript

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Citation for published version (APA):
He, Y., Dyer, J., Butler, J., & Jia, J. (2019). An additive model of decision making under risk and ambiguity. *Journal of Mathematical Economics*, 85, 78-92. <https://doi.org/10.1016/j.jmateco.2019.10.002>

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An Additive Model of Decision Making under Risk and Ambiguity

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Abstract: We extend the mean-variance (risk-value) tradeoff model to decision making under both risk and ambiguity. This model explicitly captures the tradeoff between the magnitude of risk and the magnitude of ambiguity. A measure that ranks lotteries in terms of the magnitude of ambiguity can also be obtained using this separation. By applying our model to asset pricing under ambiguity, we show that the equity premium can be decomposed into two parts: the risk premium and the ambiguity premium. Further, combining this model with the standard risk-value model, we build on the risk-ambiguity tradeoff to provide the value-risk-ambiguity preference model that does not rely on an approximation argument as the mean-variance model.

Key words: ambiguity measure, second order probability, asset pricing, equity premium puzzle

1. Introduction

Markowitz (1952)'s mean-variance analysis is the foundation of portfolio selection theory and the capital asset pricing model (CAPM). For any act f , the mean-variance preference is captured by a utility function based on the tradeoff between the mean $\mathbb{E}_\mu f$ and the variance $\sigma_\mu^2(f)$ of the distribution induced by f under probability measure μ as below.

$$U_{mv}(f) = \mathbb{E}_\mu f - \frac{\theta}{2} \sigma_\mu^2(f)$$

It is well-known that the mean-variance preference fails to be monotone in general (Dybvig and Ingersoll 1982, Jarrow and Madan 1997, Maccheroni et al 2009), which is inconsistent with von Neumann and Morgenstern's EU model. Feldstein (1969) and Baron (1977) have shown that the mean-variance preference model is consistent with EU only when the utility function is quadratic.

Despite these limitations, the interest in mean-variance preference and its applications in finance and other areas has continued due to its mathematical tractability and intuitive foundation. In the economics and finance literatures, a typical approach is to restrict the distribution of the random variable to a small but interesting class without restrictions on preferences (Bigelow 1993). In the decision analysis literature, an alternative approach is adopted. Rather than restricting distributions, scholars extended the mean-variance analysis framework to a risk-value model where value is measured by the utility of the mean and risk is measured by a more general concept than variance. Following the intuition of the mean-variance analysis, different models have been proposed to trade

off risk and value in both the expected utility (EU) framework (Sarin and Weber 1993, Jia and Dyer 1996, Schmidt 2003) and non-expected utility framework (Jia, Dyer, and Butler 1999, Jia and Dyer 2009).

In most real world decisions, objectively known probabilities are not available and many people exhibit an aversion to the consequences associated with subjective probability that cannot be uniquely specified and are therefore considered to be ambiguity averse. The objective of this paper is to extend the risk-value model to the context of decision making under ambiguity in an additive form.

Ambiguity may be created by missing information, by concerns about source credibility, or by expert disagreement about probabilities. As a result, the DM may be unable to specify a unique probability distribution for the uncertain consequence she is facing (Cerreia-Vioglio et al 2013). Given this interpretation, we model the ambiguous choice problem by using a two-stage lottery where the DM assigns subjective probabilities to different possible probability distributions. The two-stage lottery is represented by an Anscombe-Aumann (AA hereafter) act f , which maps states to probability measures over consequences. In each state there is a unique distribution representing the lottery faced by the DM. In the Subjective Expected Utility (SEU hereafter) framework, a subjective probability measure P over states can be derived from preferences, which is the second order probability. Under P , each f induces a two-stage compound lottery, which represents the ambiguous lottery in this paper.

In §3 we develop the concept of a “standardized act” f^A to separate ambiguity from risk. As the main result of the paper, we obtain a two-component additive model to separate the evaluation $\mathcal{V}(f)$ as

$$\mathcal{V}(f) = u(\mathbb{E}_P f) + \mathcal{U}(f^A)$$

where $\mathbb{E}_P f$ represents the expected risk of f obtained by reducing the two-stage lottery induced by f under P , which is called a reduced lottery in this paper; f^A represents the ambiguity of act f and $\mathcal{U}(f^A)$ is the utility loss/gain from bearing the ambiguity which shares the same recursive structure as other second order expected utility (SOEU) models in the literature (Klibanoff, Marinacci, and Mukerji 2005, Nau 2006, Grant et al 2009, and Neilson 2010). Our approach to separating risk and ambiguity is new to the literature.

In §5 we utilize the risk-value model (Jia and Dyer 1996) to express the utility from expected risk as: $u(\mathbb{E}_P f) = u(\mathbb{E}_\mu \mathbb{E}_P f) - \psi(\mathbb{E}_\mu \mathbb{E}_P f) \mathcal{R}(\mathbb{E}_P f - \mathbb{E}_\mu \mathbb{E}_P f)$ where $\mathbb{E}_\mu \mathbb{E}_P f$ is the mean of this reduced lottery under the first order probability μ , and $\psi(\mathbb{E}_\mu \mathbb{E}_P f)$ is a tradeoff factor. Substituting this decomposition into the above two-component additive separable model gives us a three-component model below.

$$\mathcal{V}(f) = u(\mathbb{E}_\mu \mathbb{E}_P f) - \psi(\mathbb{E}_\mu \mathbb{E}_P f) \mathcal{R}(\mathbb{E}_P f - \mathbb{E}_\mu \mathbb{E}_P f) - \mathcal{A}(f)$$

In this extended model, the utility of an act $\mathcal{V}(f)$ is evaluated in an additive separable model with three components, the utility of value $u(\mathbb{E}_\mu \mathbb{E}_P f)$, a measure of standardized risk $\mathcal{R}(\mathbb{E}_P f - \mathbb{E}_\mu \mathbb{E}_P f)$,

and a measure of standardized ambiguity $\mathcal{A}(f)$. This also introduces another novel contribution to the literature in §4 which is called the standard ambiguity measure defined as $\mathcal{A}(f) := -\mathcal{U}(f^A)$.

The preference represented by the above additive model and its three-component extension can be interpreted in the following way. Under ambiguity, the DM does not have a unique probability distribution over consequences. The expected distribution $\mathbb{E}_p f$ obtained from applying the Reduction of Compound Lottery (ROCL) to the two-stage lottery represents the risk component of f . After using this to evaluate the base utility level $u(\mathbb{E}_p f)$, the DM adjusts this base utility by the extra utility/disutility derived from bearing the ambiguity represented by $\mathcal{U}(f^A)$ to form her overall evaluation. This two-component additive separable model is consistent with the *anchoring and adjustment* process proposed to model decision making under ambiguity in both the vector expected utility (VEU) model (Siniscalchi 2009) and the mean-dispersion (MD) preference (Grant and Polak 2013). The base utility $u(\mathbb{E}_p f)$ in our model is consistent with the baseline expected utility in VEU and the mean utility in MD. In contrast to VEU and MD, the adjustment based on ambiguity in our model differs from these models, which is consistent with the smooth ambiguity model in Klibanoff, Marinacci, and Mukerji (2005, hereafter KMM) and other SOEU models (Nau 2006, Grant et al 2009, and Neilson 2010).¹

In the three-component additive separable model, the base utility $u(\mathbb{E}_p f)$ is further decomposed into the value and risk components to reflect the intuition that the DM adjusts the value $u(\mathbb{E}_\mu \mathbb{E}_p f)$ based on the mean of $\mathbb{E}_p f$ by the standard risk $\mathcal{R}(\mathbb{E}_p f - \mathbb{E}_\mu \mathbb{E}_p f)$ to evaluate the expected distribution $\mathbb{E}_p f$. The three-component model may be interpreted as a two-step adjustment model in the family of the *anchoring and adjustment* preferences. Dawes and Corrigan (1974) have shown that additive linear models are often consistent with the actual outcomes of the decision making process when DMs make tradeoffs between factors. Therefore, our additive model and its three-component extension are intuitive models which reflect a decision making process when a DM makes tradeoffs among different factors. These models also enjoy mathematical tractability as the components in both models are as easy to compute as the parameters of the mean-variance model. In particular, we show that the ambiguity measure $\mathcal{A}(f)$ can be approximated as the second order variance of the two-stage lottery induced from f , which has similar computational tractability as the use of variance to measure risk in the finance literature. As a result, our three-component additive model facilitates the analysis of portfolio selection and asset pricing when ambiguity is involved in a manner that has been shown to be generally descriptive.

The notion of separating risk and ambiguity is not totally new and there have been some other efforts toward such a goal. Maccheroni, Marinacci, and Ruffino (2013, hereafter MMR) applied the

¹ The smooth ambiguity model by KMM (2005), the SOEU by Grant et al (2009), and Neilson's model (2010) are axiomatized in different settings. But all of them share the same numerical representation when modeling ambiguity by two-stage lotteries. Therefore, we simply refer all these models as SOEU in this paper.

Taylor expansion to the KMM smooth ambiguity model to obtain a mean-variance-ambiguity three component separable approximation to the certainty equivalent of an act evaluated by KMM. Although the KMM model achieves a separation between *attitudes* toward risk and ambiguity (Chakravarty and Roy 2009), the *act and the utility over it* are combined under integration. MRR (2013) applied this approximation to a portfolio selection problem involving assets with ambiguous returns to study how trading off between ambiguity and risk influences the allocation of money to different assets. The extension of our additive model can achieve the same three component separation effect without relying on an approximation based on the Taylor expansion. Further, our three-component model is based on an axiomatic approach, which clearly provides the conditions on preferences that justify the use of this model.

In another series of studies, Izhakian (2017^a, 2017^b) proposed a decision-making model which achieves the separability between risk and ambiguity and derives an ambiguity measure that can be applied in the asset pricing model. As highlighted in a recent Wall Street Journal article (Eisen 2017), Berner and Izhakian (2018) are considering extending the VIX model of the implied volatility of options on the S&P500 – a so-called fear gauge – with a measure of the ambiguity of the market returns to capture the “fear” of ambiguity. A separable ambiguity model can be used to derive such an ambiguity measure and to decompose the equity premium into a risk premium and an ambiguity premium in a consumption-based capital asset pricing model (CCAPM). These models explain investor behavior (e.g. Brener and Izhakian 2018, Izhakian’s 2017^a, 2017^b) and can be applied to investigate how ambiguity influences investors’ stock option exercise decisions (Izhakian and Yermack 2017).

Our additive separable model under ambiguity can provide similar insights to Izhakian (2017^a, 2017^b) model but is both more tractable and general. Separation of risk and ambiguity is achieved by extending the SOEU (Grant et al 2009, Neilson 2010) and the final model is related to the well-known smooth ambiguity model in KMM (2005). Izhakian’s ambiguity measure is obtained by assuming that payoffs are “symmetrically distributed and probabilities of consequence are uniformly or elliptically distributed with the same expectation” (see Theorem 4, Izhakian 2017^a). In contrast, our ambiguity measure is derived for acts that induce general (first order) distributions of payoffs. Our ambiguity measure is also more intuitive as it can be approximated by the variance of the second order probability distribution over the first order utility induced by P , which shares the same spirit as using the variance of first order distribution to measure risk.

Our paper contributes to the literatures of decision theory and financial economics in the following ways. First, our separable model provides an intuitive and tractable model to disentangle the influences of risk and ambiguity in decision making under ambiguity, which results in the standard ambiguity measure proposed in the paper. This separation is useful in financial economics. For instance, we show that the equity premium of an asset with an ambiguous return can be

decomposed into the standard equity premium under risk plus an ambiguity premium relative to the standard ambiguity measure defined in the paper.

Second, our paper contributes to the family of decision-making models based on second order probability (SOP) with a recursive structure. The idea of using a SOP to represent ambiguity goes back to Marschak (1975) and Segal (1987) and has reappeared in the recent literature (KMM 2005, Nau 2006, Ergin and Gul 2009, Seo 2009, Grant et al 2009, Neilson 2010). All of these models share a similar recursive utility representation to accommodate ambiguity aversion. But, none of them are additive separable, and therefore they cannot disentangle the ambiguity from risk in the sense proposed in this paper. By assuming extra conditions beyond the SOEU model, we can obtain an additive structure to model a linear tradeoff between utility from risk and from ambiguity. Such a separation in the SOEU model has not been studied in the extant literature. Also, our model allows preferences failing to satisfy the uncertainty aversion axiom (Gilboa and Schmeidler 1989), which, however, is an axiom subscribed to by KMM and other SOEU models as we highlight in subsection 3.1. In subsection 3.2, we also point out that, in contrast to KMM and other SOEU models, one of the advantages of our model is that it explicitly reveals how DM trades off the risk component and the ambiguity component of an act in the spirit of the mean-variance analysis, which is not possible in the KMM.

Third, our model adds to the family of models with an additive form that captures decision making under ambiguity as *an anchoring and adjustment* process including both VEU model in Siniscalchi (2009) and the MD preference in Grant and Polak (2013). Our model shares a similar motivation with these two models as the tradeoff between the components in our model can also be interpreted as an anchoring and adjustment process. However, it distinguishes itself from these two models by using the SOEU model to capture ambiguity, which results in an ambiguity measure that is intuitive and tractable.

The rest of paper is organized as follows. In §2, we motivate the additive risk-ambiguity model by considering a simplified Ellsberg problem and define two concepts that separate an AA act into a risky component and an ambiguous component. §3 presents the main theorem in the paper for the additive model. In §4, we discuss the standard ambiguity measure obtained from our additive model and show that under a Taylor approximation, this measure reduces to the variance of the second order probability distribution. We also show that, in the context of ambiguous returns, the equity premium consists of two parts: the risk premium and the ambiguity premium. In §5, we further decompose our model of risk and ambiguity to obtain the standard value-risk-ambiguity tradeoff model. §6 concludes the paper.

2. Decomposing a two-stage lottery

In §2.1, we review the SOEU model which provides the underlying theory for our additive model. §2.2 covers the decision theory setup adopted in the paper, and §2.3 motivates the concepts of a risky-

reduced lottery and a standardized act which correspond to the risk component and the ambiguity component respectively in our separable model.

2.1 Review of related models

Grant et al (2009) axiomatized the SOEU model representing preferences over AA acts

$$u(f) = \sum_{s \in S} \phi(u(f(s)))P(s) \quad (1)$$

where $u(f(s))$ is the vNM utility of probability measure $f(s)$, namely $u(f(s)) = \sum_{x \in \text{supp}(f(s))} u(x)f_s(x)$ and $f_s = f(s)$ is the probability measure over payoffs delivered by act f in state s . In this model, act f specifies a different probability measure over consequences in different states, which represents the DM's inability to identify a unique probability distribution over consequences. When $f(s)$ specifies the same probability measure in every state s , the above model reduces to $\phi(u(f(s)))$, a monotonic transformation of vNM utility. When f specifies different probability measures in different states and ϕ is linear, model (1) is consistent with the concept of the reduction of compound lotteries (ROCL) to evaluate the two-stage compound lottery induced by the act f under P . But, when ϕ is concave, model (1) decreases the evaluation of f compared to the case of a linear ϕ to capture DM's aversion to the non-uniqueness of a first order probability measure over payoffs. The ambiguity aversion discussed in this paper is consistent with this interpretation.

Neilson (2010) also axiomatized a model for more general acts with infinite consequences where the expectation is calculated by integration to simplify the axiomatization of ambiguity aversion in KMM (2005). We adopt the AA framework used in Neilson (2010) to obtain the SOEU model. In KMM (2005), both first order and second order utilities are modeled as Savage's SEU, which is more general than the SOEU in Neilson (2010). However, if one induces an objective probability distribution under the implied first order subjective belief in KMM, the KMM model reduces to a model consistent with SOEU and Neilson's model.

Following the publication of KMM (2005), their smooth ambiguity model and the SOEU have been empirically tested and applied to asset pricing. Halevy (2007) conducted an experimental study to compare the descriptive power and predictive abilities of some popular models for decision making under ambiguity, including the recursive expected utility model (1). This study also demonstrates a close association between ambiguity neutrality and the reduction of compound lotteries. More recently, Cubitt et al (2019) provided stronger empirical support for KMM. Chakravarty and Roy (2009) used the smooth ambiguity model to investigate the relationship between risk and ambiguity attitudes and the presence of domain effects on ambiguous preference. Ahn et al (2014) estimated (1) in a portfolio-choice experiment while Ju and Miao (2012) applied this model in a dynamic setting to study asset pricing under ambiguity. Borgonovo and Marinacci (2015) also applied the KMM in the context of decision analysis by incorporating ambiguity aversion into a decision tree.

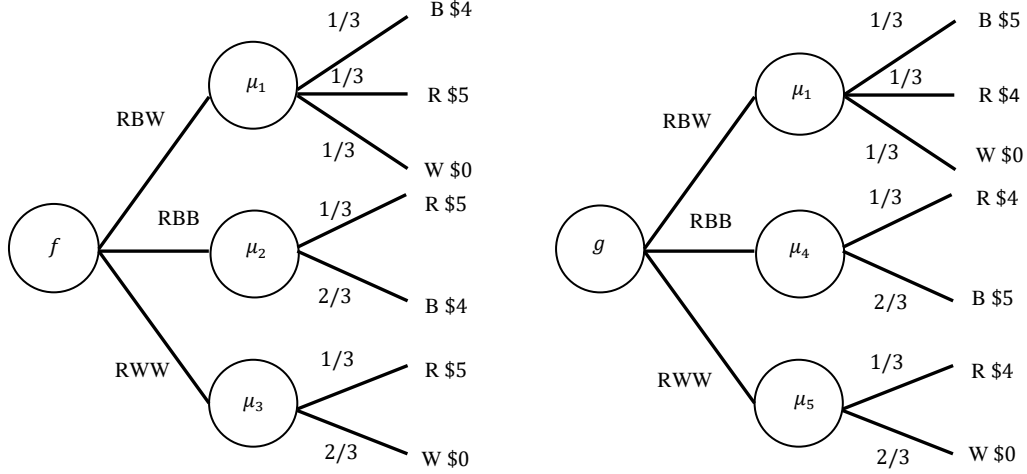
2.2 Model setup

Following the SOEU model of Grant et al (2009) and Neilson (2010), we adopt Anscombe and Aumann's (1963) framework in this paper. We consider an infinite state space S ; a generic event is a subset of S . Let \mathcal{L} designate the set of simple probability measures (roulette lotteries) denoted by μ, λ defined on a set of consequences X , which represent first order lotteries with objective probabilities. The set of consequences X is a bounded interval of real numbers containing zero and may be interpreted as monetary payoffs.

There are two types of uncertainty in our model. For the first type, the DM has enough information to identify a unique distribution (objective probability measure) of payoffs of a choice object (represented by μ) which is interpreted as pure risky lottery in our model. Ambiguity arises due to the other type of uncertainty when the DM does not have enough information to identify a unique distribution. The second type of uncertainty is represented by a simple AA act f , which is a function that maps the state space S to finite simple lotteries \mathcal{L} , i.e. $f: S \rightarrow \mathcal{L}$. These AA acts $f, g \in \mathcal{F}$ contain all simple acts except those for which $\forall s \in S, f(s) = \delta_x$ for some $x \in X$, where δ_x is a degenerate probability measure on X that assigns probability 1 to point $x \in X$. In our context, $f(s) = \delta_x$ does not represent a choice object, as it is neither an ambiguous lottery nor a risky lottery as defined in this paper. This does not exclude acts which may assign a degenerate probability measure δ_x to some s , but a non-degenerate measure μ to other s . Such an act can still be interpreted as an ambiguous lottery for which the DM is uncertain about the objective distribution in some states, but believes that in other states the objective distribution degenerates to a point.

For any $f \in \mathcal{F}$, when a state $s \in S$ occurs, a simple probability measure $f(s) = \mu, \mu \in \mathcal{L}$ is obtained. If $f(s) \equiv \mu$ for any $s \in S$, then f is referred to as a constant act. The convex combination of acts is performed pointwise as in the literature, i.e., $\alpha f + (1 - \alpha)g = h$, where $h(s) = \alpha f(s) + (1 - \alpha)g(s) = f(s)\alpha g(s)$ for all $s \in S$, which represents the lottery compounding operation. In this paper, an "act" refers to a function that maps from states to probability measures. In contrast, a "lottery" refers to a probability measure defined either on consequences or probability measures, where the latter case corresponds to a two-stage lottery. A comprehensive list of notation used in this paper is available in the Appendix.

Figure 1. Illustration of acts f and g



To motivate our additive model, we consider a DM who faces a simplified Ellsberg choice problem. Suppose there is an urn that contains three balls, where one is known to be red (R) but the color of the other two balls is unknown and each ball could be either white (W) or black (B). One ball will be drawn randomly from the urn and the DM will be paid based on the color that is selected. The DM is offered a choice between two options depicted in Figure 1. In the first, the DM will be paid \$5 for a red ball, \$4 for a black ball, and \$0 for a white ball. In the second, the DM will be paid \$4 for a red ball, \$5 for a black ball, and \$0 for a white ball. Since the DM cannot identify a unique payoff distribution for each case, the two options are ambiguous lotteries. These two ambiguous lotteries are modeled by two AA's acts f and g respectively. There are three possible states for each urn's composition: RBW, RBB, or RWW. In each state, the acts f and g assign different probability distributions to payoffs depending on the composition of the urn. These two acts f and g are illustrated in Figure 1, where μ is used to designate the objective lottery faced by the DM. For each μ , probabilities can be unambiguously determined by the DM given that the composition of the urn is known in each state.

To obtain the infinite state space used in Savage's SEU theory, we can augment the natural state space to $\Omega = \{RBW, RBB, RWW\} \times [0,1]$ and redefine the acts as $f(RBW \times [0,1]) = \mu_1$, $f(RBB \times [0,1]) = \mu_2$, and $f(RWW \times [0,1]) = \mu_3$. As this augmentation can always be done for a finite natural state space, we omit this from further discussion.

Our recursive expected utility model is based on the assumptions below which are the same as those used in Neilson (2010).

Assumption 1. (Savage's Axioms) *The preference order over \mathcal{F} satisfies the axioms required by Savage's SEU theory.*

Under assumption 1, we know that preferences \succeq over simple acts can be represented by SEU in the following form for simple acts (see theorem 14.4, Fishburn 1970), where \mathcal{P} is a unique finite additive probability measure over the state space.

$$\mathcal{U}(f) = \sum_{s \in \mathcal{S}} u(f(s)) \mathcal{P}(s) \quad (2)$$

Assumption 2. (vNM's Axioms). *The preference order over constant acts (roulette lotteries) satisfies the axioms required by von Neumann and Morgenstern's utility theory.*

Under Assumption 2, a vNM EU model (3) defines the preference order over all constant acts or simple objective probability measures, i.e., $f(s) \equiv \mu$

$$u(\mu) = \sum_{x \in \text{supp}\{\mu\}} u(x) \mu(x) \quad (3)$$

As shown in Neilson (2010), the utility function $\mathcal{U}(f(s))$ in model (2) also represents the preference order over simple objective probability measures. Therefore, there exists a monotonic transformation ϕ between $\mathcal{U}(f(s))$ and $u(\mu)$ for $f(s) = \mu$, namely $\mathcal{U}(f(s)) = \phi(u(\mu)) = \phi(\sum_{x \in \text{supp}\{\mu=f(s)\}} u(x) \mu(x))$. This results in the recursive utility model below that represents the preference order \succeq on \mathcal{F} .

$$\mathcal{U}(f) = \sum_{s \in \mathcal{S}} \phi \left(\sum_{x \in \text{supp}\{\mu=f(s)\}} u(x) \mu(x) \right) \mathcal{P}(s) \quad (4)$$

As we noted earlier, model (4) achieves descriptive power by relaxing the ROCL principle implicitly imposed in both EU and SEU. Kreps (pp. 50-52, 1989) referred to this principle as axiom zero in EU. Luce and von Winterfeldt (1994) discussed the common ground of descriptive, prescriptive, and normative decision theories, and classified the rationality axioms imposed by decision theories into three families, namely structural rationality, preference rationality, and quasi-rationality. One of the important structural rationality axioms is the ROCL. The preference rationality axioms include all the preference conditions assumed in EU and SEU. They argued that these preference rationality axioms are more likely to be descriptively valid than many forms of structural rationality. Such an argument was confirmed in Halevy's (2007) experiment, which concluded that a descriptive theory that accommodates ambiguity aversion should also account for violations of ROCL. Consistent with these arguments in the literature, the descriptive validity and flexibility of model (4) is achieved by abandoning the ROCL while keeping the other standard axioms on preferences in both EU and SEU.

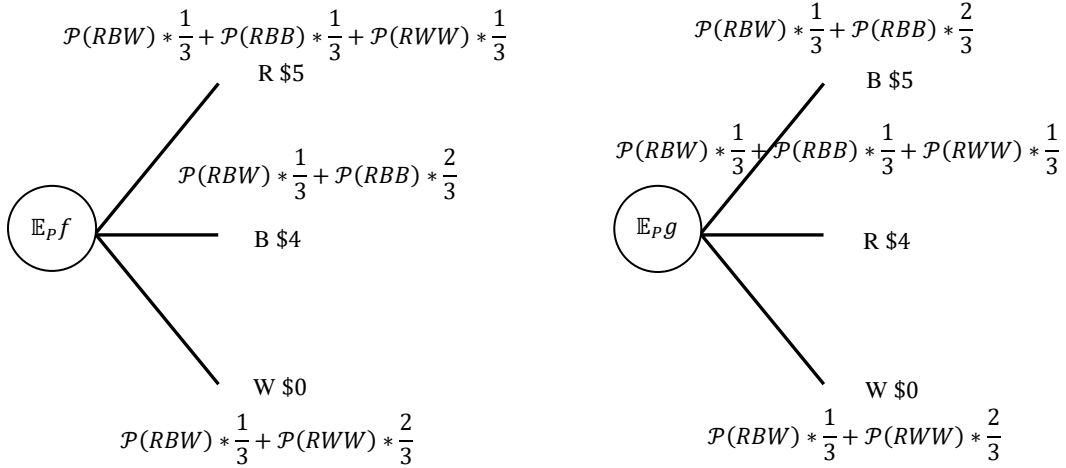
2.3 Risky reduced-lottery and standardized act

Ambiguity aversion captures the fact that the DM dislikes situations where she cannot uniquely identify an objective probability distribution, which corresponds to a concave ϕ in model (4). When ϕ is linear or equivalently when the DM applies ROCL in the model, she does not differentiate between decision making under uncertainty and decision making under risk and her choices are ambiguity neutral. Therefore, we refer to the result of applying the ROCL process to an act as the *risky reduced-*

lottery because the influence of ambiguity aversion on the act has been “reduced”, i.e. eliminated, but the influence of risk remains. Thus, our risk component – a risky reduced-lottery – should be interpreted as a component in an act over which the DM’s behavior is consistent with the SEU model.

Using the subjective probabilities from Savage’s SEU in model (4), an ambiguity-neutral decision maker would reduce an act to a risky reduced-lottery, $\mathbb{E}_P f$, by taking the expectation with respect to the second order subjective probabilities. For the acts in Figure 1, this expectation is $\mathbb{E}_P f = \mathcal{P}(BRW)\mu_1 + \mathcal{P}(RBB)\mu_2 + \mathcal{P}(RWW)\mu_3$, which is the expected objective lottery under the implied subjective beliefs \mathcal{P} . Or equivalently, when ϕ is linear, the DM’s preference represented by model (4) is ambiguity neutral (KMM 2005). If the DM has equal subjective probabilities $\mathcal{P}(BRW) = \mathcal{P}(RBB) = \mathcal{P}(RWW) = 1/3$ and applies model (4), the risky reduced-lotteries $\mathbb{E}_P f$ and $\mathbb{E}_P g$ are the same, i.e., $\mathbb{E}_P f = \mathbb{E}_P g = \left\{ \frac{1}{3}, \$5; \frac{1}{3}, \$4; \frac{1}{3}, \$0 \right\}$; see Figure 2.

Figure 2. Risky reduced-lotteries $\mathbb{E}_P f$ and $\mathbb{E}_P g$



Formally, we define the risky reduced-lottery for an act f as the expectation under the second order subjective probability \mathcal{P} .

Definition 1. *The risky reduced-lottery for $f \in \mathcal{F}$ under a second order subjective probability \mathcal{P} is the expectation given by $\mathbb{E}_P f := \sum_{s \in S} \mathcal{P}(s) f(s) \in \mathcal{L}$, which is a first order objective probability measure.*

Since both f and g have the same risky reduced-lottery under the assumed equal subjective probability distribution \mathcal{P} , these reduced lotteries cannot affect the preference between the acts f and g . The preference between the pair will be based on the DM’s comparison of these acts in terms of the magnitude of their ambiguity. Intuitively, we would like to subtract the risky reduced-lottery from the act f to isolate the ambiguity of the lottery. In general, it is not clear how to subtract one lottery from another, and so instead, we assess the willingness-to-pay (*wtp*) for the risky reduced-lottery and subtract this quantity from all the outcomes of f .

For example, assume that the DM in our example has an exponential utility function of the form $u(x) = 100 - 100 \exp(-0.05x)$ for preference over first order lotteries. We can find her willingness-to-pay (wtp) for the risky reduced-lottery $\mathbb{E}_P f = \{\frac{1}{3}, \$5; \frac{1}{3}, \$4; \frac{1}{3}, \$0\}$ under the assumed equal subjective probabilities \mathcal{P} by solving for $wtp(\mathbb{E}_P f)$ in the relationship $\mathbb{E}_P f - wtp(\mathbb{E}_P f) \sim 0$. Solving

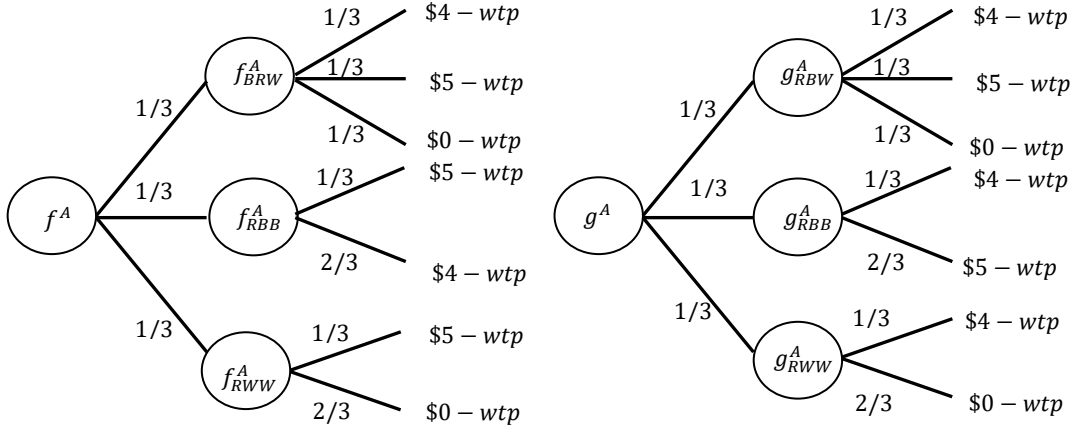
$$u(0) = 100 - \frac{100}{3} \left[\exp(-0.05 * (\$5 - wtp(\mathbb{E}_P f))) + \exp(-0.05 * (\$4 - wtp(\mathbb{E}_P f))) + \exp(-0.05 * (\$0 - wtp(\mathbb{E}_P f))) \right]$$

for $wtp(\mathbb{E}_P f)$, we obtain \$2.88 for this example. This willingness-to-pay for the risky reduced lotteries $\mathbb{E}_P f = \mathbb{E}_P g$ reflects the DM's evaluation of (only) the riskiness of the outcomes associated with f and g . Formally, for any f , a unique **willingness-to-pay** $wtp(\mathbb{E}_P f) \in X$ for the risky reduced lottery $\mathbb{E}_P f$ can be obtained from the following indifference relation

$$\mathbb{E}_P f - wtp(\mathbb{E}_P f) \sim 0 \quad (5)$$

The $wtp(\mathbb{E}_P f)$ is unique and well defined since preference is assumed to be monotonic.

Figure 3. Illustration of standardized acts



Note: $f_s^A = f(s) - wtp$, $s \in \{BRW, RBB, RWW\}$; $wtp = wtp(\mathbb{E}_P f) = wtp(\mathbb{E}_P g)$ in this example

The willingness to pay for the acts f and g can be obtained from the indifference relations $f - wtp(f) \sim 0$ and $g - wtp(g) \sim 0$, where $f - wtp(f)$ represents an act obtained by subtracting $wtp(f)$ from all payoffs of the simple objective measures of f realized in different states. Since $wtp(f)$ represents the evaluation of the act f and $wtp(\mathbb{E}_P f)$ represents the evaluation of the risky reduced-lottery associated with f , the quantity $wtp(f) - wtp(\mathbb{E}_P f)$ is the evaluation of the remaining feature, the ambiguity of the lottery. Because $f - wtp(\mathbb{E}_P f) - (wtp(f) - wtp(\mathbb{E}_P f)) \sim 0$, we can conclude that $f - wtp(\mathbb{E}_P f)$ also represents the evaluation of ambiguity in f . Motivated by this observation, we define a quantity $f^A = f - wtp(\mathbb{E}_P f)$ to represent the ambiguity in an act

after eliminating the risky component. Similar operations can also be applied to lottery g , and both f^A and g^A are shown in Figure 3.

Definition 2: *The standardized act of act $f \in \mathcal{F}$ under subjective probability \mathcal{P} , denoted by f^A , is defined as $f^A: S \rightarrow \mathcal{L}$ such that for any $s \in S$, $f^A(s) = f(s) - wtp(\mathbb{E}_P f)$.*

From this definition, it is easy to see that the risky reduced-lottery of any standardized act f^A is indifferent to zero, i.e.; $\mathbb{E}_P(f^A) := \mathbb{E}_P f - wtp(\mathbb{E}_P f) \sim 0$. Thus, this concept of a standardized act brings all acts to the same zero level in the payoff space in terms of their risk components. This allows an act f to be decomposed into its risky reduced-lottery $\mathbb{E}_P f$ and its standardized act f^A . Therefore, any $f \in \mathcal{F}$ can be uniquely represented by $(\mathbb{E}_P f, f^A)$. The standardized act f^A may be interpreted as the ambiguity after eliminating the expected first order risk.

In the example in Figure 1, f and g have the same risky reduced-lottery under the assumed equal subjective probabilities $\mathbb{E}_P f = \mathbb{E}_P g = \left\{ \frac{1}{3}, \$5; \frac{1}{3}, \$4; \frac{1}{3}, \$0 \right\}$, so the DM can compare f^A and g^A to determine her preference between f and g . If the DM is ambiguity neutral, she is indifferent between f^A and g^A and we can conclude that the DM is also indifferent between f and g . If the DM is ambiguity averse, the DM's preference of f or g can be determined by comparing only the standardized acts f^A and g^A . As before in the case of the risky reduced lottery, we represent the DM's preference over first order lotteries by the exponential utility function $u(x) = 100 - 100 \exp(-0.05x)$, which exhibits risk aversion. Suppose that the DM's second order ambiguity aversion is represented by U ; and the transformation in $U = \phi \circ u$ is a logarithmic function $\phi(x) = 20 * \log\left(1 + \frac{x}{10}\right)$. Given $wtp(\mathbb{E}_P f) = \$2.88$ as calculated previously, the utility of f^A is determined from (4) to be

$$\begin{aligned} \mathcal{U}(f^A) &= \frac{20}{3} \log\left(1 + \frac{u(5 - 2.88) + u(4 - 2.88) + u(0 - 2.88)}{3 * 10}\right) \\ &\quad + \frac{20}{3} \log\left(1 + \frac{u(5 - 2.88) + 2u(4 - 2.88)}{3 * 10}\right) \\ &\quad + \frac{20}{3} \log\left(1 + \frac{u(5 - 2.88) + 2u(0 - 2.88)}{3 * 10}\right) = -1.93 \end{aligned}$$

A similar calculation yields $\mathcal{U}(g^A) = -3.74$. The risky reduced lotteries are $\mathbb{E}_P f = \mathbb{E}_P g$, so the DM prefers f to g due to $\mathcal{U}(f^A) > \mathcal{U}(g^A)$.

Intuitively, an ambiguity averse decision maker will seek to avoid the variation of the expected utility of first order lotteries. From Figure 3, we can see that $g_{RBB}^A - wtp$ is preferred to $f_{RBB}^A - wtp$, i.e., $\frac{1}{3}u(\$4 - wtp) + \frac{2}{3}u(\$5 - wtp) > \frac{1}{3}u(\$5 - wtp) + \frac{2}{3}u(\$4 - wtp)$; $f_{RWW}^A - wtp$ is preferred to $g_{RWW}^A - wtp$, i.e., $\frac{1}{3}u(\$5 - wtp) + \frac{2}{3}u(\$0 - wtp) > \frac{1}{3}u(\$4 - wtp) + \frac{2}{3}u(\$0 - wtp)$; and $u(g_{RBW}^A - wtp) = u(f_{RBW}^A - wtp)$. Given that $\mathbb{E}_P g^A \sim \mathbb{E}_P f^A \sim 0$, i.e., $\frac{1}{3}u(g_{RBB}^A - wtp) +$

$\frac{1}{3}u(g_{RWW}^A - wtp) + \frac{1}{3}u(g_{RBW}^A - wtp) = \frac{1}{3}u(f_{RBB}^A - wtp) + \frac{1}{3}u(f_{RWW}^A - wtp) + \frac{1}{3}u(f_{RBW}^A - wtp)$, it is easy to see that g^A is a mean preserving spread of f^A in terms of the first order utility. The utilities of the first order objective lotteries in g^A are more spread out (have larger variation) than those of f^A . Thus, an ambiguity averse DM with a concave ϕ prefers f^A to g^A .

In this example, two acts have the same risky reduced-lottery and they can be ranked based solely on their standardized acts. Similarly, when the acts share the same standardized acts, we can also compare them according to their risky reduced-lotteries. This separation implies an additive utility representation for preference, and we can restrict our study of a DM's preference over acts to the smaller set of standardized acts that have a risky reduced-lottery indifferent to zero; i.e. $\mathbb{E}_p f \sim 0$. We formalize this separation concept in the next section.

3. An additive two attribute model of decision making under ambiguity

In §3.1, we introduce two assumptions to derive an additive model based on the risky reduced lottery and the standardized act, which is the first major result in the paper. In §3.2, we discuss an implied additive model under subjective beliefs.

3.1 Trading off the risky reduced lottery and the standardized act

Based on results in §2, an act $f \in \mathcal{F}$ can be uniquely represented by a pair $(\mathbb{E}_p f, f^A)$ with $\mathbb{E}_p f \in \mathcal{L}$. We assume that a DM's preference is monotonic on each component in the pair.

Assumption 3. (Monotonicity) For any $f, g \in \mathcal{F}$, if $\mathbb{E}_p f \sim \mathbb{E}_p g$, $f^A \succeq g^A$ implies $f \succeq g$; if $f^A \sim g^A$, $\mathbb{E}_p f \succeq \mathbb{E}_p g$ implies $f \succeq g$.

Intuitively, Assumption 3 says that when two acts share a common risky reduced-lottery, the preference between them should only depend on their standardized acts as described in the example in Section 2. In addition, when two acts share a common standardized act, preference should only depend on their risky reduced-lotteries. This assumption is consistent with our intuition that if a DM has an additive structure for evaluating acts, her preferences over one component of the pair $(\mathbb{E}_p f, f^A)$ should be independent of the other component. This assumption is similar to mutual preferential independence that is implied by the additive value function (Krantz et al. 2006, Keeney and Raiffa 1993). Jia and Dyer (1996, 2009) employed a similar independence condition to axiomatize their risk-value models but assume a one-way independence condition in their models.

The next assumption requires that the set of acts meets a solvability condition.

Assumption 4. (Solvability) When preference is not ambiguity neutral, for any $f \succeq g$, there exists $h \in \mathcal{F}$ such that $h \sim f$ with $h^A \sim g^A$.

This assumption says that we have a rich set of acts under consideration and that preferences change smoothly. It is motivated by the assumption of solvability in the axiomatic foundation of multi-attribute utility (value) theory (e.g., Krantz et al. 2006). In our context, the set of standardized act is bounded at zero, which represents the minimum level of ambiguity. Under the assumption of ambiguity aversion, zero is the most preferred element (upper bound) in the set of standardized acts,

namely $0 \succeq f^A$ for $\forall f \in \mathcal{F}$. Due to this upper bound for the most preferred standardized act, we may not freely make a tradeoff by decreasing the attractiveness of a risky reduced-lottery and increasing the attractiveness of a standardized act when this upper bound is reached. However, because the set of risky reduced-lotteries is unbounded, we can always find h such that $h \sim f$ with $h^A \sim g^A$.

When Assumptions 1, 2, 3, and 4 all hold, there exists an additive representation of the DM's preference over acts that captures the tradeoff between the expected risk and the standardized act. To obtain this additive model, we use Assumptions 3 and 4 to induce a preference over the product set containing all pairs of the form $(\mathbb{E}_p f, f^A)$ based on the three axioms of vNM. The induced preference over these pairs is defined according to the original preference over the set of acts \mathcal{F} . Fishburn (Theorem 1, Chap 9.1, 1982) shows that if preference over the product of mixture sets satisfies vNM's axioms, there exists an additive vNM utility to represent the preference order. We can show that the sets of the risky reduced-lottery and standardized act are both mixture sets, so it follows that there exists an additive representation for this induced preference order.

Theorem 1 (Additive Risk Ambiguity Tradeoff Model) *Assumptions 1, 2, 3, and 4 imply that there exists an additive representation for \succeq, \mathcal{V} , such that for any $f, g \in \mathcal{F}$, $f \succeq g$ iff $\mathcal{V}(f) \geq \mathcal{V}(g)$ with*

$$\mathcal{V}(f) = u(\mathbb{E}_p f) + \mathcal{U}(f^A) \quad (6)$$

where u is von Neumann Morgenstern utility function for $\mu = \mathbb{E}_p f$ given by (3); and \mathcal{U} is the recursive expected utility model (4) with $\mathcal{U}(0) = 0$.

Proof: The proof is provided in the appendix.

As we discussed in the introduction, this additive separable model can be related to two streams of research. On the one hand, it is an extension of a mean-variance (risk-value model) from decision making under risk to decision making under ambiguity. Such a model explicitly captures how the DM makes a tradeoff between “risk” and “ambiguity” components defined in this paper. On the other hand, the model is also related to the VEU (Siniscalchi 2009) and MD (Grant and Polak 2013), which can be interpreted in an anchoring and adjustment framework. In this framework, the DM evaluates an act based on its SEU, which is then adjusted (distorted) according to a measure of ambiguity. In this additive model (6), the utility of the risky reduced lottery $u(\mathbb{E}_p f)$ is also used as the mean utility in the MD (Grant and Polak 2013). However, our model (6) differs from the MD by applying the SOEU (Grant 2009, Neilson 2010) to the standardized act f^A when evaluating the dispersion (ambiguity). This separation implies that the ambiguity obtained from eliminating the expected risk can be measured by the SOEU model over our standardized act, which is discussed in section 4.

As reviewed in Section 2.3, the recursive model (4) may be assessed by assuming a functional form from a parametric family to obtain $\mathbb{E}_p f$ and f^A for any f . With explicit values for $\mathbb{E}_p f$ and f^A , it is possible to numerically verify Assumptions 3 and 4 that imply the additive model (6). Even when

such an assessment is difficult, if the DM believes that the definitions of $\mathbb{E}_P f$ and f^A capture properties that are consistent with her preferences, model (6) can be adopted as a representation of her preferences. In the smooth ambiguity model (KMM 2005), the second order act may also be difficult to observe. KMM (2005) argue that “even when verifiability is an issue, preference axioms provide a useful conceptual underpinning to choice criteria. For example, economists often apply the subjective expected utility model to a variety of situations characterized by limited verifiability”. We agree with the theme of this argument and interpret our assumptions as highlighting what must be true to support an additively separable risk ambiguity model.

Our additive risk ambiguity tradeoff model (6) can accommodate three types of ambiguity attitudes, namely ambiguity averse, ambiguity seeking, and ambiguity neutral. Consistent with KMM (2005), a concave (linear, convex) ϕ defined in model (4) in the recursive subjective utility \mathcal{U} in the model (6) corresponds to an ambiguity averse (neutral, seeking) DM (proposition 1, KMM 2005). Also, when (6) is applied to a constant act $f \equiv \mu$, the utility of a standardized act becomes zero as $f^A = \mu - wtp(\mu) \sim 0$. In this case, $\mathcal{U}(f^A) = \phi(u(\mu - wtp(\mu))) = \phi(0) = 0$, as the utility is scaled such that $u(0) = \phi(0) = 0$; and the additive risk ambiguity tradeoff model (6) reduces to the expected utility model in the absence of ambiguity. For a non-degenerate act and an ambiguity averse DM, $u(\mathbb{E}_P(f^A)) = 0$ and a concave ϕ implies that $\mathcal{U}(f^A)$ is non-positive. Thus, an ambiguity averse DM derives disutility from the standardized act of f^A . A more concave ϕ increases the disutility from $\mathcal{U}(f^A)$ in model (6) and thus reduces $\mathcal{V}(f)$. Model (6) focuses on separating the disutility of ambiguity from the total utility of an act, which has not been accomplished in the existing literature on decision making under ambiguity.

Finally, we relate our model to the uncertainty aversion axiom proposed in Gilboa and Schmeidler (1989) to axiomatize Maxmin Expected Utility. This axiom says that for any acts (horse-roulette lotteries) $f, g \in \mathcal{F}$ and $\alpha \in [0,1]$, $f \sim g$ implies that $\alpha f + (1 - \alpha)g \succeq f$, which is also satisfied by other models for decision making under ambiguity such as the variational preference model (Maccheroni et al 2006), ambiguity based on a confidence function (Chateauneuf and Faro 2009), as well as the KMM (2005) and other SOEU models (Grant et al 2009, Neilson 2010) for concave ϕ . In our model (6), the total utility of an act is the sum of two parts where the second part is a SOEU model. Therefore, uncertainty aversion in (6) holds for any two indifferent acts f and g when they share indifferent risky reduced-lotteries, i.e. $\mathbb{E}_P f \sim \mathbb{E}_P g$. To see this, notice $wtp(\mathbb{E}_P f) = wtp(\mathbb{E}_P g) = wtp(\mathbb{E}_P(f\alpha g))$, which can be denoted by WTP , our model implies

$$\mathcal{V}(f\alpha g) = u(\mathbb{E}_P(f\alpha g)) + \sum_S \phi \left(u \left(f\alpha g - wtp(\mathbb{E}_P(f\alpha g)) \right) \right) \mathcal{P}(s) = u(\mathbb{E}_P f \alpha \mathbb{E}_P g) + \sum_S \phi(u(f\alpha g - WTP)) \mathcal{P}(s)$$

$$\begin{aligned}
&= \alpha u(\mathbb{E}_P f) + (1 - \alpha)u(\mathbb{E}_P g) + \sum_S \phi(\alpha u(f - WTP) + (1 - \alpha)u(g - WTP))\mathcal{P}(s) \\
&\geq \alpha u(\mathbb{E}_P f) + (1 - \alpha)u(\mathbb{E}_P g) \\
&\quad + \sum_S (\alpha \phi(u(f - WTP)) + (1 - \alpha)\phi(u(g - WTP)))\mathcal{P}(s)
\end{aligned}$$

where we use the concavity of ϕ to obtain the inequality. The RHS of this inequality equals to

$$\begin{aligned}
&\alpha \left(u(\mathbb{E}_P f) + \sum_S \phi(u(f - wtp(\mathbb{E}_P f)))\mathcal{P}(s) \right) \\
&\quad + (1 - \alpha) \left(u(\mathbb{E}_P g) + \sum_S \phi(u(g - wtp(\mathbb{E}_P g)))\mathcal{P}(s) \right) = \alpha \mathcal{V}(f) + (1 - \alpha)\mathcal{V}(g) \\
&= \mathcal{V}(f)
\end{aligned}$$

However, in the case when $f \sim g$ with either $\mathbb{E}_P f > \mathbb{E}_P g$ or $\mathbb{E}_P f < \mathbb{E}_P g$, uncertainty aversion may not hold in our model. To see how model (6) fails to satisfy uncertainty aversion, suppose $f \sim g$ with $\mathbb{E}_P f > \mathbb{E}_P g$, then $wtp(\mathbb{E}_P f) > wtp(\mathbb{E}_P(f\alpha g)) > wtp(\mathbb{E}_P g)$. In this case,

$$\begin{aligned}
\mathcal{V}(\alpha f + (1 - \alpha)g) &= u(\mathbb{E}_P(\alpha f + (1 - \alpha)g)) + \sum_S \phi(u[\alpha f + (1 - \alpha)g - wtp(\mathbb{E}_P(f\alpha g))])\mathcal{P}(s) \\
&= u(\alpha \mathbb{E}_P f + (1 - \alpha)\mathbb{E}_P g) \\
&\quad + \sum_S \phi(u[\alpha(f - wtp(\mathbb{E}_P(f\alpha g))) + (1 - \alpha)(g - wtp(\mathbb{E}_P(f\alpha g))])\mathcal{P}(s) \\
&= \alpha u(\mathbb{E}_P f) + (1 - \alpha)u(\mathbb{E}_P g) \\
&\quad + \sum_S \phi(\alpha u[f - wtp(\mathbb{E}_P(f\alpha g))] + (1 - \alpha)u[g - wtp(\mathbb{E}_P(f\alpha g))])\mathcal{P}(s)
\end{aligned}$$

Notice that $wtp(\mathbb{E}_P f) > wtp(\mathbb{E}_P(f\alpha g)) > wtp(\mathbb{E}_P g)$, which implies $f - wtp(\mathbb{E}_P(f\alpha g)) > f - wtp(\mathbb{E}_P f)$ and $g - wtp(\mathbb{E}_P(f\alpha g)) < g - wtp(\mathbb{E}_P g)$. Therefore, the comparison between utilities $\alpha u(f - wtp(\mathbb{E}_P(f\alpha g))) + (1 - \alpha)u(g - wtp(\mathbb{E}_P(f\alpha g)))$ and $\alpha u(f - wtp(\mathbb{E}_P f)) + (1 - \alpha)u(g - wtp(\mathbb{E}_P g))$ needs to be determined by specific α , g , f , and u . If in every state s ,

$$\begin{aligned}
&\alpha u(f - wtp(\mathbb{E}_P(f\alpha g))) + (1 - \alpha)u(g - wtp(\mathbb{E}_P(f\alpha g))) \\
&\quad < \alpha u(f - wtp(\mathbb{E}_P f)) + (1 - \alpha)u(g - wtp(\mathbb{E}_P g))
\end{aligned}$$

and the ϕ is utility function with small degree of risk aversion, it is possible that in each state s ,

$$\begin{aligned}
&\phi(\alpha u[f - wtp(\mathbb{E}_P(f\alpha g))] + (1 - \alpha)u[g - wtp(\mathbb{E}_P(f\alpha g))]) \\
&\quad < \alpha \phi(u(f - wtp(\mathbb{E}_P f))) + (1 - \alpha)\phi(u(g - wtp(\mathbb{E}_P g)))
\end{aligned}$$

which results in $\mathcal{V}(\alpha f + (1 - \alpha)g) < \alpha\mathcal{V}(f) + (1 - \alpha)\mathcal{V}(g) = \mathcal{V}(f)$, i.e. the DM exhibits uncertainty seeking with concave ϕ . As a result, our model allows preferences exhibiting uncertainty seeking that cannot be represented by KMM and other SOEU models.

3.2 Implied additive model under subjective belief

In the additive tradeoff model (6), the subjective probability over states $\mathcal{P}(s)$ is implied from the DM's preferences. Given these subjective probabilities, model (6) implies an additive expected utility form without referring to the subjective state space of the problem. Such a form is useful when a state space does not need to be explicitly specified. For instance, in section 4 we apply the model to explain the equity premium puzzle where the act (return) is represented by a compound lottery without referring to the subjective state space.

For any $f \in \mathcal{F}$, we define the induced probability measure P_f over simple objective probability measures under \mathcal{P} as

$$P_f(\mu) = \mathcal{P}(\{s: f(s) = \mu\}) \text{ for any } \mu \in \mathcal{L}$$

This definition is similar in spirit to Fishburn (see equation 14.8, Fishburn 1970). Each induced probability measure corresponds to a *two-stage compound lottery* (or an *ambiguous lottery*) under the subjective probability measure \mathcal{P} . We denote the set of these induced probability measures by $\mathbb{Q} := \{P_f: \exists f \in \mathcal{F}\}$. Similarly, under \mathcal{P} , we can also define the induced *standardized ambiguous lottery* P_f^A corresponding to the standardized act f^A as $P_f^A(\mu) = \mathcal{P}(\{s: f^A(s) = \mu\})$ for any $\mu \in \mathcal{L}$, which is also contained in \mathbb{Q} , i.e., $P_f^A \in \mathbb{Q}$. Applying the definition of f^A , $P_f^A = P_f - wtp(\mathbb{E}_P f)$. With this notation, under the implied subjective probability \mathcal{P} , model (6) can also be written as $f \succeq g$ iff $\mathcal{V}(P_f) \geq \mathcal{V}(P_g)$ as shown in the appendix, where

$$\mathcal{V}(P_f) = \underbrace{\sum_x u(x)\mathbb{E}_P f(x)}_{u(\mathbb{E}_P f)} + \underbrace{\sum_\mu \phi\left(\sum_x u(x)\mu(x)\right)P_f^A(\mu)}_{u(P_f^A)} \quad (7)$$

In this implied additive model (7), the utility of the risky reduced-lottery $u(\mathbb{E}_P f)$ and the utility of the standardized ambiguous lottery $u(P_f^A)$ are expected utility models. However, $\mathcal{V}(P_f)$ is the sum of these two utilities and is no longer an expected utility function of the induced probability measure P_f , and therefore it is not a linear function of the probability measure P_f . This is due to the fact that P_f^A is not linear in f as $P_{f\gamma g}^A = P_f^A \gamma P_g^A$ may not hold for $\gamma \in (0,1)$, namely the standardized act of the convex combination of two acts may not equal the convex combination of the standardized acts. This distinguishes our additive tradeoff model from other existing models for decision making under ambiguity (e.g., KMM 2005, Nau 2006, Seo 2009).

In the appendix, $\mathcal{V}(P_f)$ is obtained as a two attribute utility function over the utility of the risky reduced-lottery and the utility of the standardized ambiguous lottery. Given a subjective probability

measure over states, using model (7) to evaluate an act is straightforward. Consider the example in section 2 and the assumed first order utility function $u(x)$ and the second order transformation function $\phi(x)$. Recall that in section 2 the wtp for $\mathbb{E}_P f = \mathbb{E}_P g$ was calculated as \$2.88. The evaluation of f in Figure 1 using model (7) with equal subjective probabilities for the three states can be calculated as

$$\begin{aligned}
\mathcal{V}(P_f) &= u(\mathbb{E}_P f) + \mathcal{U}(P_f^A) \\
&= \frac{1}{3}(100 - 100 \exp(-0.05 * 5)) + \frac{1}{3}(100 - 100 \exp(-0.05 * 4)) \\
&\quad + \frac{1}{3}(100 - 100 \exp(-0.05 * 0)) \\
&\quad + \frac{20}{3} \log \left(1 + \frac{u(5 - 2.88) + u(4 - 2.88) + u(0 - 2.88)}{3 * 10} \right) \\
&+ \frac{20}{3} \log \left(1 + \frac{u(5 - 2.88) + 2u(4 - 2.88)}{3 * 10} \right) + \frac{20}{3} \log \left(1 + \frac{u(5 - 2.88) + 2u(0 - 2.88)}{3 * 10} \right) \\
&= 13.42 - 1.93 = 11.48
\end{aligned}$$

where $u(\mathbb{E}_P f) = 13.42$ is the evaluation of the risky reduced-lottery of P_f and -1.93 is the disutility from the standardized ambiguous lottery of P_f^A . Similarly, we can calculate $\mathcal{V}(P_g) = u(\mathbb{E}_P g) + \mathcal{U}(P_g^A) = 13.42 - 3.74 = 9.68$. As mentioned earlier, the ambiguity averse DM prefers f to g under the assumed functions u and ϕ , because g is more ambiguous than f . In contrast, if one uses KMM to compare these same alternatives, the result is that $U_{KMM}(f) > U_{KMM}(g)$. But, this evaluation by KMM only indicates that $f \succ g$ without providing insights on why the DM has such a preference. Our model explicitly shows that $f \succ g$ because both acts share the same expected risk, but f has a smaller magnitude of standard ambiguity, which is preferable. Therefore, the separable additive model can be used to reveal explicitly that how DM trades off the utility of the risky reduced-lottery and the utility of the standardized ambiguous lottery when evaluating acts.

4. The standard measure of ambiguity and its application

4.1 The standard measure of ambiguity

Beyond providing an additive tradeoff model between the risky reduced-lottery and the standardized act, model (7) also implies the existence of a preference based subjective ambiguity measure. Assuming that the DM is ambiguity averse, we define a “more ambiguous” relation \succcurlyeq^A on \mathcal{F} as follows.

Definition 3. Assuming ambiguity aversion, for any $f, g \in \mathcal{F}$, if $f^A \succeq g^A$ we say g is more ambiguous than f for a given ambiguity aversion function ϕ denoted by $g \succcurlyeq_\phi^A f$.

Under the assumption of ambiguity aversion, we know ϕ is concave. For any $f, g \in \mathcal{F}$, applying model (7) to $f^A \succeq g^A$, we have $0 \geq \mathcal{U}(P_f^A) \geq \mathcal{U}(P_g^A)$, where both $\mathcal{U}(P_f^A)$ and $\mathcal{U}(P_g^A)$ are always non-positive under ambiguity aversion. This definition of a “more ambiguous” relation is based on the DM’s preference order \succeq , and therefore depends on the ambiguity aversion function ϕ . Under model (7), this relation can be represented by $\mathcal{A}(f) := -\mathcal{U}(P_f^A)$, which we define as the ambiguity measure based on a standardized ambiguous lottery, or simply the *standard ambiguity measure* analogous to the standard measure of risk in Jia and Dyer (1996). For any unambiguous lottery $\mu \in \mathcal{L}$, $\mathcal{A}(\mu) = -\mathcal{U}(\mu^A) = 0$. Since, $\mathcal{U}(P_f^A)$ is always non-positive and $\mathcal{A}(f)$ is always non-negative the larger the value of $\mathcal{A}(f)$, the more ambiguous the act f is perceived to be.

Given both functions ϕ and u , we can calculate the numerical value of $\mathcal{A}(f) = -\mathcal{U}(P_f^A)$ by using the recursive subjective expected utility model (6) under subjective probability \mathcal{P} . However, the assessment of ϕ may not be easy in practice. This requires assessing the utility over the second order probability distributions implied by \mathcal{P} to determine U in order to imply ϕ from the relation $U = \phi \circ u$. In the real world, such a higher order belief is generally hard to observe. By designing the way ambiguity occurs in an experiment, however, it is relatively easy to manipulate the experiment so that it is reasonable to assume that the DM has a uniform second order belief over possible states, such as in the Ellsberg type of urn choice problem. See the discussion on verification in KMM (2005) and on testing of the KMM model in Halevy (2007).

Instead of calculating the standard ambiguity measure $\mathcal{A}(f)$ based on ϕ , we can approximate $\mathcal{A}(f)$ via a Taylor series expansion. For this purpose, we consider ambiguous lotteries with small variations on payoffs of first order lotteries and small variations on possible first order lotteries so that the possible first order lotteries are, loosely speaking, “close”. As shown in the appendix, if we assume that $u''(0) = 0$,

$$\mathcal{A}(f) \approx -\frac{1}{2} \phi''(0) (u'(0))^2 \mathbb{E}_{\mathcal{P}} [\mathbb{E}_{\mu} f - \mathbb{E}_{\mathcal{P}} \mathbb{E}_{\mu} f]^2 \quad (8)$$

where \mathbb{E}_{μ} denotes expectation with respect to the first order probability μ , namely $\mathbb{E}_{\mu} f: S \rightarrow X$ such that $\mathbb{E}_{\mu} f(s) = \sum_{x \in \text{supp}(\mu=f(s))} x \mu(x)$. In approximation (8), both $\phi''(0)$ and $u'(0)$ are parameters that are easier to assess than a second order utility function $U = \phi \circ u$. If one can assess u over objective lotteries, finding $u'(0)$ is trivial. To assess the value $\phi''(0)$, one can use an Ellsberg type choice problem to assess U around very small payoffs (around zero) as described in Halevy (2007). We conjecture that this is easier than assessing the function ϕ in a given domain as required in the original formula.

When $u'(0)$ is positive and ϕ is concave under ambiguity aversion, (8) implies that the ambiguity measure $\mathcal{A}(f)$ derived from the utility model is proportional to a quantity that is independent of the utility function, i.e., $\mathcal{A}(f) \propto \mathbb{E}_{\mathcal{P}} [\mathbb{E}_{\mu} f - \mathbb{E}_{\mathcal{P}} \mathbb{E}_{\mu} f]^2$. Therefore, we define an ambiguity measure

$$\begin{aligned} \mathcal{U}^2(f) &:= \mathbb{E}_P[\mathbb{E}_\mu f - \mathbb{E}_P \mathbb{E}_\mu f]^2 \\ &= \sum_{\mu_i \in \text{supp}(P_f)} P_f(\mu_i) \left(\sum_{x \in \text{supp}(\mu_i)} x \mu_i(x) - \sum_{\mu_i \in \text{supp}(P)} P_f(\mu_i) \sum_{x \in \text{supp}(\mu_i)} x \mu_i(x) \right)^2 \quad (9) \end{aligned}$$

$\mathcal{U}^2(f)$ gives the same ambiguity ranking as the standard ambiguity measure $\mathcal{A}(f)$ for “small” lotteries, but it can be calculated without knowing the component utility functions. In addition, (9) has a very natural interpretation. The term $\mathbb{E}_P \mathbb{E}_\mu f$ is the total expectation of f taken over both levels of probability distributions, and $\mathbb{E}_\mu f$ is the expectation of a first order lottery obtained in different states of f . Therefore, this ambiguity measure $\mathcal{U}^2(f)$ is the variance of the second order distribution relative to the total expectation. We formalize this argument in the following theorem.

Theorem 2. *Under the assumptions of model (7), for any small lottery, if $u''(0) = 0$, $\phi''(0) \neq 0$, $u'(0) \neq 0$ holds, an objective ambiguity measure $\mathcal{U}^2(f)$ defined by (9) represents the ambiguity order \succsim_ϕ^A .*

Proof: By definition 3, $g \succsim_\phi^A f$ if and only if $f^A \succeq g^A$. Under model (7), $f^A \succeq g^A$ can be represented by $\mathcal{V}(P_f^A) = u(\mathbb{E}_P f^A) + \mathcal{U}(P_f^A) \geq \mathcal{V}(P_g^A) = u(\mathbb{E}_P g^A) + \mathcal{U}(P_g^A)$. Since, by definition, $u(\mathbb{E}_P f^A) = u(\mathbb{E}_P g^A) = 0$, we can conclude that $\mathcal{U}(P_f^A) \geq \mathcal{U}(P_g^A)$. This is equivalent to $\mathcal{A}(f^A) \leq \mathcal{A}(g^A)$. By (8), we further conclude that $\mathcal{A}(f^A) \leq \mathcal{A}(g^A)$ if and only if $\mathcal{U}^2(f^A) \leq \mathcal{U}^2(g^A)$, which represents $g \succsim_\phi^A f$. ■

This result is consistent with our intuition that ambiguity is defined as the subjective uncertainty over a first order distribution and provides a natural progression from using variance to measure risk in portfolio selection theory (Markowitz 1952) to using the variance of the second order distribution to measure ambiguity. Halevy (2007)’s experimental study reveals that people exhibit ambiguity aversion together with an *aversion to mean preserving spreads* in the second order distribution indicating an association between the variance of the second order distribution and ambiguity. Our axiomatic model (7) provides a theoretical foundation for identifying the variance of the second order distribution as an ambiguity measure.

4.2 Explaining the equity premium puzzle

The equity premium puzzle was first formalized by Mehra and Prescott (1985) who observed that accounting for the historical returns on risky securities requires unrealistic high levels of risk aversion. Detailed reviews on this problem as well as its explanations can be found in Mehra (2003, 2008) and Mehra and Prescott (2003). Following the consumption-based-capital-asset-pricing model (CCAPM) for decision making under risk (Cochrane 2005), we consider an investor who can freely buy or sell an (ambiguous) asset ξ with a real distribution of returns that is uncertain to her. The price of this

asset at time t is denoted by p_t . The future payoff of the asset, i.e. the value of the asset in period $t + 1$, is denoted by $P_{t+1} \in \mathbb{Q}$.

The investor tries to maximize her utility over her consumption from period t to $t + 1$ by determining how much of asset ξ to buy in period t . Formally, her decision problem can be represented as

$$\begin{aligned} \max_{\xi} & u(c_t) + \rho \mathcal{V}(C_{t+1}) \\ \text{s.t. } & c_t = e_t - p_t \xi; \quad C_{t+1} = e_{t+1} + P_{t+1} \xi \end{aligned} \quad (10)$$

where c_t and C_{t+1} denote the consumption levels in periods t and $t + 1$; e_t and e_{t+1} denote the endowed consumption; ρ is a time discount factor; and $\mathcal{V}(C_{t+1})$ is of the form of model (7). In this problem, the investor makes a decision in period t , e_{t+1} is certain and the uncertainty arises from P_{t+1} . Given that P_{t+1} is a lottery induced by an act f under \mathcal{P} , i.e., $P_{t+1} \in \mathbb{Q}$, the consumption level C_{t+1} is also a lottery in \mathbb{Q} .

The CCAPM can be obtained from the first order condition (FOC) below,

$$p_t u'(c_t) = \rho \frac{d\mathcal{V}(e_{t+1} + P_{t+1} \xi)}{d\xi}.$$

To obtain the asset pricing model, we need to calculate the derivative on the right hand side of the FOC.

As demonstrated in the appendix, we can represent the equity premium associated with the investor's decision problem as

$$\underbrace{\mathbb{E}_{\mu}[\mathbb{E}_P R] - R_f}_{\text{Equity Premium}} \approx \underbrace{-\frac{\text{Cov}[u'(\mathbb{E}_P C_{t+1}), \mathbb{E}_P R]}{\mathbb{E}_{\mu}[u'(\mathbb{E}_P C_{t+1})]}}_{\text{Risk Premium}} + \underbrace{\rho R_f \xi \frac{\kappa p_t \mathcal{U}^2(R)}{u'(c_t)}}_{\text{Ambiguity Premium}} \quad (11)$$

where R_f is the risk-free rate, $R = P_{t+1}/p_t$ is the ambiguous return of the asset, and $\kappa = -\phi''(0)(u'(0))^2 \geq 0$. The left hand side of (11) is the equity premium, the return earned by a security in excess of that earned by a risk-free asset (US T-bill). The right hand side consists of two terms. The first term appears in the classic consumption based asset pricing model (Cochrane 2005), where both $\mathbb{E}_P C_{t+1}$ and $\mathbb{E}_P R$ are risky lotteries that are not ambiguous and captures the premium required by the risk of the asset $\mathbb{E}_P R$. The second term captures the equity premium required by the ambiguity of the asset $\mathcal{U}^2(R)$. Therefore, under our additive tradeoff model (7), the **equity premium** can be decomposed into two components corresponding to the **risk premium** and to the **ambiguity premium**. When the payoff of the asset is risky but not ambiguous, $\mathcal{U}^2(R) = 0$, and (11) reduces to the classic asset pricing model.

This development does depend on using the ambiguity measure $\mathcal{U}^2(R)$ developed for ‘‘small’’ ambiguous lotteries, so the results will only be approximate. However, this measure can be used without any need to estimate a market second order ‘‘utility function’’ $U = \phi \circ u$, and may provide a good approximation of the equity premium associated with equity risks given realistic scenarios.

Olsen and Troughton (2000), Erbas and Mirakhor (2007), as well as Rieger and Wang (2012) empirically verified that ambiguity aversion has a significant influence on the equity premium. Guidolin and Rinaldi (2013) provide a more complete review on this topic. A pricing model (11) based on our model (7) can achieve a separation between the premium from risk aversion and that from ambiguity aversion. Pinar (2014) studied a portfolio selection problem under ambiguity aversion in mean-variance analysis framework, where the ambiguity is modeled by an ellipsoidal set of the mean return of a portfolio. Pinar (2014) also obtained an CAPM-like formula where the risk premium is adjusted by a parameter capturing ambiguity aversion, but ambiguity and risk are entangled in the model. Ju and Miao (2012) applied the smooth ambiguity model (KMM 2005) to an asset pricing problem in a dynamic context. However, their pricing kernel confounds the aversion attitude and ambiguity aversion, making it impossible to separate the ambiguity premium from the risk premium. In contrast, model (11) achieves such a separation in the equity premium, making it easy to quantify how much risk and ambiguity contribute to the total equity premium. Moreover, (11) can explain the equity premium puzzle: the fact that we observe an equity premium that is higher than the premium due solely to risk aversion is due to an additional premium produced by ambiguity aversion.

As the second term in (11) has a closed form expression, our ambiguous asset pricing model can be tested empirically. Compared with the classical CCAPM under risk, model (11) appends an ambiguity premium to the risk premium. Therefore, we need to compute our ambiguity measure $\mathcal{U}^2(R)$ based on market data, which can be achieved by following the idea used in Brenner and Izhakian (2015) to construct a two-stage lottery to represent the ambiguous asset return to apply our definition to compute the ambiguity measure $\mathcal{U}^2(R)$. If we assume the commonly adopted constant risk aversion utility function (Mehra and Prescott 1985), model (11) can be tested in a similar way to test CCAPM based on consumption data (Breedon et al. 1989) where the unknown parameters in the model, i.e., risk aversion in u , ξ , and κ in ambiguity premium, can be estimated by the Generalized Method of Moments (Cochrane 2005). Thus, the key in this effort is to model second order belief over possible distributions of returns. One possible way is to model the returns of an asset using a Bayesian approach as discussed in Ju and Miao (2012). The effort to quantify our ambiguity measure and to explore the use of separable model will be the subject of further research.

5. Value-Risk-Ambiguity tradeoff model

In decision making under ambiguity, it may be desirable to compare acts in terms of their ambiguity, risk, and value. This idea is a natural extension of the risk-value approach in decision making under risk (Markowitz 1952, Sarin and Weber 1993, Jia and Dyer 1996, Jia, Dyer, and Butler 1999, Schmidt 2003). In this section, we extend our two-component additive model (7) by using *standard risk-value* model (Jia and Dyer 1996, 2009) to a three-component additive model that captures the tradeoff between these three components in decision making under ambiguity, i.e., value, risk, and ambiguity, where the value is measured by the expectation, the risk is measured by the standard risk measure

proposed by Jia and Dyer (1996, 2009), and the ambiguity is measured by the standard ambiguity measure $\mathcal{A}(f)$.

As $\mathbb{E}_P f$ is an expected first order distribution, i.e., $\mathbb{E}_P f \in \mathcal{L}$, we employ the logic behind the risk-value model (Jia and Dyer 1996) to further separate value from risk in $\mathbb{E}_P f$. A standard risk measure for $\mathbb{E}_P f$ is defined as $\mathbb{E}_P f - \mathbb{E}_\mu \mathbb{E}_P f$, a first order distribution obtained by subtracting its expectation from each payoff. We denote a set of all such *standardized risky lotteries* by $\mathcal{L}^0 := \{\mu \mid \mu \in \mathcal{L}, \mathbb{E}_\mu \mu = 0\}$. The value of lottery $\mathbb{E}_P f$ is measured by its expectation $\mathbb{E}_\mu \mathbb{E}_P f \in X$. Thus, the standardized risky lottery has the expected value $\mathbb{E}_\mu(\mathbb{E}_P f - \mathbb{E}_\mu \mathbb{E}_P f) = 0$. Therefore, for any ambiguous lottery P_f induced by act f , there are three components, $\mathbb{E}_\mu \mathbb{E}_P f$, $\mathbb{E}_P f - \mathbb{E}_\mu \mathbb{E}_P f$, and P_f^A , representing the value, the risk, and the ambiguity of the lottery, respectively. We refer to this vector $(\mathbb{E}_\mu \mathbb{E}_P f, \mathbb{E}_P f - \mathbb{E}_\mu \mathbb{E}_P f, P_f^A)$ as the *value-risk-ambiguity* representation of an act f . The risk independence condition proposed by Jia and Dyer (1996) requires that for any $\mu, \lambda \in \mathcal{L}^0$ and $w \in X$, $\mu \succeq \lambda$ implies $\mu + w \succeq \lambda + w$. Assuming this condition, we can further decompose the utility over the risky reduced-lottery in the model (7) to obtain (12).

$$\mathcal{V}(P_f) = u(\mathbb{E}_\mu \mathbb{E}_P f) - \psi(\mathbb{E}_\mu \mathbb{E}_P f) \mathcal{R}(\mathbb{E}_P f - \mathbb{E}_\mu \mathbb{E}_P f) - \mathcal{A}(f) \quad (12)$$

Model (12) gives a preference representation that consists of three components of an act f , namely the value, the risk, and the ambiguity. As pointed out in Jia and Dyer (Theorem 4, 1996), if the utility function for expected risk u is continuously differentiable, the only utility functions that are consistent with the standard risk-value model are quadratic utility, exponential utility, and linear plus exponential utility. Thus, these three utility functions are the only functions that are consistent with model (12).

A similar three-component model was recently proposed by MMR (2013), which applies the idea of the Taylor series used in the mean-variance analysis by Markowitz (1956) to the KMM model to obtain a mean-variance-ambiguity approximation to the certainty equivalent of an act. For any act f , its certainty equivalent at wealth level w , denoted by $CE(w + f)$, is approximated by MMR as

$$CE(w + f) \approx w + \mathbb{E}_{\bar{\mu}}(f) - \frac{1}{2} \lambda_u(w) \sigma_{\bar{\mu}}^2(f) - \frac{1}{2} (\lambda_U(w) - \lambda_u(w)) \sigma_P^2(\mathbb{E}_{\bar{\mu}}(f)) \quad (13)$$

In this approximation, $\bar{\mu}$ is the expected first order probability under the second order subjective prior P , i.e., $\bar{\mu} = \int_{\Delta(S)} \mu dP(\mu)$, and $\mathbb{E}_{\bar{\mu}}(f) = \int_S f(s) d\bar{\mu}(s)$ is the expected value of f under a specific $\bar{\mu}$, which is a random variable itself under the subjective second order prior P . Thus, $\mathbb{E}_{\bar{\mu}}(f)$ is the expected value of f under $\bar{\mu}$, $\sigma_{\bar{\mu}}^2(f)$ is the variance of f under $\bar{\mu}$, and $\sigma_P^2(\mathbb{E}_{\bar{\mu}}(f))$ is the variance of the expected value $\mathbb{E}_{\bar{\mu}}(f)$ under P , which is the second-order variance.

From (13), it is clear that the certainty equivalent of f at wealth level w consists of three components: value $w + \mathbb{E}_{\bar{\mu}}(f)$, risk represented by first order variance $\sigma_{\bar{\mu}}^2(f)$, and ambiguity represented by second order variance $\sigma_P^2(\mathbb{E}_{\bar{\mu}}(f))$. The two wealth level dependent factors $\lambda_u(w)$ and

$\lambda_U(w) - \lambda_u(w)$ capture the tradeoff between these components. As the mean-variance tradeoff model simplifies the decision making under risk (Markowitz 1952, 1987), this model is more tractable than KMM for decision making under ambiguity. However, as an approximation, (13) is only exact when utility functions on both levels exhibit constant absolute risk aversion (CARA) and the act f has a normal cumulative distribution $\Phi(m, \sigma)$ with unknown mean m and variance σ^2 (see p1085, MMR 2013). In contrast to (13), our value-risk-ambiguity tradeoff model (12) is obtained based on preference assumptions and provides an exact decomposition model that does not depend on an approximation argument. Therefore, our development not only provides an axiomatic foundation for the three-component model (12), it also allows the three-component tradeoff model to be applied in more general situations as an exact model.

6. Conclusion

In this paper, using an AA act to represent an ambiguous lottery, we first isolate the ambiguity in an AA act by defining a risky-reduced lottery and a standardized act. Based on this conceptual separation, we propose some assumptions to obtain an additive risk-ambiguity tradeoff model to extend the concept of mean-variance analysis to decision making under ambiguity. The existing literature of SOEU models does not include this variation. The separation also provides a standard ambiguity measure based on preference. Since the utility over standard ambiguity in our model shares the same representation with SOEU model, it can be tested in the same way to test models of SOEU family (Halevy 2007, Cubitt et al 2019). Using a Taylor series expansion, we show that the standard ambiguity measure is proportional to the variance of the second order probability distribution of a two-stage lottery induced by an act. Thus, the variance of the second order probability can be used to rank lotteries according to their ambiguity. This provides a theoretical foundation for using such an ambiguity measure in other applied study when ambiguity is involved.

We apply the model to the consumption-based asset pricing model for assets with ambiguous payoffs and show that the equity premium of an ambiguous asset can be decomposed into the sum of a risk premium and an ambiguity premium. This provides a testable and intuitive explanation for the equity premium puzzle. By combining our additive risky reduced-lottery ambiguity model with the standard risk-value model, we obtain an extension of the model as a three-factor separation model over value, risk, and ambiguity. This provides an axiomatic foundation for the risk-ambiguity-value tradeoff model that is similar to the mean-variance-ambiguity approximation model proposed by MMR (2013).

Finally, recall that our model relaxes the ROCL assumption adopted in the EU\SEU framework to obtain descriptive flexibility. Such an approach is standard in the literature of SOEU models reviewed in §1. However, our relaxation only applies to a restricted smaller set of two-stage lotteries where the reduced risky part of the act is indifferent to zero. For general AA acts, our approach deviates from standard SOEU by separating the expected first order distribution from the act first to

obtain two components, then relaxing ROCL for all standard acts. Such an approach may give some descriptive flexibility different from the models in the SOEU family. As we discussed in §3.1, the uncertainty aversion axiom does not apply to our model in some cases, but all models in the SOEU family satisfy this axiom and are therefore more restrictive.

Acknowledgement:

Jianmin Jia would like to thank the National Natural Science Foundation of China for partially supporting the research as a Major Program of the National Natural Science Foundation of China (#71490722).

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Appendix

Table of Notations

Notation	Description
S	State space
X	Set of consequences
\mathcal{L}	Set of simple probability measures defined on X
μ	Simple probability measure (first order lottery) defined over X
\mathcal{F}	Set of horse-roulette lotteries
f	Defined as $f: S \rightarrow \mathcal{L}$, and $f \in \mathcal{F}$
\mathcal{P}	Subjective Probability measure over S derived from preference in model
$\mathbb{E}_P f$	Risky reduced lottery: the expected first order distribution of act f $\mathbb{E}_P f := \sum_{s \in S} \mathcal{P}(s) f(s) \in \mathcal{L}$
$wtp(\mathbb{E}_P f)$	Willingness to Pay for $\mathbb{E}_P f$ which satisfies $\mathbb{E}_P f - wtp(\mathbb{E}_P f) \sim 0$
f^A	Standardized act: $f^A(s) = f(s) - wtp(\mathbb{E}_P f) \in \mathcal{F}$
P_f	Induced ambiguous lottery (two stage compound lottery) by act f under \mathcal{P} defined as $P_f(\mu) = \mathcal{P}(\{s: f(s) = \mu\})$
\mathbb{Q}	The set of ambiguous lotteries defined as $\mathbb{Q} := \{P_f: \exists f \in \mathcal{F}\}$
P_f^A	Standardized ambiguous lottery: $P_f^A(\mu) = \mathcal{P}(\{s: f^A(s) = \mu\})$
u	Utility of first order lotteries
\mathcal{U}	Utility of AA acts
ϕ	Transformation from u to \mathcal{U} , namely $\mathcal{U} = \phi \circ u$
\mathcal{V}	Additive Utility of acts for both two/three component models
$\mathcal{A}(f)$	Standard ambiguity measure of act f defined as $\mathcal{A}(f) := -\mathcal{U}(P_f^A)$
$\mathcal{U}^2(f)$	An objective ambiguity measure defined as $\mathcal{U}^2(f) := \mathbb{E}_P[\mathbb{E}_\mu f - \mathbb{E}_P \mathbb{E}_\mu f]^2$
$CE(f)$	Certainty equivalent of act f in MMR

Proof of Theorem 1

First, when preference is ambiguity neutral, model (6) implies that $\mathcal{U}(f^A) = 0$ as $\mathbb{E}_P f^A \sim 0$. Thus, model (6) holds in a trivial case. Therefore, we do not consider the ambiguity neutral case in the proof from now on.

Following the idea in Fishburn (pp 203, 1970), we define a preference order \succeq on $\mathbb{Q} := \{P_f: \exists f \in \mathcal{F}\}$ by $P_f \succeq P_g \Leftrightarrow f \succeq g$. Thus, the preference \succeq over acts in \mathcal{F} can be equivalently considered as the preference order \succeq over induced compound lotteries P_f in the set of \mathbb{Q} . So, we can work on the preference order over \mathbb{Q} to obtain model (7) first. Then, (6) is equivalent to model (7) under the implied subjective probability \mathcal{P} . When we do not stress that P_f is the compound lottery induced by f , defined as $P_f(\mu) := \mathcal{P}(\{s: f(s) = \mu\})$, we use $P, Q \in \mathbb{Q}$ to denote such a compound lottery for simplicity. Then, $\mathbb{E}_P P = \mathbb{E}_P f = \sum_{s \in S} \mathcal{P}(s) f(s)$ is used to denote the expectation with respect to the second order probability, which is the expected first order objective probability measure. To obtain an additive two attribute model, we extend the preference order \succeq on \mathbb{Q} to the product set $\mathcal{L} \times \mathbb{Q}^0$ by defining an induced preference order \succeq^I on this product set, where $\mathbb{Q}^0 :=$

$\{P: \mathbb{E}_P P \sim 0 \text{ and } P \in \mathbb{Q}\}$, namely \mathbb{Q}^0 is a set of compound lotteries whose reduced lottery (the expected first order lottery) is indifferent to zero. Similarly, we use P^A to denote the standardized ambiguous lottery $P_f^A \in \mathbb{Q}^0$, when we do not stress it is induced by act f . Then, by using a theorem from Fishburn (Theorem 1, chapter 9, 1982), we show that Assumptions 1, 2, 3, and 4 imply an additive representation for the induced preference order \succeq^I on this product set. Finally, we use the definition of the induced preference order to show the additive form also represents the original preference order \succeq .

To implement the above idea, we notice that the product set $\mathcal{L} \times \mathbb{Q}^0$ is a larger set than \mathbb{Q} . For arbitrary $(\mu, P) \in \mathcal{L} \times \mathbb{Q}^0$, there may not exist a $Q \in \mathbb{Q}$ such that $(\mathbb{E}_P Q, Q^A) = (\mu, P)$. Therefore, to compare any pair of vectors in $\mathcal{L} \times \mathbb{Q}^0$, we need a mapping from $\mathcal{L} \times \mathbb{Q}^0$ to \mathbb{Q} so that we can compare any vector in $\mathcal{L} \times \mathbb{Q}^0$ in term of its corresponding element in \mathbb{Q} . This mapping can be constructed in the following way. For any $(\mu, P) \in \mathcal{L} \times \mathbb{Q}^0$, we can find a unique act ${}_{\mu}P \in \mathbb{Q}$ such that $({}_{\mu}P)^A = P \in \mathbb{Q}^0$ and $\mathbb{E}_P({}_{\mu}P) \sim \mu$. In other words, we can find ${}_{\mu}P \in \mathbb{Q}$ such that it has the same standardized ambiguous lottery as P and the equivalent risky reduced-lottery as μ . To find this ${}_{\mu}P$, first, in a similar manner used to determine wtp in (3), we find a unique Δ_{μ} such that $\mathbb{E}_P P - \Delta_{\mu} \sim \mu$. Subtracting Δ_{μ} from the payoffs in P , we can obtain a compound lottery ${}_{\mu}P := P - \Delta_{\mu}$. In this way, ${}_{\mu}P \in \mathbb{Q}$ is uniquely determined by the vector $(\mu, P) \in \mathcal{L} \times \mathbb{Q}^0$. Thus, ${}_{\mu}P$ is a well-defined function from $\mathcal{L} \times \mathbb{Q}^0$ to \mathbb{Q} .

From the above process we can also see why the product set $\mathcal{L} \times \mathbb{Q}^0$ is larger than \mathbb{Q} . In the indifference relation $\mathbb{E}_P P - \Delta_{\mu} \sim \mu$, Δ_{μ} always exists. But, there may not exist Δ' such that $\mathbb{E}_P P - \Delta' = \mu$, for instance $supp(\mathbb{E}_P P)$ may have more elements than $supp(\mu)$. Finally, by the definition above for any $Q \in \mathbb{Q}$ we have $\mathbb{E}_P Q(Q^A) = Q$, namely, if we take the standardized ambiguous lottery Q^A and its risky reduced-lottery $\mathbb{E}_P Q$ of Q , we can use the definition ${}_{\mu}P$ to recover Q from its two components. Moreover, for any $(\mu, P) \in \mathcal{L} \times \mathbb{Q}^0$, ${}_{\mu}P$ defined above has a risky reduced-lottery indifferent to μ and a standardized ambiguous lottery equal to P . To see this, by definition we have $\mathbb{E}_P({}_{\mu}P) = \mathbb{E}_P(P - \Delta_{\mu}) = \mathbb{E}_P(P) - \Delta_{\mu} \sim \mu$. By definition again, we have $({}_{\mu}P)^A = (P - \Delta_{\mu})^A = P - \Delta_{\mu} - wtp(\mathbb{E}_P(P - \Delta_{\mu}))$. Since $P \in \mathbb{Q}^0$, $\mathbb{E}_P(P) - \Delta_{\mu} - (-\Delta_{\mu}) \sim 0$, which implies that $\mathbb{E}_P(P - \Delta_{\mu}) - (-\Delta_{\mu}) \sim 0$ and $wtp(\mathbb{E}_P(P - \Delta_{\mu})) = -\Delta_{\mu}$. Thus, we also have $({}_{\mu}P)^A = (P - \Delta_{\mu})^A = P - \Delta_{\mu} - wtp(\mathbb{E}_P(P - \Delta_{\mu})) = P - \Delta_{\mu} - (-\Delta_{\mu}) = P$.

Now, we extend the preference order \succeq over \mathbb{Q} to an induced preference order \succeq^I over the product set $\mathcal{L} \times \mathbb{Q}^0$ in the following way. For any $(\mu, P), (\lambda, Q) \in \mathcal{L} \times \mathbb{Q}^0$, define $(\mu, P) \succeq^I (\lambda, Q)$ iff ${}_{\mu}P \succeq {}_{\lambda}Q$.

Under the monotonicity Assumption 3 and the solvability Assumption 4 assumed for \succeq on \mathcal{F} , it is easy to see that these assumptions also apply to \succeq on \mathbb{Q} .

Assumption A3. (Monotonicity) For any $P, Q \in \mathbb{Q}$, if $\mathbb{E}_P P \sim \mathbb{E}_P Q$; $P^A \succeq Q^A$ implies $P \succeq Q$; if $P^A \sim Q^A$, $\mathbb{E}_P P \succeq \mathbb{E}_P Q$ implies $P \succeq Q$.

Assumption A4. (Solvability) When preference is ambiguity neutral, for any $P \succeq Q$, there exists $W \in \mathbb{Q}$ such that $W \sim P$ with $W^A \sim Q^A$.

We show that the induced preference order \succeq^I over the product set $\mathcal{L} \times \mathbb{Q}^0$ satisfies the vNM's axioms. For this purpose, we use the fact that the preferences \succeq over the compound lottery set \mathbb{Q} induced by subjective probability \mathcal{P} satisfies the three vNM's axioms (see Lemma 14.3 and 14.4 Fishburn 1970). Since $\mathbb{Q}^0 \subset \mathbb{Q}$, the preferences \succeq restricted to \mathbb{Q}^0 also satisfy these vNM's axioms.

Lemma 1. (vNM Axioms on Risk-Ambiguity Space). Under Assumptions 1, 2, 3, and 4, the induced preference order \succeq^I over $\mathcal{L} \times \mathbb{Q}^0$ also satisfies the von Neumann Morgenstern's Axioms.

Proof: Fishburn (1970, see page 203 Lemma 14.3 and 14.4) showed that \succeq over \mathbb{Q} satisfies the three vNM's axioms under Assumption 1. Assumption 2 also states that \succeq over \mathcal{L} satisfies these vNM's axioms. We show that Assumption 3 and 4 imply that these three vNM's axioms are satisfied by \succeq^I over $\mathcal{L} \times \mathbb{Q}^0$.

Weak Order: Based on the fact that \succeq on \mathbb{Q} and on \mathcal{L} satisfies the vNM's axioms, we can show that the induced order \succeq^I over $\mathcal{L} \times \mathbb{Q}^0$ is also a weak order. For any $(\mu, P), (\gamma, Q) \in \mathcal{L} \times \mathbb{Q}^0$, we know either $(\mu, P) \succeq^I (\gamma, Q)$ or $(\mu, P) \preceq^I (\gamma, Q)$ because either $\mu P \succeq \gamma Q$ or $\mu P \preceq \gamma Q$. Similarly, if $(\mu, P) \succeq^I (\gamma, Q)$ and $(\gamma, Q) \succeq^I (\lambda, X)$, then from $\mu P \succeq \gamma Q$ and $\gamma Q \succeq \lambda X$ implies $\mu P \succeq \lambda X$, we can conclude that $(\mu, P) \succeq^I (\lambda, X)$.

Independence: For any $(\mu, P), (\gamma, Q), (\lambda, X) \in \mathcal{L} \times \mathbb{Q}^0$, suppose $(\mu, P) \succeq^I (\gamma, Q)$. There are only three cases. We show in each of them, for any $\alpha \in (0,1)$, we have $(\mu, P)\alpha(\lambda, X) \succeq^I (\gamma, Q)\alpha(\lambda, X)$. We first show that independence holds for the case where (μ, P) dominates (γ, Q) on both the risky reduced-lottery and the standardized ambiguous lottery dimension. Then, we can use Assumption A4 above to show that the case where the dominance relation does not exist can always be converted to the dominance case by trading off the two components of an act such that a vector in $\mathcal{L} \times \mathbb{Q}^0$ is indifferent to (μ, P) and dominates (γ, Q) . Assumption A4 guarantees such a tradeoff is possible.

(1) $\mu \succeq \gamma$ and $P \succeq Q$: In this case, from the independence of the original preference order, we know for any $\lambda \in \mathcal{L}, X \in \mathbb{Q}^0, \alpha \in (0,1)$, $\mu\alpha\lambda \succeq \gamma\alpha\lambda$ and $P\alpha X \succeq Q\alpha X$. From the monotonicity in Assumption A3, we can conclude if one ambiguous lottery dominates the other one on both the risky-reduced lottery and the standardized ambiguous lottery dimensions, then the first one is preferred to the second one. Thus, we can conclude that $\mu\alpha\lambda P\alpha X \succeq \gamma\alpha\lambda Q\alpha X$, as $\mathbb{E}_P(\mu\alpha\lambda P\alpha X) \sim \mu\alpha\lambda \succeq \gamma\alpha\lambda \sim \mathbb{E}_P(\gamma\alpha\lambda Q\alpha X)$ and $(\mu\alpha\lambda P\alpha X)^A = P\alpha X \succeq Q\alpha X = (\gamma\alpha\lambda Q\alpha X)^A$. Then, by the

definition of an induced order, ${}_{\mu\alpha\lambda}P\alpha X \succeq {}_{\gamma\alpha\lambda}Q\alpha X$ implies $(\mu\alpha\lambda, P\alpha X) \succeq^I (\gamma\alpha\lambda, Q\alpha X)$, which further implies that $(\mu, P)\alpha(\lambda, X) \succeq^I (\gamma, Q)\alpha(\lambda, X)$.

- (2) $\mu \preceq \gamma$ and $P \succeq Q$: As ${}_{\mu}P \succeq {}_{\gamma}Q$, using the solvability Assumption A4, there exists ${}_{\lambda}W \sim {}_{\mu}P$, such that $W \sim Q$. The monotonicity in A3 implies that $\lambda \succeq \gamma$. From the definition of an induced preference order, the indifference ${}_{\mu}P \sim {}_{\lambda}W$ implies $(\lambda, W) \sim^I (\mu, P)$. Using independence of the preference order on \mathbb{Q}^0 and \mathcal{L} for $\lambda \succeq \gamma$ and $W \succeq Q$, we know for any $\xi \in \mathcal{L}, X \in \mathbb{Q}^0$ and $\alpha \in (0,1)$, $\lambda\alpha\xi \succeq \gamma\alpha\xi$ and $W\alpha X \succeq Q\alpha X$. Then, from monotonicity, we have ${}_{\lambda\alpha\xi}W\alpha X \succeq {}_{\gamma\alpha\xi}Q\alpha X$, which implies $(\lambda\alpha\xi, W\alpha X) \succeq^I (\gamma\alpha\xi, Q\alpha X)$ or equivalently $(\lambda, W)\alpha(\xi, X) \succeq^I (\gamma, Q)\alpha(\xi, X)$. Finally, substituting the indifference $(\lambda, W) \sim^I (\mu, P)$ into this preference relation, we have $(\mu, P)\alpha(\xi, X) \succeq^I (\gamma, Q)\alpha(\xi, X)$.

- (3) $\mu \succeq \gamma$ and $P \preceq Q$: This can be proved by using the same idea used in proving case (2).

Continuity: Suppose $(\mu, P) \succeq^I (\gamma, Q) \succeq^I (\lambda, X)$. If $P \succeq Q \succeq X$ and $\mu \succeq \gamma \succeq \lambda$, we used the continuity of the original preference order to conclude that there exists $\alpha_1, \alpha_2, \beta_1, \beta_2 \in (0,1)$ such that $P\alpha_1 X \succeq Q \succeq P\beta_1 X$ and $\mu\alpha_2 \lambda \succeq \gamma \succeq \mu\beta_2 \lambda$. Let $\bar{\alpha} = \max\{\alpha_1, \alpha_2\}$, $\underline{\beta} = \min\{\beta_1, \beta_2\}$, by Lemma 5.6 (a) in Kreps (1988), we know $P\bar{\alpha} X \succeq P\alpha_1 X \succeq Q \succeq P\beta_1 X \succeq P\underline{\beta} X$ and $\mu\bar{\alpha} \lambda \succeq \mu\alpha_2 \lambda \succeq \gamma \succeq \mu\beta_2 \lambda \succeq \mu\underline{\beta} \lambda$. Using monotonicity, we have ${}_{\mu\bar{\alpha}\lambda}P\bar{\alpha} X \succeq {}_{\gamma}Q \succeq {}_{\mu\underline{\beta}\lambda}P\underline{\beta} X$. By the definition of an induced preference order, we have $(\mu\bar{\alpha}\lambda, P\bar{\alpha} X) \succeq^I (\gamma, Q) \succeq^I (\mu\underline{\beta}\lambda, P\underline{\beta} X)$ or the equivalent $(\mu, P)\bar{\alpha}(\lambda, X) \succeq^I (\gamma, Q) \succeq^I (\mu, P)\underline{\beta}(\lambda, X)$.

If $P \succeq Q \succeq X$ and $\mu \succeq \gamma \succeq \lambda$ are not true, as we did in the proof for independence, we can find ${}_{\mu'}P' \sim {}_{\mu}P$, ${}_{\gamma'}Q' \sim {}_{\gamma}Q$, ${}_{\lambda'}X' \sim {}_{\lambda}X$ such that $P' \succeq Q' \succeq X'$ and $\mu' \succeq \gamma' \succeq \lambda'$. Using the same idea, we can show there exists $\bar{\alpha}, \underline{\beta} \in (0,1)$ such that $(\mu', P')\bar{\alpha}(\lambda', X') \succeq^I (\gamma', Q') \succeq^I (\mu', P')\underline{\beta}(\lambda', X')$. By the definition of the induced preference order, we have $(\mu', P') \sim^I (\mu, P)$, $(\gamma', Q') \sim^I (\gamma, Q)$, $(\lambda', X') \sim^I (\lambda, X)$. Using these results, we obtain $(\mu, P)\bar{\alpha}(\lambda, X) \succeq^I (\gamma, Q) \succeq^I (\mu, P)\underline{\beta}(\lambda, X)$. ■

To obtain the additive model, we employ a result from Fishburn (Theorem 9.1, 1982), which says that a preference order over a product of mixture sets can be represented by an additive vNM utility function if this preference order satisfies the vNM axioms.

Lemma 2. *Assumptions 1, 2, 3 and 4 imply that the induced preference order \succeq^I over $\mathcal{L} \times \mathbb{Q}^0$ can be represented by the following additive utility model, $(\mu, P) \succeq^I (\lambda, Q)$ iff $v(\mu, P) \geq v(\lambda, Q)$, where $v(\mu, P) = u(\mu) + \mathcal{U}(P)$*

$$u(\mu) = \sum_{x_i \in \text{supp}\{\mu\}} u(x_i)\mu(x_i) \quad (\text{A1})$$

$$\mathcal{U}(P) = \sum_{\lambda_i \in \text{supp}\{P\}} P(\lambda_i)\phi\left(\sum_{x_j \in \text{supp}\{\lambda_i\}} \lambda_i(x_j)u(x_j)\right) \quad (\text{A2})$$

for $(\mu, P) \in \mathcal{L} \times \mathbb{Q}^0$.

Proof: First, we need to show that both sets \mathcal{L} and \mathbb{Q}^0 are mixture sets (Fishburn 1980). Apparently, \mathcal{L} is a mixture set and \mathbb{Q} is a mixture set. We only need to show the mixture operation is closed in \mathbb{Q}^0 to show that $\mathbb{Q}^0 \subset \mathbb{Q}$ is also a mixture set. This can be observed from

- If $P, Q \in \mathbb{Q}^0$ and $\alpha \in (0,1)$, we have $\mathbb{E}_P(P\alpha Q) = \mathbb{E}_P P\alpha \mathbb{E}_P Q \sim 0\alpha 0 = 0$, thus $P\alpha Q \in \mathbb{Q}^0$;

In the above argument, we used lemma 5.6 c (Kreps 1988).

Given that both \mathcal{L} and \mathbb{Q}^0 are mixture sets, the additive representation can be obtained directly by using Theorem 1 in chapter 9 (Fishburn 1980) together with results from Lemma 1. This theorem implies that both $u(\mu)$ and $\mathcal{U}(P)$ are expected utility functions over each mixture set. Since Assumption 1 and 2 imply the recursive expected utility representation (4), we rewrite this representation under \mathcal{P} using the induced lottery P_f . Omitting the subscript f , we obtain the result that the expected utility form $\mathcal{U}(P)$ is of the form (A2) ■

Lemma 2 gives us an additive representation for the induced preference order \succeq^I over $\mathcal{L} \times \mathbb{Q}^0$. The expected utility u is defined on \mathcal{L} and the utility over \mathbb{Q}^0 has a recursive expected utility form. Based on this result, we can obtain an additive representation for the original order \succeq over \mathbb{Q} . From the definition of the induced order, we have that for any $P, Q \in \mathbb{Q}$, $P \succeq Q$ if and only if $(\mathbb{E}_P P, P^A) \succeq^I (\mathbb{E}_P Q, Q^A)$, which implies the model (7) in the main text.

Lemma 3, Assumptions 1, 2, 3, and 4 imply that there exists an additive representation for \succeq such that for any $P, Q \in \mathbb{Q}$, $P \succeq Q$ iff $\mathcal{V}(P) \geq \mathcal{V}(Q)$ with

$$\mathcal{V}(P) = u(\mathbb{E}_P P) + \mathcal{U}(P^A) \quad (7)$$

Where u is von Neumann Morgenstern utility given by (3) and \mathcal{U} is of the form of model (4) with $\mathcal{U}(0) = 0$.

Proof: From Lemma 2 and the definition of the induced preference order \succeq^I , we have $P \succeq Q$ iff $(\mathbb{E}_P P, P^A) \succeq^I (\mathbb{E}_P Q, Q^A)$ iff $v(\mathbb{E}_P P, P^A) \geq v(\mathbb{E}_P Q, Q^A)$. Define the vector valued function $\mathcal{J}(P) := (\mathbb{E}_P P, P^A)$ and the composite function $\mathcal{V}(P) := v(\mathcal{J}(P))$ for any $P \in \mathbb{Q}$. We have $P \succeq Q$ iff $\mathcal{V}(P) \geq \mathcal{V}(Q)$ where $\mathcal{V}(P) = v(\mathcal{J}(P)) = u(\mathbb{E}_P P) + \mathcal{U}(P^A)$. As \mathcal{U} is the expected utility function that is unique under affine transformation, we can rescale it such that $\mathcal{U}(0) = 0$. ■

Finally, let $P = P_f$ and $Q = P_g$ in Lemma 3, we can obtain model (6) in Theorem 1 immediately.

Note on the necessary conditions of the representation (6):

Assumption 1 is assumed on the set of all acts \mathcal{F} , which is not necessary for model (6). In the proof, it can be seen we only need this assumption to hold on a smaller set of \mathcal{F}^0 where $\mathcal{F}^0 := \{f \in \mathcal{F} \mid \mathbb{E}_P f \sim 0\}$. However, restricting the Assumption 1 to this smaller set \mathcal{F}^0 may not be intuitive as this small set

cannot be perceived as easily as the whole set \mathcal{F} .² Also, Assumption 4 is a technical assumption, which requires the richness of the set \mathcal{F} . Such a condition becomes necessary for model (6) if one makes extra structure assumptions on \mathcal{F} . Given (6), Assumption 4 holds if the utility is continuous in \mathcal{F} .³ With the restriction and extra assumption, one can make conditions in the theorem become both necessary and sufficient for model (6). However, we feel such a work adds no more intuition to the model and the sufficient conditions stated in the theorem are more intuitive in term of verification.

Approximation of $\mathcal{A}(f)$

For the ambiguity measure $\mathcal{A}(f) = -\mathcal{U}(P_f^A) = -\mathbb{E}_P \phi \left(\mathbb{E}_\mu u(f - wtp(\mathbb{E}_P f)) \right)$, we first expand ϕ at zero, which gives

$$\begin{aligned} & \mathbb{E}_P \phi \left(\mathbb{E}_\mu u(f - wtp(\mathbb{E}_P f)) \right) \\ & \approx \mathbb{E}_P \phi(0) + \phi'(0) \mathbb{E}_P \mathbb{E}_\mu u(f - wtp(\mathbb{E}_P f)) + \frac{1}{2} \phi''(0) \mathbb{E}_P \left[\mathbb{E}_\mu u(f - wtp(\mathbb{E}_P f)) \right]^2 \end{aligned}$$

By definition $\mathbb{E}_P f - wtp(\mathbb{E}_P f) \sim 0 \Leftrightarrow \mathbb{E}_P \mathbb{E}_\mu u(f - wtp(\mathbb{E}_P f)) = 0$. Substitute this together with $\phi(0) = 0$ into the approximation above, we have

$$\mathcal{A}(f) = -\mathbb{E}_P \phi \left(\mathbb{E}_\mu u(f - wtp(\mathbb{E}_P f)) \right) \approx -\frac{1}{2} \phi''(0) \mathbb{E}_P \left[\mathbb{E}_\mu u(f - wtp(\mathbb{E}_P f)) \right]^2 \quad (\text{A3})$$

Now, we expand u by Taylor expansion at zero to obtain

$$\mathbb{E}_\mu u(f - wtp(\mathbb{E}_P f)) \approx u'(0) (\mathbb{E}_\mu f - wtp) + \frac{1}{2} u''(0) \mathbb{E}_\mu (f - wtp)^2$$

Substitute the approximation above into (A3), we have

$$\mathcal{A}(f) \approx -\frac{1}{2} \phi''(0) \mathbb{E}_P \left[u'(0) (\mathbb{E}_\mu f - wtp) + \frac{1}{2} u''(0) \mathbb{E}_\mu (f - wtp)^2 \right]^2 \quad (\text{A4})$$

We can further simplify (A4) by noticing that $u''(0)$ should be equal to zero under general assumptions on $u(x)$. If the utility function $u(x)$ is concave on the domain of gains and convex on the domain of losses (Kahneman and Tversky 1979), then $u''(x) \leq 0$ for $x \in (0, +\infty)$ and $u''(x) \geq 0$ for $x \in (-\infty, 0)$. Thus, we can reasonably obtain $u''(0) = 0$ by assuming $u''(x)$ is continuous. By applying the condition $u''(0) = 0$ in (A4), we have (8) in subsection 4.1.

Derivation of Equity Premium

As described in the body of the paper, to obtain the asset pricing model, we need to calculate the derivative on the right hand side of FOC

$$p_t u'(c_t) = \rho \frac{d\mathcal{V}(e_{t+1} + P_{t+1}\xi)}{d\xi}$$

² It is easier for people to think about axioms assumed on a larger set than a more restricted one. See discussion of this issue in axiomatization in Gilboa et al (2019).

³ This requires endowing the set of functionals \mathcal{F} with a topological structure and adding some conditions to make utility become continuous.

For this purpose, we apply model (7) to $\mathcal{V}(e_{t+1} + P_{t+1}\xi)$

$$\mathcal{V}(e_{t+1} + P_{t+1}\xi) = u(\mathbb{E}_P(e_{t+1} + P_{t+1}\xi)) + \mathcal{U}((e_{t+1} + P_{t+1}\xi)^A) \quad (\text{A5})$$

From Definition 1, $\mathbb{E}_P(e_{t+1} + P_{t+1}\xi) = e_{t+1} + \xi\mathbb{E}_P P_{t+1}$. Thus, we have $du(e_{t+1} + \xi\mathbb{E}_P P_{t+1})/d\xi = \mathbb{E}_\mu[u'(e_{t+1} + \xi\mathbb{E}_P P_{t+1})\mathbb{E}_P P_{t+1}]$. From Definition 2, we have $(e_{t+1} + P_{t+1}\xi)^A = (P_{t+1}\xi)^A$, but $(P_{t+1}\xi)^A \neq (P_{t+1})^A \xi$ in general. By Definition 2, $(P_{t+1}\xi)^A = (P_{t+1}\xi) - wtp(\mathbb{E}_P(P_{t+1}\xi))$. However, the derivative $dwtp(\mathbb{E}_P(P_{t+1}\xi))/d\xi$ may not be explicitly known. Therefore, we apply the approximation (8) in this problem to further simplify the FOC. By substituting (8) into (A5) and using the definition of $\mathcal{U}^2(P)$ given by (9), we have the following approximation for (A5)

$$\mathcal{V}(e_{t+1} + P_{t+1}\xi) \approx u(\mathbb{E}_P(e_{t+1} + P_{t+1}\xi)) + \frac{1}{2}\phi''(0)(u'(0))^2\mathcal{U}^2(P_{t+1}\xi) \quad (\text{A6})$$

Because $\mathcal{U}^2(P_{t+1}\xi) = \xi^2\mathcal{U}^2(P_{t+1})$ from (9), we can simplify the FOC to

$$p_t u'(c_t) \approx \rho \mathbb{E}_\mu[u'(\mathbb{E}_P C_{t+1})\mathbb{E}_P P_{t+1}] + \rho \xi \phi''(0)(u'(0))^2 \mathcal{U}^2(P_{t+1})$$

Define $\kappa = -\phi''(0)(u'(0))^2 \geq 0$ and, we have

$$p_t u'(c_t) \approx \rho \mathbb{E}_\mu[u'(\mathbb{E}_P C_{t+1})\mathbb{E}_P P_{t+1}] - \rho \xi \kappa \mathcal{U}^2(P_{t+1}).$$

This leads to

$$1 \approx \rho \mathbb{E}_\mu \left[\frac{u'(\mathbb{E}_P C_{t+1})}{u'(c_t)} \mathbb{E}_P \frac{P_{t+1}}{p_t} \right] - \rho \xi \frac{\kappa p_t \mathcal{U}^2 \left(\frac{P_{t+1}}{p_t} \right)}{u'(c_t)} \quad (\text{A7})$$

Define $R = \frac{P_{t+1}}{p_t}$ as the return of the asset, which is also an act. Using the identity $Cov(X, Y) = E(XY) - E(X)E(Y)$, (A7) reduces to

$$1 \approx \rho Cov \left[\frac{u'(\mathbb{E}_P C_{t+1})}{u'(c_t)}, \mathbb{E}_P R \right] + \mathbb{E}_\mu \left[\rho \frac{u'(\mathbb{E}_P C_{t+1})}{u'(c_t)} \right] \mathbb{E}_\mu [\mathbb{E}_P R] - \rho \xi \frac{\kappa p_t \mathcal{U}^2(R)}{u'(c_t)} \quad (\text{A8})$$

We use R_f to denote the risk-free rate. Applying (A7) to price the risk-free bond, we obtain $R_f =$

$\frac{1}{\mathbb{E}_\mu \left[\rho \frac{u'(\mathbb{E}_P C_{t+1})}{u'(c_t)} \right]}$. Substituting this expression back into (A8), we have

$$R_f \approx \frac{\rho Cov \left[\frac{u'(\mathbb{E}_P C_{t+1})}{u'(c_t)}, \mathbb{E}_P R \right]}{\mathbb{E}_\mu \left[\rho \frac{u'(\mathbb{E}_P C_{t+1})}{u'(c_t)} \right]} + \mathbb{E}_\mu [\mathbb{E}_P R] - \rho R_f \xi \frac{\kappa p_t \mathcal{U}^2(R)}{u'(c_t)}$$

which implies (11) in the body of the paper.