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Early cosmological evolution of primordial electromagnetic fields

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It is usually assumed that when Weyl invariance is unbroken in the electromagnetic sector, the energy density of primordial magnetic fields will redshift as radiation. Here we show that primordial magnetic fields do *not* exhibit radiationlike redshifting in the presence of stronger electric fields, as a consequence of Faraday's law of induction. In particular for the standard Maxwell theory, magnetic fields on superhorizon scales can redshift as $B^2 \propto a^{-6}H^{-2}$, instead of the usually assumed a^{-4}. Taking into account this effect for inflationary magnetogenesis can correct previous estimates of the magnetic field strength by up to 37 orders of magnitude. This opens new possibilities for inflationary magnetogenesis, and as an example we propose a scenario where femto-Gauss intergalactic magnetic fields are created on Mpc scales, with high-scale inflation producing observable primordial gravitational waves, and reheating happening at low temperatures.

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I. INTRODUCTION

The origin of the magnetic fields in our universe is a mystery. There are several known astrophysical and cosmological mechanisms for producing the galactic magnetic fields. On the other hand for intergalactic magnetic fields which are suggested by recent gamma ray observations to be of femto-Gauss strength, their large correlation length (typically of megaparsec scales or larger) indicates a cosmological origin [1–3]. Theories of primordial magnetic field generation have been widely studied, and the proposed mechanisms include magnetogenesis during the inflationary epoch [4,5], the postinflationary epoch [6], and upon cosmological phase transitions [7,8]. See also reviews such as [9–12] and references therein. These proposed mechanisms are however not without challenges. For example, in order for the description to stay perturbative and avoid backreaction [13], working models of inflationary magnetogenesis leading to femto-Gauss magnetic fields on the megaparsec scale have so far required a very low scale of inflation [14–17] or a combination of mechanisms [6].

Here, in order to connect the magnetic fields produced in the early universe with those (indirectly) observed in the present universe, it is crucial to understand the evolution of magnetic fields along the cosmological history. In most of the literature on cosmological magnetogenesis, it is assumed that magnetic fields on superhorizon scales undergo a radiationlike redshifting with the cosmological scale factor a as

$$B^2 \propto \frac{1}{a^4}. \tag{1.1}$$

This rather rapid decay has been considered as the main obstacle against magnetic fields produced in the primordial universe from surviving until today and seeding the observed fields.

However, it is actually the sum of the magnetic and electric fields $B^2 + E^2$ which redshifts as radiation, whereas the individual B^2 and E^2 can have different redshift behaviors; the goal of our paper is to explicitly show this. In particular when the electric field is stronger than the magnetic field, we show that the magnetic field outside the horizon can evolve in time as

$$B^2 \propto \frac{1}{a^6 H^2}, \tag{1.2}$$

where H is the Hubble rate. In a decelerating universe, this yields less redshift to the magnetic fields compared to (1.1). For instance, if the universe is effectively matter-dominated, i.e., $H^2 \propto a^{-3}$, the magnetic field would redshift as $B^2 \propto a^{-3}$. Such a behavior of cosmological magnetic fields was seen in [6] in the context of postinflationary magnetogenesis scenarios.[1] In this paper, we show that superhorizon magnetic fields in a decelerating universe

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[1]Most of the analyses in [6] are based on directly solving the gauge field's equation of motion, arriving at the correct scaling behavior of the magnetic field. However their Sec. III 2 assumes the redshifting (1.1) and thus can be modified by taking into account the proper magnetic scaling.

generically follow the scaling (1.2) in the presence of stronger electric fields.[2]

Many of the previously proposed inflationary magnetogenesis scenarios, including the well-studied I^2FF model [5], produce much stronger primordial electric fields than magnetic fields during the inflationary epoch. The electric fields continue to exist after inflation until the universe turns into a good conductor. This can happen any time from the end of inflation until the end of reheating depending on the details of the reheating mechanism [4]. It is usually assumed that conductivity turns on already at the end of inflation erasing the electric field, but if the conductivity remains small during this epoch between the end of inflation and the end of reheating, the strong electric field induces the magnetic field evolution of (1.2), which yields less redshift compared to the usually assumed (1.1). As a consequence, the present-day amplitude of magnetic fields arising from inflationary magnetogenesis can actually be much larger than what has been claimed in previous studies. The difference is drastic especially when there is a hierarchy between the inflation and reheating scales; this implies that a higher inflation scale can help produce stronger magnetic fields today, as opposed to the widespread belief based on (1.1) that high-scale inflation is incompatible with efficient inflationary magnetogenesis. While the conclusions of [15,16], that femto-Gauss magnetic fields on the Mpc scale require inflation to happen below the TeV scale,[3] remains true under their assumption of instantaneous reheating or high conductivity throughout reheating, a prolonged period of reheating with vanishing conductivity can significantly alter these conclusions—opening a new space for inflationary magnetogenesis phenomenology and model building. As an example, we propose a toy model of inflationary magnetogenesis that can produce the femto-Gauss intergalactic magnetic fields during high-scale inflation, while being free from strong couplings [13,26] or affecting the background cosmology [14,17,27,28] and curvature perturbations [15,16,24,29–35]. We demonstrate how the various constraints on primordial magnetogenesis claimed in the literature such as those cited here are relaxed when the electric field-induced scaling (1.2) is taken into account.

We also study the cosmological consequences of primordial electric fields. By analyzing their gravitational backreaction, we derive constraints on magnetic fields produced from generic Weyl symmetry-breaking scenarios during the inflationary and postinflationary epochs. Primordial electric fields can also raise the conductivity of the universe even before reheating by producing charged particles via the Schwinger process [36]; this issue will also be discussed.

The paper is organized as follows: In Sec. II we provide a simple argument for the electromagnetic scalings based on Faraday's law of induction, without specifying the gauge field action. In Sec. III we focus on I^2FF theories and give a more rigorous derivation using Bogoliubov coefficients. We study a toy model of inflationary magnetogenesis in Sec. IV, where we see how the induction effect impacts the final magnetic field strength; here we also propose a scenario capable of producing femto-Gauss intergalactic magnetic fields during high-scale inflation. We further provide model-independent constraints on primordial magnetic fields in Sec. V, and then briefly discuss the possibility of electric field quenching due to the Schwinger process in Sec. VI. We summarize our findings in Sec. VII.

We will occasionally use the conversions of $1\ \mathrm{G} \approx 2\times 10^{-20}\ \mathrm{GeV}^2$ (in Heaviside-Lorentz units), and $1\ \mathrm{Mpc} \approx 2\times 10^{38}\ \mathrm{GeV}^{-1}$. Moreover, we use Greek letters for the spacetime indices $\mu, \nu = 0, 1, 2, 3$, and Latin letters for spatial indices $i, j = 1, 2, 3$.

II. FARADAY'S LAW OF INDUCTION OUTSIDE THE HUBBLE HORIZON

The magnetic field scaling of (1.2) can be simply understood from Faraday's law of induction. In this section we present general arguments that capture the essence of the physics without specifying the details of the vector field theory.

The electric and magnetic fields measured by a comoving observer with 4-velocity u^μ ($u^i = 0$, $u_\mu u^\mu = -1$) are given by

$$E_\mu = u^\nu F_{\mu\nu}, \qquad B_\mu = \frac{1}{2}\eta_{\mu\nu\rho\sigma} u^\sigma F^{\nu\rho}, \tag{2.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and $\eta_{\mu\nu\rho\sigma}$ is a totally antisymmetric tensor with $\eta_{0123} = -\sqrt{-g}$.

Throughout this paper we fix the metric to a flat Friedmann-Robertson-Walker (FRW),[4]

$$ds^2 = a(\tau)^2(-d\tau^2 + d\boldsymbol{x}^2). \tag{2.2}$$

Then Faraday's law of induction follows from the electromagnetic fields' definitions (2.1) as

$$(aB_i)' = -\hat{\varepsilon}_{ijl}\partial_j(aE_l). \tag{2.3}$$

[2]Nonradiationlike redshifting of magnetic fields has also been claimed for anisotropic [18] or open [19] universes, although the mechanism for the open universe was strongly questioned in [20]. Other proposals exist as well, e.g., [21]. However we stress that the effect discussed in the current paper is different from those.

[3]One can also consider other options, like the generation of helical magnetic fields from a coupling of the type $I^2F_{\mu\nu}\tilde{F}^{\mu\nu}$, where $\tilde{F}$ is the dual field strength [22]. Such mechanisms however suffer from their own backreaction, anisotropy and perturbativity constraints [23,24] yielding similar problems for magnetogenesis [25].

[4]Gravitational backreaction on the metric from the gauge field will be discussed later on.

Here, a prime represents a conformal time τ derivative, $\hat{\varepsilon}_{ijl}$ is totally antisymmetric with $\hat{\varepsilon}_{123} = 1$, and a sum over repeated spatial indices is implied irrespective of their positions. Integrating both sides of the equation yields

$$aB_i = -\hat{\varepsilon}_{ijl} \int \frac{da}{a} \frac{\partial_j E_l}{H}, \tag{2.4}$$

where we have rewritten the τ-integral in terms of the scale factor a and Hubble rate $H = a'/a^2$.

Now let us go to momentum space, and focus on modes larger than the Hubble length, i.e., on comoving wave numbers that satisfy $k < aH$. For such wave modes, the timescales of the electric field oscillations are longer than the Hubble time, and thus the integrand of (2.4) can in many cases be approximated by some power-law function of a. Hence Faraday's law implies a relation of

$$\tilde{B}(\tau, k) \sim \frac{k}{aH} \tilde{E}(\tau, k) + \frac{C(k)}{a}, \tag{2.5}$$

where $\tilde{B}$ and $\tilde{E}$ are the Fourier components of B_i and E_i, respectively, and we have neglected the spatial indices as well as $\hat{\varepsilon}_{ijl}$ since we are interested in order-of-magnitude estimates. C is a time-independent integration constant. Since the electromagnetic field strengths are written in terms of the vector components as

$$E^2 \equiv E^\mu E_\mu = \frac{E_i E_i}{a^2}, \qquad B^2 \equiv B^\mu B_\mu = \frac{B_i B_i}{a^2}, \tag{2.6}$$

one sees from (2.5) that the part of the magnetic field expressed as the integration constant undergoes a radiation-like redshifting (1.1). However, there is another part which is related to the electric field as

$$\Delta B^2 \propto \frac{E^2}{(aH)^2}. \tag{2.7}$$

This magnetic component grows relative to the electric field in a decelerating universe, which can be understood as the electric fields sourcing the magnetic fields. In particular when the electric field is strong enough for this magnetic component to dominate over the integration constant part, and further if the electric field redshifts as $E^2 \propto a^{-4}$, then the magnetic field would evolve in time as (1.2).[5]

In the above discussions we made some rough approximations upon obtaining (2.5), however we stress that the argument itself followed directly from the definitions of the electromagnetic fields. In particular, we have not specified the gauge field action, and thus the result applies to the standard Maxwell theory, as well as to modified electromagnetic theories often invoked in magnetogenesis scenarios. In the following sections we give more rigorous arguments for a certain class of gauge field theories.

Once the gauge field action is specified, one obtains the (generalized) Ampère-Maxwell law [e.g., (6.2)], which can be integrated to yield an equation similar to (2.5) but with E and B flipped, and with some dependence on the details of the action. This is useful for studying the relation between the electromagnetic fields in the presence of strong magnetic fields. However we should also remark that going between cases of $E^2 \gg B^2$ and $E^2 \ll B^2$ can be more than just flipping the role of the electric and magnetic fields; this reflects the fact that the Ampère-Maxwell law depends on the gauge field action while Faraday's law is independent.

III. ELECTROMAGNETIC FIELDS AND PHOTON NUMBER

Hereafter we focus on U(1) gauge field theories described by an effective action of the form

$$S = -\frac{1}{4} \int d^4x \sqrt{-g} I(\tau)^2 F_{\mu\nu} F^{\mu\nu}, \tag{3.1}$$

with a time-dependent coefficient $I(\tau)^2$ of the kinetic term.

The standard Maxwell theory corresponds to the case of $I^2 = 1$, where the action is invariant under a Weyl transformation,

$$g_{\mu\nu} \to \Omega^2 g_{\mu\nu}, \qquad A_\mu \to A_\mu. \tag{3.2}$$

Hence with a Weyl-flat background metric such as the flat FRW (2.2), the gauge field is simply a sum of plane waves.

On the other hand when I^2 depends on time, the Weyl invariance is generically violated and thus the gauge field can be excited even in a flat FRW universe. The time-dependent coefficient arises, for instance, from the Weyl anomaly of quantum electrodynamics [37,38]. Further time dependence may arise from beyond-the-Standard-Model physics, such as via couplings of the gauge field to (nearly) homogeneous degrees of freedom such as the inflaton field [5]; such explicit violation of the Weyl invariance has been invoked in most primordial magnetogenesis models in the literature.

Below we canonically quantize the theory (3.1), and write down various quantities in terms of time-dependent Bogoliubov coefficients. This will be useful for analyzing the redshifting behaviors of electromagnetic fields, as well as for studying explicit examples in the following sections.

A. Canonical quantization

We decompose the spatial components of the gauge field into irrotational and incompressible parts,

[5] We have not discussed the cross term between the two terms of (2.5) since it only becomes marginally important while $k\tilde{E}/aH$ and C/a are comparable to each other.

$$A_\mu = (A_0, \partial_i S + V_i) \quad \text{with} \quad \partial_i V_i = 0. \tag{3.3}$$

A_0 is a Lagrange multiplier in (3.1), and its constraint equation under proper boundary conditions gives $A_0 = S'$. This can be used to eliminate both A_0 and S from the action to yield, up to surface terms,

$$S = \frac{1}{2}\int d\tau d^3x I(\tau)^2 (V_i' V_i' - \partial_i V_j \partial_i V_j). \tag{3.4}$$

We promote V_i to an operator,

$$V_i(\tau, \boldsymbol{x}) = \sum_{p=1,2} \int \frac{d^3k}{(2\pi)^3} \epsilon_i^{(p)}(\boldsymbol{k}) \Big\{ e^{i\boldsymbol{k}\cdot\boldsymbol{x}} a_{\boldsymbol{k}}^{(p)} u_{\boldsymbol{k}}^{(p)}(\tau) + e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} a_{\boldsymbol{k}}^{\dagger(p)} u_{\boldsymbol{k}}^{*(p)}(\tau) \Big\}, \tag{3.5}$$

where $\epsilon_i^{(p)}(\boldsymbol{k})$ ($p = 1, 2$) are two orthonormal polarization vectors satisfying

$$\epsilon_i^{(p)}(\boldsymbol{k}) k_i = 0, \qquad \epsilon_i^{(p)}(\boldsymbol{k}) \epsilon_i^{(q)}(\boldsymbol{k}) = \delta_{pq}. \tag{3.6}$$

It follows from these conditions that

$$\sum_{p=1,2} \epsilon_i^{(p)}(\boldsymbol{k}) \epsilon_j^{(p)}(\boldsymbol{k}) = \delta_{ij} - \frac{k_i k_j}{k^2}, \tag{3.7}$$

where $k \equiv |\boldsymbol{k}|$. Unlike the spacetime indices, we do not assume implicit summation over the polarization index (p).

The time-independent annihilation and creation operators, $a_{\boldsymbol{k}}^{(p)}$ and $a_{\boldsymbol{k}}^{\dagger(p)}$, satisfy the commutation relations:

$$\Big[a_{\boldsymbol{k}}^{(p)}, a_{\boldsymbol{h}}^{(q)}\Big] = \Big[a_{\boldsymbol{k}}^{\dagger(p)}, a_{\boldsymbol{h}}^{\dagger(q)}\Big] = 0,$$
$$\Big[a_{\boldsymbol{k}}^{(p)}, a_{\boldsymbol{h}}^{\dagger(q)}\Big] = (2\pi)^3 \delta^{pq} \delta^{(3)}(\boldsymbol{k} - \boldsymbol{h}). \tag{3.8}$$

Moreover, for V_i and its conjugate momentum obtained from the action $S = \int d\tau d^3x \mathcal{L}$ of (3.4) as

$$\Pi_i = \frac{\partial \mathcal{L}}{\partial V_i'} = I^2 V_i', \tag{3.9}$$

we impose commutation relations as

$$[V_i(\tau, \boldsymbol{x}), V_j(\tau, \boldsymbol{y})] = [\Pi_i(\tau, \boldsymbol{x}), \Pi_j(\tau, \boldsymbol{y})] = 0,$$
$$[V_i(\tau, \boldsymbol{x}), \Pi_j(\tau, \boldsymbol{y})] = i\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}) \left(\delta_{ij} - \frac{\partial_i \partial_j}{\partial_l \partial_l} \right)$$
$$= i \sum_{p=1,2} \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} \epsilon_i^{(p)}(\boldsymbol{k}) \epsilon_j^{(p)}(\boldsymbol{k}), \tag{3.10}$$

where the equality in the third line follows from (3.7).

The mode function $u_{\boldsymbol{k}}^{(p)}$ obeys the equation of motion:

$$u_{\boldsymbol{k}}''^{(p)} + 2\frac{I'}{I} u_{\boldsymbol{k}}'^{(p)} + k^2 u_{\boldsymbol{k}}^{(p)} = 0. \tag{3.11}$$

Choosing the polarization vectors such that $\epsilon_i^{(p)}(\boldsymbol{k}) = \epsilon_i^{(p)}(-\boldsymbol{k})$, one can check that the commutation relations (3.8) and (3.10) are equivalent to each other when the mode function is independent of the direction of $\boldsymbol{k}$, i.e.,

$$u_{\boldsymbol{k}}^{(p)} = u_k^{(p)}, \tag{3.12}$$

and also obeys the normalization condition,

$$I^2 (u_k^{(p)} u_k'^{*(p)} - u_k^{*(p)} u_k'^{(p)}) = i. \tag{3.13}$$

Defining the vacuum state by

$$a_{\boldsymbol{k}}^{(p)} |0\rangle = 0 \tag{3.14}$$

for $p = 1, 2$ and $^\forall \boldsymbol{k}$, then the correlation functions of the electromagnetic fields (2.1) can be computed,

$$\langle 0 | E_\mu(\tau, \boldsymbol{x}) E^\mu(\tau, \boldsymbol{y}) | 0 \rangle = \int \frac{d^3k}{4\pi k^3} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} \mathcal{P}_E(\tau, k),$$
$$\langle 0 | B_\mu(\tau, \boldsymbol{x}) B^\mu(\tau, \boldsymbol{y}) | 0 \rangle = \int \frac{d^3k}{4\pi k^3} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} \mathcal{P}_B(\tau, k), \tag{3.15}$$

where the power spectra are given in terms of the mode functions as

$$\mathcal{P}_E(k) = \frac{k^3}{2\pi^2 a^4} \sum_{p=1,2} |u_k'^{(p)}|^2,$$
$$\mathcal{P}_B(k) = \frac{k^5}{2\pi^2 a^4} \sum_{p=1,2} |u_k^{(p)}|^2. \tag{3.16}$$

We occasionally omit the argument τ, however it should be noted that the power spectra generically are time-dependent quantities.

B. Bogoliubov coefficients

Since the operators $a_{\boldsymbol{k}}^{(p)}$ and $a_{\boldsymbol{k}}^{\dagger(p)}$ do not necessarily diagonalize the Hamiltonian under the function $I(\tau)^2$ with a general time dependence, let us further introduce a set of time-dependent annihilation and creation operators (see [38,39] for similar analyses applied to cosmological field excitations),

$$b_{\boldsymbol{k}}^{(p)}(\tau) = \alpha_k^{(p)}(\tau) a_{\boldsymbol{k}}^{(p)} + \beta_k^{*(p)}(\tau) a_{-\boldsymbol{k}}^{\dagger(p)},$$
$$b_{\boldsymbol{k}}^{\dagger(p)}(\tau) = \alpha_k^{*(p)}(\tau) a_{\boldsymbol{k}}^{\dagger(p)} + \beta_k^{(p)}(\tau) a_{-\boldsymbol{k}}^{(p)}, \tag{3.17}$$

where $\alpha_k^{(p)}(\tau)$ and $\beta_k^{(p)}(\tau)$ are time-dependent Bogoliubov coefficients expressed in terms of the mode function as

$$\alpha_k^{(p)} = I\left(\sqrt{\frac{k}{2}} u_k^{(p)} + \frac{i}{\sqrt{2k}} u_k'^{(p)}\right),$$
$$\beta_k^{(p)} = I\left(\sqrt{\frac{k}{2}} u_k^{(p)} - \frac{i}{\sqrt{2k}} u_k'^{(p)}\right). \tag{3.18}$$

One can easily check that $b_{\boldsymbol{k}}^{(p)}$ and $b_{\boldsymbol{k}}^{\dagger(p)}$ satisfy equal-time commutation relations similar to (3.8) for $a_{\boldsymbol{k}}^{(p)}$ and $a_{\boldsymbol{k}}^{\dagger(p)}$, and also diagonalize the Hamiltonian,

$$\tilde{H} = \int d^3x (\Pi_i V_i' - \mathcal{L})$$
$$= \sum_{p=1,2} \int \frac{d^3k}{(2\pi)^3} k \left(b_{\boldsymbol{k}}^{\dagger(p)} b_{\boldsymbol{k}}^{(p)} + \frac{1}{2}[b_{\boldsymbol{k}}^{(p)}, b_{\boldsymbol{k}}^{\dagger(p)}]\right). \tag{3.19}$$

It follows from the normalization condition (3.13) that the Bogoliubov coefficients obey

$$|\alpha_k^{(p)}|^2 - |\beta_k^{(p)}|^2 = 1, \tag{3.20}$$

$$|\beta_k^{(p)}|^2 = \frac{I^2}{2}\left(k|u_k^{(p)}|^2 + \frac{|u_k'^{(p)}|^2}{k}\right) - \frac{1}{2}. \tag{3.21}$$

It is also worth noting that for the standard Maxwell theory where the mode function is a sum of plane waves [cf. (4.11)], the amplitudes $|\alpha_k^{(p)}|$ and $|\beta_k^{(p)}|$ are independent of time.

Now let us suppose $a_{\boldsymbol{k}}^{(p)}$ and $a_{\boldsymbol{k}}^{\dagger(p)}$ to have initially diagonalized the Hamiltonian, i.e., $\beta_k^{(p)} = 0$ in the distant past, and that the system was initially in the vacuum state (3.14). However, the photons will eventually be produced due to the time-dependent background described by $I(\tau)^2$, and the number of photons with polarization p per unit six-dimensional phase volume is computed as

$$\frac{\langle 0| b_{\boldsymbol{k}}^{\dagger(p)} b_{\boldsymbol{k}}^{(p)} |0\rangle}{V} = |\beta_k^{(p)}|^2, \tag{3.22}$$

where V is the comoving spatial volume,

$$V \equiv \int d^3x = (2\pi)^3 \delta^{(3)}(\mathbf{0}). \tag{3.23}$$

For instance, magnetogenesis models that give rise to coherent magnetic fields with comoving correlation length of k^{-1} would create a large number of photons with momentum k, thus yield $|\beta_k^{(p)}|^2 \gg 1$.

In terms of the Bogoliubov coefficients, the electromagnetic spectra (3.16) are written as

$$\mathcal{P}_E(k) = \frac{k^4}{4\pi^2 a^4 I^2} \sum_{p=1,2} \left|\alpha_k^{(p)} - \beta_k^{(p)}\right|^2,$$
$$\mathcal{P}_B(k) = \frac{k^4}{4\pi^2 a^4 I^2} \sum_{p=1,2} \left|\alpha_k^{(p)} + \beta_k^{(p)}\right|^2. \tag{3.24}$$

Here, using (3.20), it can be checked that

$$\left|\alpha_k^{(p)} \mp \beta_k^{(p)}\right|^2 = 1 + 2|\beta_k^{(p)}|^2$$
$$\mp 2|\beta_k^{(p)}|\sqrt{1 + |\beta_k^{(p)}|^2} \cos\{\arg(\alpha_k^{(p)} \beta_k^{(p)*})\}, \tag{3.25}$$

which allows the electromagnetic spectra to be expressed in terms of the photon number density $|\beta_k^{(p)}|^2$, and the relative phase between $\alpha_k^{(p)}$ and $\beta_k^{(p)}$.

The energy density of the gauge field can be obtained as the vacuum expectation value of the Hamiltonian (3.19) divided by the spatial volume,

$$\rho_A = \frac{\langle 0|\tilde{H}|0\rangle}{a^4 V}$$
$$= \frac{1}{a^4} \int \frac{d^3k}{(2\pi)^3} k \sum_{p=1,2} \left(|\beta_k^{(p)}|^2 + \frac{1}{2}\right)$$
$$= \frac{I^2}{2} \int \frac{dk}{k} \{\mathcal{P}_E(k) + \mathcal{P}_B(k)\}, \tag{3.26}$$

where the third line is written in terms of the electromagnetic power spectra. In the second line, the $1/2$ inside the parentheses is the zero-point energy and can be removed by a normal ordering. (Although, when $|\beta_k^{(p)}|^2 \gg 1$, the zero-point energy is anyway tiny compared to the total ρ_A.) When I^2 is constant and thus the photon density $|\beta_k^{(p)}|^2$ is conserved, one clearly sees that the gauge field density,[6] and the sum of the electric and magnetic power, both redshift as $\propto a^{-4}$. However we stress that this is not necessarily the case for the individual electric and magnetic power, as we will explicitly see below.

C. Hierarchical electromagnetic power spectra

Many Weyl symmetry-breaking models of magnetogenesis produce much stronger electric fields compared

[6]If the k-integral is cut off at some k_{UV}, in this paragraph we are assuming k_{UV} to be time-independent.

to magnetic fields, or vice versa.[7] In terms of the expression (3.25), such a situation with a hierarchy between the electromagnetic fields is described as the case of $|\beta_k^{(p)}|^2 \gg 1$ with $\arg(\alpha_k^{(p)}\beta_k^{(p)*}) \simeq 0, \pm\pi, \pm 2\pi, \cdots$.

To see this more clearly, let us write the relative phase as

$$\arg(\alpha_k^{(p)}\beta_k^{(p)*}) \equiv \pi + \theta_k^{(p)}. \tag{3.27}$$

One can check that when

$$\frac{1}{|\beta_k^{(p)}|^2} \ll |\theta_k^{(p)}| \ll 1 \tag{3.28}$$

is satisfied, then (3.25) is approximated by

$$\left|\alpha_k^{(p)} - \beta_k^{(p)}\right|^2 \simeq 4|\beta_k^{(p)}|^2, \qquad \left|\alpha_k^{(p)} + \beta_k^{(p)}\right|^2 \simeq (\theta_k^{(p)})^2|\beta_k^{(p)}|^2. \tag{3.29}$$

This yields

$$\mathcal{P}_E(k) \simeq \frac{k^4}{4\pi^2 a^4 I^2}\sum_{p=1,2} 4|\beta_k^{(p)}|^2,$$
$$\mathcal{P}_B(k) \simeq \frac{k^4}{4\pi^2 a^4 I^2}\sum_{p=1,2} (\theta_k^{(p)})^2|\beta_k^{(p)}|^2, \tag{3.30}$$

describing a much stronger electric field strength compared to the magnetic. Cases where the magnetic field is stronger can similarly be described by $\arg(\alpha_k^{(p)}\beta_k^{(p)*})$ being close to 0.

D. Maxwell theory on superhorizon scales

For the standard Maxwell theory, i.e., $I^2 = 1$, the mode function is a sum of plane waves,

$$u_k^{(p)} = \frac{1}{\sqrt{2k}}\left\{A_k^{(p)} e^{-ik(\tau-\tau_i)} + B_k^{(p)} e^{ik(\tau-\tau_i)}\right\}. \tag{3.31}$$

Here τ_i is some arbitrary time, while $A_k^{(p)}$ and $B_k^{(p)}$ are time-independent complex numbers satisfying $|A_k^{(p)}|^2 - |B_k^{(p)}|^2 = 1$ as required by the normalization condition (3.13). The time-dependent Bogoliubov coefficients are obtained as

$$\alpha_k^{(p)} = A_k^{(p)} e^{-ik(\tau-\tau_i)}, \qquad \beta_k^{(p)} = B_k^{(p)} e^{ik(\tau-\tau_i)}, \tag{3.32}$$

yielding

$$\cos\left\{\arg\left(\alpha_k^{(p)}\beta_k^{(p)*}\right)\right\} = \cos\left\{\arg\left(A_k^{(p)}B_k^{(p)*}\right) - 2k(\tau-\tau_i)\right\}. \tag{3.33}$$

Now, supposing that the FRW universe has a constant equation of state w ($\neq -1/3$), the Hubble rate would scale as $H \propto a^{-3(w+1)/2}$. Hence the elapsed conformal time is obtained as

$$\tau - \tau_i = \int_{a_i}^{a} \frac{da}{a^2 H} = \frac{2}{3w+1}\left(\frac{1}{aH} - \frac{1}{a_i H_i}\right), \tag{3.34}$$

where quantities with the subscript i are evaluated at τ_i. Rewriting as

$$\arg(A_k^{(p)}B_k^{(p)*}) = \pi + \Theta_k^{(p)}, \tag{3.35}$$

(note that $\Theta_k^{(p)}$ is independent of time), then the phase parameter of (3.27) is

$$\theta_k^{(p)} = \Theta_k^{(p)} - \frac{4}{3w+1}\left(\frac{k}{aH} - \frac{k}{a_i H_i}\right), \tag{3.36}$$

up to the addition of integer multiples of 2π.

Let us now consider a situation where there is a hierarchy between the electric and magnetic power spectra on superhorizon scales. For this purpose we assume that $|\Theta_k^{(p)}| \ll 1$, so that $\theta_k^{(p)}$ is also tiny for modes satisfying $k \ll aH, a_i H_i$. Further supposing the photon density $|\beta_k^{(p)}|^2 = |B_k^{(p)}|^2$ to be large enough to satisfy (3.28), then the superhorizon electromagnetic power spectra are approximately obtained as

$$\mathcal{P}_E(k) \simeq \sum_p \frac{k^4}{\pi^2 a^4}|B_k^{(p)}|^2,$$
$$\mathcal{P}_B(k) \simeq \sum_p \frac{k^4}{4\pi^2 a^4}\left\{\Theta_k^{(p)} - \frac{4}{3w+1}\left(\frac{k}{aH} - \frac{k}{a_i H_i}\right)\right\}^2 |B_k^{(p)}|^2. \tag{3.37}$$

Focusing on the time dependences, one sees that the electric power redshifts as $\propto a^{-4}$. The magnetic power, on the other hand, contains a component with a similar redshifting $\propto a^{-4}$ [cf. (1.1)] arising from the $\Theta_k^{(p)}$ and $k/a_i H_i$ terms, as well as a component with $\propto a^{-6}H^{-2}$ [cf. (1.2)] arising from the k/aH term. The former corresponds to the second term in the right-hand side of (2.5), and the latter corresponds to the first term, thus manifesting Faraday's law. If the expansion of the universe is decelerating, i.e., $w > -1/3$, the magnetic power would eventually be dominated by the component with $\propto a^{-6}H^{-2}$.

In Fig. 1 we show the time evolution of the electromagnetic power spectra for the standard Maxwell theory, in

[7] This is analogous to the squeezing of inflaton and graviton fluctuations during inflation [39].

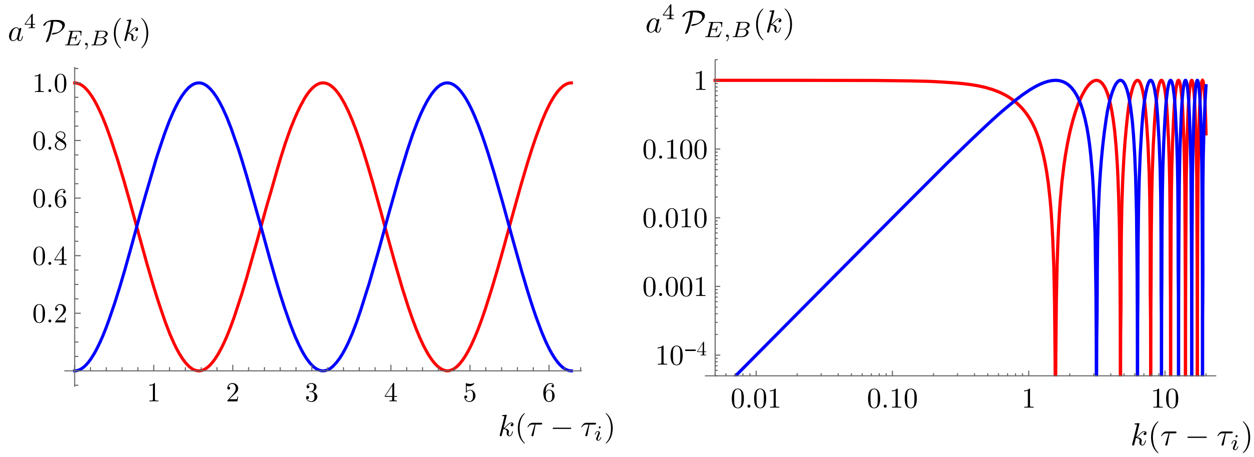

FIG. 1. Time evolution of electromagnetic fields for the standard Maxwell theory, in linear (left panel) and log scales (right panel). Shown are the electric (red) and magnetic (blue) power spectra multiplied by a^4 and normalized such that their oscillation amplitude is unity. The photon density is taken as $|\beta_k^{(p)}|^2 \gg 10^2$, and the phase as $\Theta_k^{(p)} = 0$. Time is shown in terms of the elapsed conformal time in units of k^{-1}. When the mode is outside the horizon of a decelerating universe, i.e., $k(\tau-\tau_i) \ll 1$, the magnetic spectrum grows relative to the electric spectrum which redshifts as $\mathcal{P}_E \propto a^{-4}$ (see the text for details).

terms of $k(\tau-\tau_i)$. Here, the photon density is taken as $|\beta_k^{(p)}|^2 \gg 10^2$, and the phase as $\Theta_k^{(p)} = 0$. The spectra are multiplied by a^4, and normalized such that their oscillation amplitude is unity. As shown in the left plot, the spectra undergo sinusoidal oscillations in conformal time. The superhorizon scaling behaviors of (3.37) are easier to see in the log plot in the right panel. Here, note that in a decelerating universe ($w > -1/3$), the asymptotic future corresponds to $k(\tau-\tau_i) \to \infty$. One clearly sees from the log plot that when $k(\tau-\tau_i) \ll 1$, the magnetic field grows relative to the electric field. On the other hand when $k(\tau-\tau_i) \gg 1$, the electric and magnetic fields oscillate with similar amplitudes.

Thus we have explicitly shown for the standard Maxwell theory that the magnetic power spectrum can scale as (1.2) on superhorizon scales in the presence of stronger electric power. In the next section we will see how this effect fits within magnetic field generation scenarios, by studying a specific model of inflationary magnetogenesis.

IV. EXAMPLE: INFLATIONARY POWER-LAW MAGNETOGENESIS

Let us now study the scaling behaviors of electromagnetic fields in a specific inflationary magnetogenesis model of the type postulated in [5] where the $I(\tau)$ function in the action (3.1) decreases as $\propto a^{-s}$ during inflation, and then becomes constant after inflation,

$$I = \begin{cases} \left(\frac{a_{\mathrm{end}}}{a}\right)^s & \text{for } a \leq a_{\mathrm{end}}, \\ 1 & \text{for } a \geq a_{\mathrm{end}}. \end{cases} \tag{4.1}$$

The subscript "end" denotes quantities at the end of inflation. We take the power s to be a positive integer, i.e.,

$$s = 1, 2, 3, \ldots. \tag{4.2}$$

Since I^2 does not go below unity in this model, the gauge kinetic term is never strongly suppressed and thus we do not worry about strong couplings.

In the following we analyze the cosmological evolution of the electromagnetic fields during both the inflation and postinflation epochs using the formalism based on Bogoliubov transformations developed in the previous sections. In Appendix, we reproduce the result by matching directly the classical field across the transition in the long wavelength approximation. Since the gauge field theory under consideration is symmetric between the two polarizations, hereafter we omit the polarization index (p).

A. Inflationary magnetogenesis

During the inflationary epoch $a \leq a_{\mathrm{end}}$, we consider the Hubble rate to take a time-independent value H_{inf}. Then the mode function that satisfies the equation of motion (3.11) and the normalization condition (3.13), as well as approaches a positive frequency solution in the asymptotic past (i.e., starts from a Bunch-Davies initial condition), is written in terms of the Hankel function as

$$u_k = \frac{1}{2I}\left(\frac{\pi z}{k}\right)^{\frac{1}{2}} H^{(1)}_{-s+\frac{1}{2}}(z), \tag{4.3}$$

up to an unphysical phase. Here the variable z is defined as

$$z \equiv \frac{k}{aH_{\mathrm{inf}}}. \tag{4.4}$$

The time-dependent Bogoliubov coefficients (3.18) are thus obtained as

$$\alpha_k = \left(\frac{\pi z}{8}\right)^{\frac{1}{2}} \left\{ H^{(1)}_{-s+\frac{1}{2}}(z) - iH^{(1)}_{-s-\frac{1}{2}}(z) \right\},$$
$$\beta_k = \left(\frac{\pi z}{8}\right)^{\frac{1}{2}} \left\{ H^{(1)}_{-s+\frac{1}{2}}(z) + iH^{(1)}_{-s-\frac{1}{2}}(z) \right\}. \quad (4.5)$$

The real and imaginary parts of the Hankel functions are respectively the Bessel functions of the first and second kinds,

$$H^{(1)}_\nu(z) = J_\nu(z) + iY_\nu(z), \quad (4.6)$$

where $\nu = -s \pm \frac{1}{2}$. In the superhorizon limit, i.e., $z \to 0$, these asymptote to (noting that s is a positive integer) [40],

$$J_\nu(z) \simeq \frac{1}{\Gamma(\nu+1)} \left(\frac{z}{2}\right)^\nu,$$
$$Y_\nu(z) \simeq -\frac{\Gamma(\nu)}{\pi} \left(\frac{z}{2}\right)^{-\nu}. \quad (4.7)$$

Using these expressions, one can compute the photon number density as

$$|\beta_k|^2 \simeq \frac{\Gamma(s+\frac{1}{2})^2}{4\pi} \left(\frac{2}{z}\right)^{2s}, \quad (4.8)$$

and the phase parameter defined in (3.27) as, up to the addition of integer multiples of 2π,

$$\theta_k \simeq -\frac{z}{s-\frac{1}{2}}. \quad (4.9)$$

As the wave mode goes well outside the horizon, these quantities go as $|\beta_k|^2 \to \infty$ and $\theta_k \to 0$, while satisfying the condition (3.28). Hence one can use the approximation (3.30) to obtain the electromagnetic power spectra on superhorizon scales $k \ll aH_{\rm inf}$,

$$\mathcal{P}_E(k) \simeq \frac{8\Gamma(s+\frac{1}{2})^2}{\pi^3} \frac{H^4_{\rm inf}}{I^2} \left(\frac{k}{2aH_{\rm inf}}\right)^{-2(s-2)},$$
$$\mathcal{P}_B(k) \simeq \frac{8\Gamma(s-\frac{1}{2})^2}{\pi^3} \frac{H^4_{\rm inf}}{I^2} \left(\frac{k}{2aH_{\rm inf}}\right)^{-2(s-3)}. \quad (4.10)$$

The two spectra are related via $\mathcal{P}_B \simeq (2s-1)^{-2} (k/aH_{\rm inf})^2 \mathcal{P}_E$, which is a manifestation of (2.5) implied by Faraday's law.

B. After inflationary magnetogenesis

The universe after inflation stays cold until its dominant energy component turns into heat; we refer to this time when the universe thermalizes as reheating. During the epoch between the end of inflation and reheating, let us suppose charged particles to be nonexistent, and also the universe to expand with some constant equation of state w ($>-1/3$ such that the expansion decelerates).

Such a postinflationary expansion can be supported by, for instance, an inflaton field coherently oscillating about its potential minimum. If the oscillation is (mostly) along a potential of $V \propto \phi^n$, the equation of state averaged over the oscillations would be $w = (n-2)/(n+2)$ [41]. In this picture, reheating would be induced by the decay of the inflaton.[8]

1. Between inflation and reheating

We have assumed in (4.1) that the standard Maxwell theory is recovered at the end of inflation. (Strictly speaking, even within the Standard Model, virtual charged particles in the loops yield an anomalous dependence of the effective action for quantum electrodynamics on a, and thus I is not a constant. However we ignore this since it has little effect on gauge field excitation [38].) Hence during the cold stage between the end of inflation and reheating, the gauge field would follow the Maxwell equation in vacuum, i.e., the equation of motion (3.11) with $I = 1$, whose solution is given by

$$u_k = \frac{1}{\sqrt{2k}} \{ \alpha_k(\tau_{\rm end}) e^{-ik(\tau-\tau_{\rm end})} + \beta_k(\tau_{\rm end}) e^{ik(\tau-\tau_{\rm end})} \}. \quad (4.11)$$

This expression corresponds to (3.31) with the choice of $\tau_i = \tau_{\rm end}$, where the coefficients of the positive and negative frequency solutions are fixed by requiring the Bogoliubov coefficients during inflation (4.5) and after (3.32) to match at $\tau_{\rm end}$. This is equivalent to matching u_k and u'_k in the two epochs at the end of inflation.[9]

For wave modes that have exited the horizon during inflation, the phase parameter in the postinflation epoch is obtained from (3.36) and (4.9) as

[8]As the oscillation amplitude decreases, eventually, the potential would likely be dominated by a quadratic term and thus w approaches 0. However for simplicity, we consider w to be constant all the way until reheating.

[9]The toy model under consideration involves a sudden jump at the end of inflation in the time derivatives of the I function as well as the Hubble rate H. Hence depending on whether one chooses to connect u'_k or $(Iu_k)'$ or something else, different results can be obtained. Here we choose to match the Bogoliubov coefficients since they are directly related to physical quantities. We have also verified this procedure by introducing smooth interpolation for I and H between the two epochs, and numerically solving the gauge field's equation of motion; the numerical results agree well with our analytic expressions (4.10) and (4.14) respectively in the asymptotic regimes $a \ll a_{\rm end}$ and $a \gg a_{\rm end}$.

$$\theta_k \simeq -\frac{2}{2s-1}\frac{k}{a_{\rm end}H_{\rm inf}}\left\{1+\frac{4s-2}{3w+1}\left(\frac{a_{\rm end}H_{\rm inf}}{aH}-1\right)\right\}, \tag{4.12}$$

whose amplitude monotonically increases in time. The photon number density $|\beta_k|^2$, which is now time-independent, is obtained by evaluating (4.8) at the end of inflation. The condition (3.28) continues to be satisfied while the mode is well outside the horizon, i.e., $k \ll aH$, and thus from (3.30) one can obtain the electromagnetic power spectra as[10]

$$\begin{aligned}\mathcal{P}_E(k) &\simeq \frac{8\Gamma(s+\frac{1}{2})^2}{\pi^3}H_{\rm inf}^4\left(\frac{k}{2a_{\rm end}H_{\rm inf}}\right)^{-2(s-2)}\left(\frac{a_{\rm end}}{a}\right)^4,\\ \mathcal{P}_B(k) &\simeq \frac{8\Gamma(s-\frac{1}{2})^2}{\pi^3}H_{\rm inf}^4\left(\frac{k}{2a_{\rm end}H_{\rm inf}}\right)^{-2(s-3)}\left(\frac{a_{\rm end}}{a}\right)^4\\ &\quad\times\left\{1+\frac{4s-2}{3w+1}\left(\frac{a_{\rm end}H_{\rm inf}}{aH}-1\right)\right\}^2.\end{aligned} \tag{4.14}$$

The decelerated expansion of the universe eventually renders $a_{\rm end}H_{\rm inf} \gg aH$, then the relation between the electromagnetic power becomes $\mathcal{P}_B \simeq (2/3w+1)^2(k/aH)^2\mathcal{P}_E$, being compatible with (2.5) which follows from Faraday's law. Here, it is also important to note that while the electric power redshifts as $\mathcal{P}_E \propto a^{-4}$, the magnetic power scales[11] as $\mathcal{P}_B \propto a^{-6}H^{-2} \propto a^{3(w-1)}$.

2. *After reheating*

Upon reheating, the conductivity of the universe becomes high, and thus the electric fields are shorted out while the magnetic flux is frozen in. Hence we consider large-scale magnetic fields after reheating to redshift as $\mathcal{P}_B \propto a^{-4}$ until today.

The magnetic power spectrum in the present universe is thus obtained as, for wave modes that are outside the horizon at the time of reheating,[12]

$$\begin{aligned}\mathcal{P}_{B0}(k) &= \mathcal{P}_{B\,\rm reh}(k)\left(\frac{a_{\rm reh}}{a_0}\right)^4\\ &\simeq \frac{8\Gamma(s-\frac{1}{2})^2}{\pi^3}H_{\rm inf}^4\left(\frac{k}{2a_{\rm end}H_{\rm inf}}\right)^{-2(s-3)}\left(\frac{a_{\rm end}}{a_0}\right)^4\\ &\quad\times\left\{1+\frac{4s-2}{3w+1}\left(\frac{a_{\rm end}H_{\rm inf}}{a_{\rm reh}H_{\rm reh}}-1\right)\right\}^2,\end{aligned} \tag{4.15}$$

where the subscript "reh" is used to describe quantities upon reheating, and "0" for today. The enhancement factor of

$$\frac{a_{\rm end}H_{\rm inf}}{a_{\rm reh}H_{\rm reh}} = \left(\frac{H_{\rm inf}}{H_{\rm reh}}\right)^{\frac{3w+1}{3w+3}} \tag{4.16}$$

inside the parentheses represents the effect of the electromagnetic induction during the epoch between inflation and reheating. This would be missed if one were to assume the magnetic power to redshift as a^{-4} right from the end of inflation, as has been done in most previous works. The enhancement factor becomes particularly large when there is a hierarchy between the inflation and reheating scales. The scale of inflation is bounded from above by the current observational limit on primordial gravitational waves as $H_{\rm inf} \lesssim 10^{14}$ GeV [42], while the reheating temperature needs to be higher than about 5 MeV in order not to spoil BBN [43], setting a lower bound on the reheating scale as $H_{\rm reh} \gtrsim 10^{-23}$ GeV. Hence the ratio between the inflation and reheating scales can in principle be as large as $H_{\rm inf}/H_{\rm reh} \lesssim 10^{37}$, and the postinflationary induction would significantly impact the final magnetic field amplitude. The effect is maximized for a stiff equation of state $w \gg 1$, for which the factor of (4.16) can be as large as 10^{37}. If $w = 1/3$, which is the case we will mainly consider in the example in the next subsection, the factor can be up to 10^{18}. Even with a pressureless state $w = 0$, the factor can be as large as 10^{12}.

C. Intergalactic magnetic fields from high-scale inflation

To demonstrate the importance of the postinflationary induction, let us present an example where the femto-Gauss intergalactic magnetic fields as suggested by recent gamma ray observations are produced from inflationary magnetogenesis with a high inflation scale and low reheat temperature.

The example is given by the model of (4.1) with a power

$$s = 2, \tag{4.17}$$

which produces a k-independent electric power spectrum, cf. (4.10). The gauge field's energy density (3.26) during inflation is dominated by the scale-invariant electric power, which is roughly of order

[10]If one allows for a general postinflation expansion history instead of assuming a constant w, then in (4.14), the final parentheses of $\mathcal{P}_B$ is replaced by

$$\begin{aligned}&\left\{1+\frac{4s-2}{3w+1}\left(\frac{a_{\rm end}H_{\rm inf}}{aH}-1\right)\right\}^2\\ &\quad\to\left\{1+(2s-1)\int_{a_{\rm end}}^{a}\frac{da}{a}\frac{a_{\rm end}H_{\rm inf}}{aH}\right\}^2.\end{aligned} \tag{4.13}$$

[11]If the standard Maxwell theory is recovered during inflation instead of at the very end, the magnetic power would initially redshift as $\mathcal{P}_B \propto a^{-4}$, then some time after inflation switch to $\propto a^{-6}H^{-2}$.

[12]Reheating happens before big bang nucleosynthesis (BBN), and since the comoving Hubble radius at the beginning of BBN is of $a_0/(a_{\rm BBN}H_{\rm BBN}) \sim 10$ pc, the result (4.15) applies at least for wave numbers satisfying $k/a_0 < (10\ {\rm pc})^{-1}$.

$$\rho_A \sim H_{\mathrm{inf}}^4 \log\left(\frac{aH_{\mathrm{inf}}}{k_{\mathrm{IR}}}\right). \tag{4.18}$$

Here, upon carrying out the k-integral in (3.26), we have introduced a UV cutoff and set it to the mode exiting the horizon, i.e., $k_{\mathrm{UV}} \sim aH_{\mathrm{inf}}$, since for higher k modes the gauge field fluctuations have not yet become classical and thus their contributions to the energy density should be renormalized.[13] We have also introduced an IR cutoff k_{IR}; considering it to be the wave mode that exited the horizon at the beginning of inflation, the factor $\log(aH_{\mathrm{inf}}/k_{\mathrm{IR}})$ corresponds to the number of elapsed inflationary e-folds $\mathcal{N}$. Here, from the observational limit $H_{\mathrm{inf}} \lesssim 10^{14}$ GeV, the ratio between the gauge field density (4.18) and the total density of the universe $\rho_{\mathrm{tot}} = 3M_p^2 H_{\mathrm{inf}}^2$ is bounded as $\rho_A/\rho_{\mathrm{tot}} \lesssim 10^{-9}\mathcal{N}$, being much smaller than the amplitude of the curvature perturbation $\zeta \sim 10^{-5}$ measured on CMB scales (unless the inflationary period is extraordinarily long). Thus the effect of the excited gauge field on the cosmological perturbations[14] and the inflationary background is negligible.

However we should also remark that, depending on the postinflationary equation of state w, the gauge field's backreaction may become non-negligible after inflation. Here, recall that once the standard Maxwell theory is recovered, the gauge field density redshifts as radiation, i.e., $\rho_A \propto a^{-4}$. Hence if $w \leq 1/3$, its ratio to the total density $\rho_A/\rho_{\mathrm{tot}}$ does not increase in time. However if $w > 1/3$, the ratio would grow and thus one needs to verify whether the backreaction becomes significant.

The magnetic field strength today is obtained by substituting $s = 2$ into (4.15), and let us suppose that $a_{\mathrm{end}}H_{\mathrm{inf}} \gg a_{\mathrm{reh}}H_{\mathrm{reh}}$, namely, that the universe thermalizes well after inflation ends. Considering the entropy of the universe to be conserved since reheating, the redshift and energy scale of reheating are related by (supposing the Standard Model degrees of freedom),

$$\frac{a_0}{a_{\mathrm{reh}}} \approx 3\times 10^{10}\left(\frac{H_{\mathrm{reh}}}{10^{-23}\ \mathrm{GeV}}\right)^{1/2}. \tag{4.19}$$

Also using (4.16), the magnetic field spectrum on large scales is obtained as

$$\mathcal{P}_{B0}(k) \sim \frac{(10^{-33}\ \mathrm{G})^2}{(3w+1)^2}\left(\frac{k}{a_0}\mathrm{Mpc}\right)^2\left(\frac{H_{\mathrm{inf}}}{10^{14}\ \mathrm{GeV}}\right)\left(\frac{H_{\mathrm{inf}}}{H_{\mathrm{reh}}}\right)^{\frac{9w+1}{3w+3}}. \tag{4.20}$$

[13]By "becoming classical," we mean that the classical volume of the space spanned by the gauge field fluctuation and its conjugate momentum becomes much larger than their quantum uncertainty. See [17,38,44] for detailed analyses.

[14]The gauge field sources curvature perturbations roughly of $\zeta_A \sim \rho_A/(\epsilon\rho_{\mathrm{tot}})$, where $\epsilon = -H'/(aH^2)$ is the rate of change of the Hubble parameter. See e.g., [16] for detailed analyses of CMB constraints.

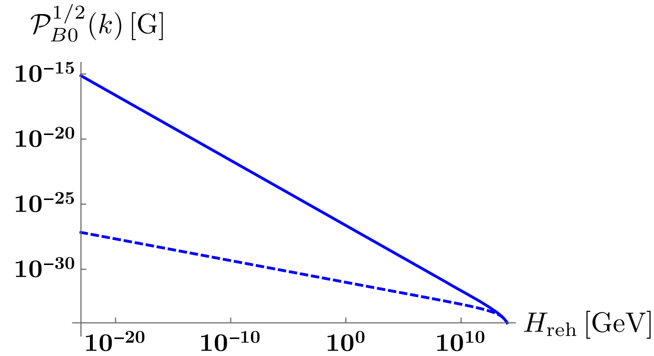

FIG. 2. Magnetic field strength today on $k/a_0 = (1\ \mathrm{Mpc})^{-1}$, generated by the inflationary magnetogenesis model (4.1) with $s = 2$. The inflation scale is fixed to $H_{\mathrm{inf}} = 10^{14}$ GeV, and the field strength is shown as a function of the Hubble scale at reheating. The postinflationary equation of state is taken as $w = 0$ (dashed line) and $w = 1/3$ (solid).

From this expression one sees that the magnetic field strength is larger for smaller length scales, higher inflation scales, and if $w > -1/9$, for larger $H_{\mathrm{inf}}/H_{\mathrm{reh}}$ ratios. In the case with the largest possible hierarchy between the inflation and reheating scales,[15] i.e., $H_{\mathrm{inf}} = 10^{14}$ GeV and $H_{\mathrm{reh}} = 10^{-23}$ GeV, the magnetic field strength on the wave number $k/a_0 = (1\ \mathrm{Mpc})^{-1}$ is $\mathcal{P}_{B0}^{1/2} \sim 10^{-27}$ G for $w = 0$, and $\mathcal{P}_{B0}^{1/2} \sim 10^{-15}$ G for $w = 1/3$. For the same parameters but with a higher reheating scale $H_{\mathrm{reh}} = 10^{-12}$ GeV (corresponding to a temperature of $T_{\mathrm{reh}} \sim 1$ TeV), then $\mathcal{P}_{B0}^{1/2} \sim 10^{-21}$ G for $w = 1/3$. In Fig. 2 we plot the magnetic field strength as a function of H_{reh}, for $k/a_0 = (1\ \mathrm{Mpc})^{-1}$ and $H_{\mathrm{inf}} = 10^{14}$ GeV. The dashed line shows the case of $w = 0$, while the solid line is for $w = 1/3$. The lines are seen to bend at $H_{\mathrm{reh}} \gtrsim 10^{13}$ GeV; here reheating happens soon after inflation and hence there is not enough time for the induction effect to become important, namely, the second term inside the $\{\}$ parentheses of (4.15) is not much greater unity and thus the result deviates from the approximation (4.20). The field strength basically increases with w, however for $w > 1/3$, the postinflation backreaction may become non-negligible as discussed above.[16]

Thus by properly taking into account electromagnetic induction after inflation, we have shown that the simple inflationary magnetogenesis model (4.1) with $s = 2$ is capable of creating femto-Gauss intergalactic magnetic fields on Mpc scales, given a high-scale inflation $H_{\mathrm{inf}} = 10^{14}$ GeV and low reheating $H_{\mathrm{reh}} = 10^{-23}$ GeV, with the

[15]The case $\rho_{\mathrm{inf}}^{1/4} = 10^{16}$ GeV ($H_{\mathrm{inf}} \sim 10^{14}$ GeV) and $s = 2$ is within the region where backreaction and anisotropy constraints are satisfied [16].

[16]An equation of state of $w > 1/3$ can also blue-tilt the primordial gravitational wave spectrum [45]. It would be interesting to study the possibility of probing w from a joint analysis of the magnetic fields and gravitational waves.

two periods connected by an equation of state $w = 1/3$. Here we stress that the equation of state $w = 1/3$ of this scenario is not due to charged relativistic particles, but instead should be realized by some substance without charge such as an oscillating inflaton condensate.

V. MODEL-INDEPENDENT CONSTRAINTS

We have shown in the previous sections that if primordial magnetogenesis creates stronger electric fields than magnetic fields, then even after the standard Maxwell theory is recovered, the electromagnetic spectra on superhorizon scales can be related by

$$\mathcal{P}_B(k) \sim \left(\frac{k}{aH}\right)^2 \mathcal{P}_E(k), \tag{5.1}$$

yielding the magnetic scaling $\mathcal{P}_B \propto a^{-6}H^{-2}$ instead of a radiationlike redshifting. In this section we derive generic bounds on primordial magnetic fields with such a behavior, by analyzing the gauge field's gravitational backreaction.

A. Generic reheating bound

We start by constraining cases where the standard Maxwell theory is recovered by the time of reheating. (Thus the Weyl invariance of the gauge field action can explicitly be violated even after inflation, as in postinflationary magnetogenesis scenarios [6], see also [46,47].) We suppose that coherent electromagnetic fields have been created on some wave modes that are outside the horizon upon reheating, and that right before reheating when the electric fields have not yet vanished, the power spectra satisfy the relation (5.1) on the wave modes of interest.

One can read off from the third line of (3.26) that in Maxwell theory ($I^2 = 1$), the electric power spectrum with wave number k contributes to the gauge field's energy density as $\Delta\rho_A \sim \mathcal{P}_E(k)/2$, given that the spectrum $\mathcal{P}_E(k)$ is smooth over a range of $\Delta k \sim k$ so that the integral $\int dk/k$ can be approximated by an order-unity factor.[17] Considering that the other contributions to the gauge field density are non-negative, an inequality of

$$\rho_A \gtrsim \frac{1}{2}\mathcal{P}_E(k) \tag{5.2}$$

is thus obtained. We further assume the magnetic power to redshift after reheating as

$$\mathcal{P}_B(k) \propto a^{-4} \quad (a \geq a_{\rm reh}). \tag{5.3}$$

[17] Sharp features localized to ranges of $\Delta k \ll k$ can be produced if rapidly time varying backgrounds give rise to resonant production of photons. This could in principle provide a way to evade the constraints in this section.

Based on these assumptions, an upper bound on the magnetic spectrum in the current universe is obtained as

$$\begin{aligned}\mathcal{P}_{B0}(k) &= \mathcal{P}_{B\,\rm reh}(k)\left(\frac{a_{\rm reh}}{a_0}\right)^4 \\ &\lesssim 6M_p^2\left(\frac{k}{a_0}\right)^2\left(\frac{\rho_A}{\rho_{\rm tot}}\bigg|_{\rm reh}\right)\left(\frac{a_{\rm reh}}{a_0}\right)^2 \\ &\sim (10^{-13}\ {\rm G})^2\left(\frac{k}{a_0}{\rm Mpc}\right)^2\left(10^5\frac{\rho_A}{\rho_{\rm tot}}\bigg|_{\rm reh}\right) \\ &\quad\times\left(\frac{10^{-23}\ {\rm GeV}}{H_{\rm reh}}\right).\end{aligned} \tag{5.4}$$

Here we have used (5.3) in the first line, then $\rho_{\rm tot} = 3M_p^2H^2$, (5.1), and (5.2) to get to the second line, and (4.19) for the third line. The reference value of $(\rho_A/\rho_{\rm tot})_{\rm reh} = 10^{-5}$ has been chosen from the amplitude of the large-scale curvature perturbation $\zeta \sim 10^{-5}$; a larger density ratio, in particular if its main contribution is on CMB scales, would source too large curvature perturbations and contradict with observations. Note also that $H_{\rm reh} = 10^{-23}$ GeV is the lowest possible reheating scale compatible with BBN. Hence the bound (5.4) shows that if the electromagnetic spectra satisfy the relation (5.1) right before reheating, then the magnetic field strength cannot exceed 10^{-13} G on Mpc or larger scales today, otherwise the gauge field fluctuations would spoil the cosmological perturbations. In particular, in order to have femto-Gauss magnetic fields on Mpc scales, the reheating scale should satisfy $H_{\rm reh} \lesssim 10^{-18}$ GeV, which in terms of the reheating temperature translates into $T_{\rm reh} \lesssim 1$ GeV.

Upon deriving the bound (5.4), we have only employed assumptions about times from reheating onward. In particular, no assumption was made regarding cosmology and the gauge field theory in epochs prior to reheating.

B. Less generic inflation bound

Let us now make some assumptions about the period between inflation and reheating, in order to obtain a magnetic field bound in terms of the inflation scale. Hereafter we assume that by the end of inflation, the standard Maxwell theory is recovered and yields (5.2) (hence the following discussions are limited to inflationary magnetogenesis scenarios). We further assume that the postinflationary universe expands with a constant equation of state w until reheating, with the electric field redshifting as $\mathcal{P}_E \propto a^{-4}$ during this period. As in Sec. V A, we suppose the relation (5.1) to hold right before reheating, and the magnetic field to redshift as (5.3) after reheating.

Then in a similar way as we derived (5.4), but now considering the backreaction at the end of inflation (note $H(a_{\rm end}) = H_{\rm inf}$), one can obtain

$$\mathcal{P}_{B0}(k) \lesssim 6M_p^2 \left(\frac{k}{a_0}\right)^2 \left(\frac{\rho_A}{\rho_{\rm tot}}\bigg|_{\rm end}\right) \left(\frac{a_{\rm reh}}{a_0}\right)^2 \left(\frac{H_{\rm reh}}{H_{\rm inf}}\right)^{\frac{2(-3w+1)}{3(w+1)}}. \tag{5.5}$$

The main difference from the reheating bound (5.4) is the presence of the ratio $H_{\rm reh}/H_{\rm inf}$. It appears in the bound with a positive (negative) power for $w < (>)\ 1/3$, reflecting the fact that the gauge density ratio $\rho_A/\rho_{\rm tot}$ decreases (increases) in time after inflation. Thus the bound would be equivalent to (5.4) if $w = 1/3$ (i.e., radiationlike background), or $H_{\rm reh} = H_{\rm inf}$ (i.e., instantaneous reheating at the end of inflation).

If, for instance, $w = 0$, then (5.5) can be rewritten as

$$\mathcal{P}_{B0}(k) \lesssim (10^{-15}\ {\rm G})^2 \left(\frac{k}{a_0}{\rm Mpc}\right)^2 \left(10^5 \frac{\rho_A}{\rho_{\rm tot}}\bigg|_{\rm end}\right) \times \left(\frac{10^{-23}\ {\rm GeV}}{H_{\rm reh}}\right)^{1/3} \left(\frac{10^{-16}\ {\rm GeV}}{H_{\rm inf}}\right)^{2/3}. \tag{5.6}$$

Hence one finds that for $w = 0$, femto-Gauss magnetic fields can exist on Mpc scales only if the inflation scale satisfies[18] $H_{\rm inf} \lesssim 10^{-16}$ GeV.

VI. COMMENTS ON SCHWINGER EFFECT

The nonradiationlike scaling of the electromagnetic fields arises in the presence of a hierarchy between the electric and magnetic field strengths. In the previous sections we considered electric fields much stronger than magnetic fields being produced in the primordial universe, which then affect the subsequent magnetic field evolution. Up until the time of reheating, we supposed a cold universe where charged particles are absent, and hence assumed the electric fields to survive. However, if the electric field is strong enough, it can give rise to Schwinger production of charged particles [48–50], which in turn would backreact significantly on the electric fields before reheating [36] (see also e.g., [51–53]).

Studying the fate of strong cosmological electric fields would require an analysis of the Schwinger process in a curved spacetime, whose behavior can differ from that in flat space due to the extra effect from the gravitational background, as was shown explicitly for de Sitter spacetimes in e.g., [36,54,55]. A complete analysis in a generic FRW spacetime is beyond the scope of this paper; instead we provide here a crude estimate of the impact of the Schwinger process in a cosmological background, and postulate the condition under which primordial electric fields are unaffected by the Schwinger effect.

We again consider the gauge field theory of (3.1), but now coupled to matter such that the equation of motion of the gauge field includes a conserved current J^μ,

$$\nabla_\mu (I^2 F^{\mu\nu}) = -J^\nu. \tag{6.1}$$

Its spatial component yields the modified Ampère-Maxwell law, which in terms of the electromagnetic fields reads [our sign convention follows from the definition (2.1) with $u^0 > 0$]

$$\hat{\varepsilon}_{ijl}\partial_j B_l = \frac{(aI^2E_i)'}{aI^2} + \frac{aJ_i}{I^2} = E_i' + \left\{\frac{(aI^2)'}{aI^2} + \frac{a\sigma}{I^2}\right\} E_i. \tag{6.2}$$

Here in the far right-hand side, the current is considered to be carried by particles produced via the Schwinger process, and thus we have rewritten it using the conductivity σ introduced as

$$J_i = \sigma E_i. \tag{6.3}$$

Let us now assume the first term inside the {} parentheses to be of

$$\left|\frac{(aI^2)'}{aI^2}\right| \sim aH, \tag{6.4}$$

as is the case for the standard Maxwell theory ($I^2 = 1$), as well as the power-law magnetogenesis model in Sec. IV. Then the condition under which the induced current has a negligible effect on the evolution of the electric field can be read off as

$$|\sigma| \lesssim I^2 H. \tag{6.5}$$

In Minkowski space, the conductivity induced by a background electric field through the Schwinger pair production is of (see e.g., [56])

$$\sigma \sim (t - t_{\rm on}) e^3 E \exp\left(-\frac{\pi m^2}{eE}\right), \tag{6.6}$$

where t is time, $t_{\rm on}$ is when the electric field was turned on, E is the electric field strength, and m and e are respectively the mass and amplitude of the charge of the produced pairs. The linear dependence on time reflects the fact that the produced particles accumulate until their backreaction to the electric field becomes non-negligible.

On the other hand in a cosmological background, the expansion of the universe dilutes away the particles produced by the electric field, thus introducing the time scale H^{-1}. Hence, supposing that the rate of change of the electric field is comparable to or smaller than H, we crudely

[18] In [17], constraints on inflationary magnetogenesis were derived for general gauge field theories with a two-derivative kinetic term, under the assumption of the postinflationary redshifting $\mathcal{P}_B \propto a^{-4}$. Their bound (3.20), for instance, is modified by instead adopting $\mathcal{P}_B \propto a^{-6}H^{-2}$; further multiplying by 10^{-5} considering the curvature perturbation, the modified bound matches with our (5.6).

estimate the induced conductivity in a FRW background by replacing the elapsed time in the flat space result (6.6) by the Hubble time,[19]

$$\sigma \sim \frac{e^3 E}{H} \exp\left(-\frac{\pi m^2}{eE}\right). \tag{6.7}$$

Here the electric field strength is understood as $E = (E_\mu E^\mu)^{1/2}$. In an inflationary de Sitter space, the conductivity induced by a time-independent electric field actually does take this form in the strong electric field regime $eE \gg H^2$; while with weak electric fields, gravitational particle production renders σ to take a different form [36]. Since now we are interested in strong primordial electric fields, let us adopt (6.7) for the moment as the induced conductivity in a generic FRW universe.

Then the condition (6.5) reads

$$\frac{e^3 E}{H^2} \exp\left(-\frac{\pi m^2}{eE}\right) \lesssim I^2. \tag{6.8}$$

This can be understood as an upper bound on the electric field strength for which the backreaction from the produced particles can be neglected. In particular if the charged particle is light enough such that $m^2 \ll eE$, then the bound is simplified to

$$\frac{e^3 E}{H^2} \lesssim I^2, \tag{6.9}$$

which implies that if light charged particles exist in the theory, then electric fields exceeding the Hubble scale multiplied by I^2 would receive significant backreaction from the Schwinger process.[20]

When applying the above discussion to the inflationary magnetogenesis scenario of Sec. IV by the substitution $E \to \mathcal{P}_E(k)^{1/2}$, one can check that the condition (6.9) for $e \sim 1$ is either saturated or violated (depending on the value of s) at some k-mode towards the end of inflation. Moreover in the postinflation epoch, the condition is strongly violated since the ratio $P_E(k)^{1/2}/H^2 \propto a^{1+3w}$ grows in time. Hence our crude estimate suggests that the evolution of the electric field is affected by the produced light charged particles before reheating, and thus the magnetic scaling would deviate from $P_B(k) \propto a^{-6} H^{-2}$. The Schwinger production, however, could be avoided if there is some mechanism in the early universe giving sufficiently large masses to charged particles.[21]

We stress that the analyses in this section rely on the very rough estimate of the conductivity (6.7) induced by the Schwinger effect in a FRW universe. Clearly a more precise calculation would be necessary in determining detailed bounds on primordial electric fields. It is also important to study what actually happens when the Schwinger effect becomes relevant; whether the electric fields quickly decay, or the field decay balances the Schwinger production and thus allows the electric field to survive. Other than from the Schwinger process, the electric field may also be affected by a gradual decay of the inflaton before it completely thermalizes the universe, depending on the decay process [4]. We leave a careful exploration of these issues for future work.

VII. CONCLUSIONS

We showed that primordial electric and magnetic fields do not necessarily redshift in a radiationlike manner on superhorizon scales. This is a simple consequence of Maxwell's equations allowing exchange of power between the two fields. Given that electric fields stronger than magnetic fields are produced in the early universe, the electric fields can render the magnetic fields to redshift slowly, or even blueshift. In particular for the standard Maxwell theory, we showed that the magnetic power scales as $B^2 \propto a^{-6} H^{-2}$ in the postinflationary universe until the electric fields disappear.

The implication of the induction effect for primordial magnetogenesis is that the produced magnetic fields continue to be sourced by the electric fields up until the time of reheating, thus leading to stronger magnetic strengths than were previously estimated. The effect is particularly large if the inflation and reheating scales are well separated, and/or the postinflationary universe has a stiff equation of state, in which cases the previous estimates are corrected by up to 37 orders of magnitude. As an example, we presented a toy model of inflationary magnetogenesis which produces femto-Gauss magnetic fields on Mpc scales, combined with a high inflation scale of $H_{\mathrm{inf}} = 10^{14}$ GeV and low reheating temperature just above the BBN scale, with the postinflation epoch possessing an equation of state $w = 1/3$. This offers a counterexample to the common lore that high-scale inflation is incompatible with efficient inflationary magnetogenesis; moreover it opens up the possibility of producing both observable magnetic fields

[19]We assume the I^2 function to multiply only the photon kinetic term but not the photon-matter coupling terms, therefore the induced conductivity would not explicitly depend on I^2.

[20]Given that $e^2/I^2 \lesssim 1$, then the regime affected by the Schwinger process, i.e., $e^3 E \gg I^2 H^2$, would fall into the strong field regime $eE \gg H^2$ where the approximation (6.7) is expected to hold. This justifies our use of (6.7) for constraining electric fields.

[21]In [36], Schwinger effect constraints on inflationary magnetogenesis were derived by analyzing the Schwinger process during inflation, and assuming $\mathcal{P}_B \propto a^{-4}$ after inflation. The constraints can be relaxed in the presence of the postinflation induction; however the estimate in this section indicates that even if the Schwinger process during inflation is negligible, it may become important afterwards.

and gravitational waves from inflation. It would also be interesting to explore other scenarios of primordial magnetogenesis by taking into account the correct scaling behavior of the electromagnetic fields.

We also derived model-independent bounds on primordial magnetic fields that are supported by the induction effect, setting a consistency relation between the magnetic field strength and the reheating scale. Finally, we briefly commented on the possibility that primordial electric fields may quench prior to reheating via the Schwinger production of charged particles, in which case the magnetic fields would lose support from the electric fields and thus obey the radiationlike redshifting. We crudely estimated the condition for the Schwinger process to be important; a more precise calculation of this effect is an important task for the future.

Although we have focused on electromagnetic fields throughout this paper, the induction effect can also be important for addressing the fate of other gauge fields, such as dark photons, that could have been excited in the early universe.

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APPENDIX: SUPERHORIZON MATCHING OF CLASSICAL FIELD

Here we match the classical gauge field on superhorizon scales across the inflationary and postinflationary epochs in the Ratra model, and show that it agrees with the result in Sec. IV, which was obtained using the more generally applicable method of Bogoliubov transformations.

Normalizing the mode function as $\tilde{u}_k = I u_k$ (we drop polarization indices), the equation of motion (3.11) can be written as

$$\tilde{u}_k'' + \left(k^2 - \frac{I''}{I}\right)\tilde{u}_k = 0. \tag{A1}$$

The superhorizon regime and the long wavelength regime of $k^2 \ll |I''/I|$ approximately coincide for reasonable power-law functions $I \propto a^{-s}$, and in this regime the equation has the general solution

$$\tilde{u}_k \sim C_1 I + C_2 I \int \frac{d\tau}{I^2}. \tag{A2}$$

In any regime where I is constant (as in the postinflation regime of the model discussed in Sec. IV), the solution will have a constant term and one proportional to $\tau \supset C_3/(aH)$ [cf. (3.34)], which will be growing in a decelerated expansion phase.

To be more precise, let us consider the model with $I \propto a^{-s}$ during inflation, which by setting the conformal time as $\tau = -1/(aH_{\rm inf})$ leads to

$$\tilde{u}_k'' + \left(k^2 - \frac{s(s-1)}{\tau^2}\right)\tilde{u}_k = 0. \tag{A3}$$

The solution starting from the Bunch-Davies vacuum is given in (4.3). In the superhorizon limit, this can be expanded in terms of $(-k\tau)$ as [15] (given that s is not a half-integer),

$$\begin{aligned} u_k &= \frac{\tilde{u}_k}{I} \\ &= \tilde{C}_1(k,s)\left\{1 - \frac{1}{s+\frac{1}{2}}\left(\frac{-k\tau}{2}\right)^2 + \cdots\right\} \\ &\quad + \tilde{D}_1(k,s)\left\{\left(\frac{-k\tau}{2}\right)^{-2s+1} + \cdots\right\} \end{aligned} \tag{A4}$$

where the dots indicate higher order terms in the long wavelength approximation and

$$\tilde{C}_1(k,s) = -\frac{i\Gamma(-s+\frac{1}{2})}{(2\pi k)^{1/2}}\left(\frac{-k\tau_{\rm end}}{2}\right)^s, \tag{A5}$$

$$\tilde{D}_1(k,s) = -\frac{e^{is\pi}\Gamma(s-\frac{1}{2})}{(2\pi k)^{1/2}}\left(\frac{-k\tau_{\rm end}}{2}\right)^s. \tag{A6}$$

Here the subscript "end" denotes the end of inflation, and we have set $I_{\rm end} = 1$.

The mode function after inflation ends and I becomes a constant is a sum of plane waves, i.e., (4.11), which can be expanded in terms of $k(\tau - \tau_{\rm end})$,

$$\begin{aligned} u_k = \frac{1}{(2k)^{1/2}}[&\alpha_k(\tau_{\rm end}) + \beta_k(\tau_{\rm end}) \\ &- i\{\alpha_k(\tau_{\rm end}) - \beta_k(\tau_{\rm end})\}k(\tau-\tau_{\rm end}) + \cdots]. \end{aligned} \tag{A7}$$

The coefficients $\alpha_k(\tau_{\rm end})$ and $\beta_k(\tau_{\rm end})$ are determined by the matching conditions at $\tau_{\rm end}$. Here, due to considerations pertaining to energy conservation, it is u_k and u_k' that has to be matched across the transition. Focusing on the case of $s > 1/2$, then the mode function during inflation (A4) is dominated by the $\tilde{D}_1$ term and thus we obtain the coefficients as

$$\alpha_k(\tau_{\rm end}) + \beta_k(\tau_{\rm end}) \simeq -\frac{e^{is\pi}\Gamma(s-\frac{1}{2})}{\pi^{1/2}}\left(\frac{-k\tau_{\rm end}}{2}\right)^{-s+1}, \tag{A8}$$

$$\alpha_k(\tau_{\rm end}) - \beta_k(\tau_{\rm end}) \simeq -\frac{ie^{is\pi}\Gamma(s+\frac{1}{2})}{\pi^{1/2}}\left(\frac{-k\tau_{\rm end}}{2}\right)^{-s}. \tag{A9}$$

For the power spectrum of magnetic fields on superhorizon scales after the end of inflation, we then find

$$\begin{aligned}\mathcal{P}_B(k) &= \frac{k^5}{\pi^2 a^4}|u_k|^2\\ &\simeq \frac{\Gamma(s-\frac{1}{2})^2 k^4}{2\pi^3 a^4}\left(\frac{-k\tau_{\rm end}}{2}\right)^{-2(s-1)}\\ &\quad\times\left\{1+(2s-1)\frac{\tau-\tau_{\rm end}}{-\tau_{\rm end}}\right\}^2.\end{aligned} \tag{A10}$$

It is easy to verify that this is equivalent to the result in Eq. (4.14) when using $\tau_{\rm end} = -1/(a_{\rm end}H_{\rm inf})$, and (3.34) for the elapsed time $(\tau - \tau_{\rm end})$.

An important observation compared with [15] is that, when connected to an epoch of $I \propto |\tau|^{\tilde{s}}$ with a different power satisfying $\tilde{s} < -1/2$ in [15], the growing solution got matched to the decaying solution, resulting in loss of power at the transition and thus no enhanced magnetic fields. On the other hand in the present case of connecting to an epoch with a constant I, the growing solution gets matched directly on to the growing solution after the transition with no loss of power.

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