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# Accepted Manuscript

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# The parameterized complexity landscape of finding 2-partitions of digraphs

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## Abstract

Given a network modeled by a directed graph  $D = (V, A)$ , it is natural to ask whether we can partition the vertex set of  $D$  into two disjoint subsets  $V_1, V_2$  (called a 2-partition), such that the digraphs  $D[V_1], D[V_2]$  induced by each of these has one of the two properties of interest. This question gives rise to a rich realm of combinatorial problems. The complexity of many such problems was determined in [2, 3]. We analyze a subset of those problems from the viewpoint of parameterized complexity, and present a complete dichotomy of basic, natural properties. More precisely, given a directed graph  $D = (V, A)$  and two non-negative integers  $k_1$  and  $k_2$ , we seek a 2-partition  $(V_1, V_2)$  of the vertex set  $V$  such that  $|V_1| \geq k_1$ ,  $|V_2| \geq k_2$ , and each of the subdigraphs induced by  $V_1$  and  $V_2$  has a structural property as defined by the problem at hand—for example,  $D[V_1]$  is acyclic and  $D[V_2]$  is strongly connected. Specifically, we consider the following eight structural properties: being strongly connected; being connected; having an out-branching; having an in-branching; having minimum degree at least one; having minimum semi-degree at least one; being acyclic; and being complete.

**Keywords:** Parameterized complexity, directed graphs, digraph partitioning, 2-partition.

## 1 Introduction

A **2-partition**,  $(V_1, V_2)$  of a digraph  $D = (V, A)$  is a partition of  $V$  into two disjoint sets  $V_1, V_2$ . Significant interest in problems dealing with partitions of the vertex set of a (di)graph has been shown in the literature [1, 2, 3, 7, 8, 9]. One of the natural questions is whether a digraph  $D$  has a 2-partition  $(V_1, V_2)$  such that the induced sub(di)graphs  $D[V_1], D[V_2]$  on the partitions have some prescribed structural properties. A variant of this problem is to decide for a digraph  $D = (V, A)$  and positive integers  $k_1, k_2$  if it is possible to partition the vertex set into two disjoint vertex set  $V_1$  and  $V_2 = V \setminus V_1$  such that each of the induced subdigraphs on the partitions have prescribed properties, and  $|V_1| \geq k_1$  and  $|V_2| \geq k_2$ . In this paper, we study this problem from the viewpoint of parameterized complexity where we consider the following structural properties; being strongly connected, being connected, having an out-branching; having an in-branching, having minimum degree at least one, having minimum semi-degree at least one, being acyclic, and being complete. Some of the properties above imply others, e.g. every strongly connected digraph has an out-branching and an in-branching, is connected and has minimum semi-degree at least 1 (in fact 2). Also a digraph  $D$  has an out-branching if and only the digraph that we obtain by reversing all arcs has an in-branching. Bang-Jensen et al. [2, 3] studied the complexity of these problems when  $k_1$  and  $k_2$  are considered fixed constants. However, in this paper, we treat  $k_1$  and  $k_2$  as part of the input, and from this viewpoint, the analysis given

$D[V_1] \setminus D[V_2]$	Strong	Connected	$B^+$	$B^-$	$\delta \geq 1$	$\delta^0 \geq 1$	Acyclic	Complete
Strong	NPc	NPc	NPc	NPc	NPc	NPc	$W[1]^3$	$W[1]^4$
Connected	NPc	FPT	$W[1]^1$	$W[1]^1$	FPT	NPc	$W[1]^3$	$W[1]^4$
$B^+$	NPc	$W[1]^2$	FPT	NPc	$W[1]^1$	NPc	$W[1]^3$	$W[1]^4$
$B^-$	NPc	$W[1]^2$	NPc	FPT	$W[1]^1$	NPc	$W[1]^3$	$W[1]^4$
$\delta \geq 1$	NPc	FPT	$W[1]^2$	$W[1]^2$	FPT	NPc	$W[1]^3$	$W[1]^4$
$\delta^0 \geq 1$	NPc	NPc	NPc	NPc	NPc	NPc	$W[1]^3$	$W[1]^4$
Acyclic	$W[1]^3$	$W[1]^3$	$W[1]^3$	$W[1]^3$	$W[1]^3$	$W[1]^3$	NPc	P
Complete	$W[1]^4$	$W[1]^4$	$W[1]^4$	$W[1]^4$	$W[1]^4$	$W[1]^4$	P	P

Table 1: The parameterized complexity of the partition problems where the structural properties for the induced subdigraphs on  $V_1$  and  $V_2$  are given by the rows and columns respectively. **Properties:** Strong: strongly connected; Connected: being a connected graph;  $B^+$ : having an out-branching;  $B^-$ : having an in-branching;  $\delta \geq 1$ : having minimum degree at least one;  $\delta^0 \geq 1$ : having minimum semi-degree at least one; Acyclic: being and acyclic digraph; and Complete: being a complete digraph. **Complexities:** P: polynomial time solvable; FPT: Fixed-Parameter Tractable;  $W[1]^1$ :  $W[1]$ -hard for  $k_1 > 3$ ;  $W[1]^2$ :  $W[1]$ -hard for  $k_2 > 3$ ,  $W[1]^3$ :  $W[1]$ -hard for  $k_1 \geq 2$ ;  $W[1]^4$ :  $W[1]$ -hard for  $k_1 \geq 3$ ; NPc: NP-complete as determined in [2, 3].

in [2, 3] only divide the problems into being NP-complete or in  $XP^1$ . We consider the 23 problems which are not NP-complete and give a more fine-grained view of their complexity. We show that 2 are polynomial time solvable, 5 are Fixed-Parameter Tractable, and 16 are  $W[1]$ -hard. The proofs can be found in Sections 3, 4 and 5 respectively, and Table 1 gives an overview of the results. In Table 1 the problems above the diagonal and the problems below the diagonal are symmetrical, so for each such pair, e.g. (Strong,  $B^+$ ) and ( $B^+$ , Strong), we will only prove the complexity of one of them.

## 2 Notation and definitions

We start providing some terminology and complexity definitions.

### 2.1 Graph Theory.

The notation we use is consistent with that of [4]. For a digraph  $D = (V, A)$  we say that two vertices  $u$  and  $v$  are **adjacent** if at least one of the arcs  $uv, vu$  is in  $A$ . Furthermore, if  $D$  contains both of the arcs  $uv, vu$  then these induce a **2-cycle** between  $u$  and  $v$ . For a set of vertices  $U \subseteq V$ , the **subdigraph induced by  $U$** , denoted  $D[U]$ , is the digraph obtained from  $D$  by deleting all vertices  $V \setminus U$  and all arcs adjacent to those vertices. A  $(u, v)$ -path is a directed path from  $u \in V$  to  $v \in V$  and the digraph  $D$  is **strongly connected** if it contains a  $(u, v)$ -path for all ordered pairs of vertices  $u, v \in V$ . The **underlying graph  $U(D)$**  is the graph obtained from  $D$  by replacing every 2-cycle with one arc and then suppressing all the directions of the arcs. A digraph  $D$  is **connected** if  $U(D)$  is connected and the **connected components** of  $D$  are the connected components of  $U(D)$ . For a vertex  $u \in V$ , we denote by  $N(u)$  its **neighbours**, that is, the set of vertices that are adjacent to  $u$ . The **out-degree  $d^+(u)$**  is the number of arcs going out of  $u$ . Similarly, the **in-degree  $d^-(u)$**  is the number of arcs going into  $u$ . The **minimum degree  $\delta(D)$**  of a digraph  $D$  is the minimum of  $d^+(u) + d^-(u)$  over all  $u \in V$ , and the **minimum semi-degree  $\delta^0(D)$**  is the minimum of  $d^+(u)$  and  $d^-(u)$  over all  $u \in V$ . Sometimes we write  $d_D^+(v)$  to specify that we are talking about the out-degree in the subdigraph  $D'$  of  $D$ .

An **out-tree** rooted in  $s$  is a connected digraph  $T$  such that  $d_T^-(s) = 0$  and  $d_T^-(u) = 1$  for all  $u \in V(T) \setminus \{s\}$ . Similarly, an **in-tree** rooted in  $t$  is a connected digraph  $T$  such that  $d_T^+(t) = 0$  and

<sup>1</sup>A parameterized problem  $Q$  is in the complexity class  $XP$  if there exist computable functions  $f, g$  so that  $Q$  can be solved in time  $O(f(k)n^{g(k)})$  where  $n$  is the size of the input.

$d_T^+(u) = 1$  for all  $u \in V(T) \setminus \{t\}$ . An **out-branching (in-branching)** of a digraph  $D$  is an out-tree (in-tree) such that  $V(T) = V(D)$ .

A digraph  $D$  is **acyclic** if it contains no induced directed cycles, and it is **complete** if for every pair of vertices  $u, v \in V$  induce a 2-cycle  $uvu$ . The **complement** of  $D$  is the digraph  $\bar{D} = (V, A')$  where  $uv \in A'$  if and only if  $uv \notin A$ . A graph  $G = (V, E)$  is **bipartite** if there is a 2-partition  $(V_1, V_2)$  of  $V$  such that every edge in  $E$  has one end in  $V_1$  and the other in  $V_2$ . A graph  $G = (V, E)$  is a **star** if  $G$  is connected and there exists a vertex  $v \in V$  such that  $G[V \setminus \{v\}]$  is an independent set. We call  $v$  the **center** of the star and denote the star on  $p$  vertices with center  $v$  by  $S_{v,p}$ .

## 2.2 Parameterized Complexity.

For a more extensive explanation we refer to [5, 6]. An instance of a parameterized problem  $\Pi$  consists of a main part  $I$  and one or more parameters  $k_1, \dots, k_p$  and is denoted by  $(I, k_1, \dots, k_p)$ . An instance  $(I, k_1, \dots, k_p)$  of a parameterized decision problem  $\Pi$  is a **yes-instance** if there is an affirmative answer and a **no-instance** otherwise. A parameterized decision problem  $\Pi$  is **fixed-parameter tractable (FPT)** if it admits an algorithm  $\mathcal{A}$  which can decide, for every instance  $(I, k_1, \dots, k_p) \in \Pi$ , if it is a yes-instance or no-instance in time  $O(f(k_1, \dots, k_p)|I|^c)$ , where  $c$  is a constant and  $f$  is a computable function. Under common complexity-theoretical assumptions, it is possible to establish that a problem  $\Pi$  is not FPT by giving a **parameterized reduction** from a  $W[1]$ -complete problem  $\Pi'$  (for the definition of  $W[1]$  see [5]). A parameterized reduction from  $\Pi'$  to  $\Pi$  is a many-to-one reduction where given an  $(I', k'_1, \dots, k'_p) \in \Pi'$  then the output is an instance  $(I, k_1, \dots, k_p) \in \Pi$  such that

1.  $(I', k'_1, \dots, k'_p)$  is a yes-instance if and only if  $(I, k_1, \dots, k_p)$  is a yes-instance,
2.  $k_i < g_i(k'_i)$  for all  $i \in [1, p]$  where  $g_i$  is some computable function,
3. the complexity of the reduction is  $O(f(k_1, \dots, k_p)|I|^c)$  for some constant  $c$  and a computable function  $f$ .

## 2.3 Problem definitions.

Below  $k_1$  and  $k_2$  always denote two integers such that  $k_1 + k_2 \leq |V|$  and  $k_1, k_2 \geq 1$ . Given structural properties  $\mathcal{E}_1$  and  $\mathcal{E}_2$  we denote by  $(\mathcal{E}_1, \mathcal{E}_2)$ -**partition** the problem of deciding for a given input consisting of a digraph  $D$  and integers  $k_1, k_2$ , whether there exists a vertex partition  $V_1 \subseteq V$  and  $V_2 = V \setminus V_1$  such that  $|V_1| \geq k_1$ ,  $|V_2| \geq k_2$ ,  $D[V_1]$  has the property  $\mathcal{E}_1$ , and  $D[V_2]$  has the property  $\mathcal{E}_2$ .

## 3 The polynomial time solvable cases

In this subsection we show that *(Complete, Acyclic)*- and *(Complete, Complete)*-partition can be solved in polynomial time. The algorithms will take as input a digraph  $D = (V, A)$  and two integers  $k_1$  and  $k_2$ . They will either find a 2-partition  $(V_1, V_2)$  of  $V$  such that  $|V_i| \geq k_i$ , for  $i = 1, 2$  and  $D[V_1]$  is complete and  $D[V_2]$  is acyclic, respectively, complete, or decide that no such partition exists.

Recall that an undirected graph  $G = (V, E)$  is a **split graph** if there exists a 2-partition  $(V_1, V_2)$  (called a **split-partition**) such that  $G[V_1]$  has no edges ( $V_1$  is an independent set) and  $G[V_2]$  is complete.

For an given digraph  $D = (V, A)$  we denote by  $G_D = (V, E)$  the undirected graph whose vertex set is the same as that of  $D$  and the edge set  $E$  consists of those pairs of vertices which do not induce a 2-cycle in  $D$  (so they have at most one arc between them). Then the following is clear.

**Proposition 1.** *Let  $G_D$  be defined from  $D$  as above.*

- $D$  has a *(Complete, Complete)*-partition  $(V_1, V_2)$  with  $|V_i| \geq k_i$ ,  $i = 1, 2$  if and only if  $G_D$  is bipartite and has a bipartition  $(V_1, V_2)$  with  $|V_i| \geq k_i$   $i = 1, 2$ .
- If  $D$  has a *(Complete, Acyclic)*-partition  $(V_1, V_2)$ , then  $(V_1, V_2)$  is a split partition of  $G_D$ .

**Corollary 2.** *The *(Complete, Complete)*-partition can be solved in polynomial time.*

**Proof.** Let  $(D = (V, A), k_1, k_2)$  be an instance of  $(Complete, Complete)$ -partition. By Proposition 1 we just have to check whether  $G_D$  has a bipartition  $(V_1, V_2)$  with  $|V_i| \geq k_i$  for  $i = 1, 2$ . If  $G_D$  is connected this is trivial to check so assume that  $G_D$  has  $r \geq 2$  connected components and let  $(a_1, b_1), \dots, (a_r, b_r)$  denote the pairs of sizes of the partite sets in each connected component. As  $\sum_{i=1}^r (a_i + b_i) = |V|$  it is easy to check in polynomial time, using dynamic programming, whether there exists a bipartition  $(V_1, V_2)$  of  $G_D$  with  $|V_i| \geq k_i$  for  $i = 1, 2$ .  $\square$

**Corollary 3.** *The  $(Complete, Acyclic)$ -partition can be solved in polynomial time.*

**Proof.** Let  $(D = (V, A), k_1, k_2)$  be an instance of  $(Complete, Acyclic)$ -partition and let  $G_D$  be as defined above. It is well-known that one can check in polynomial time whether a graph is a split graph. For convenience we provide a simple reduction to 2-SAT: Given  $G = (V, E)$  let  $\mathcal{F}$  be the 2-SAT instance which has variables  $\{x_i | v_i \in V\}$  and a clause  $(x_i \vee x_j)$  for each edge  $v_i v_j \in E$  as well as a clause  $(\bar{x}_a \vee \bar{x}_b)$  for each pair  $v_a, v_b$  such that  $v_a v_b \notin E$ . It is easy to check that  $\mathcal{F}$  is satisfiable if and only if  $G$  is a split graph by letting the true variables correspond to the vertices in the complete part of a split-partition. As 2-SAT is solvable in polynomial time [4, Section 17.5] we get that split graphs can be recognized in polynomial time. Observe that a split graph may have several split partitions but for any pair of such partitions  $(S_1, S_2)$  and  $(S'_1, S'_2)$  the size of  $S_i$  and  $S'_i$  differ by at most one for  $i = 1, 2$ . Hence we can check whether  $D$  has a  $(Complete, Acyclic)$ -partition  $(V_1, V_2)$  with  $|V_i| \geq k_i$  for  $i = 1, 2$  as follows: If  $G_D$  is not a split graph, then by Proposition 1,  $D$  has no  $(Complete, Acyclic)$ -partition so assume that  $G_D$  is a split graph with split partition  $(S_1, S_2)$ . If  $|S_i| < k_i - 1$  for  $i = 1$  or  $i = 2$ , then  $(D = (V, A), k_1, k_2)$  is a no-instance so assume  $|S_i| \geq k_i - 1$  for  $i = 1, 2$ . If  $|S_i| \geq k_i$  for  $i = 1, 2$  and  $D[S_2]$  is acyclic, then  $(D = (V, A), k_1, k_2)$  is a yes-instance. If not then there are at most  $O(n^2)$  other partitions to check and each of these can be obtained by moving one vertex from  $S_i$  to  $S_{3-i}$  for  $i = 1$  or  $i = 2$  or both. For each such new partition we can verify in polynomial time whether this is a good partition.  $\square$

## 4 The FPT-time solvable cases

In this subsection, we first argue that the FPT-time solvable partition problems are not solvable in polynomial time (unless  $NP = P$ ) and then we give FPT-algorithms to solve them. We show that the partition problems are  $NP$ -complete by reducing from the Hitting Set problem which is a well-known  $NP$ -complete problem. Here we use the following formulation of the problem: Given a bipartite graph  $G = (U_1, U_2, E)$  and an integer  $k \geq 1$  the problem is to decide if there exists a set of vertices  $H \subseteq U_2$  such that  $|H| \leq k$  and all vertices in  $U_1$  have a neighbour in  $H$ .

**Lemma 4.** *The  $(Connected, Connected)$ -,  $(B^+, B^+)$ -,  $(B^-, B^-)$ -,  $(Connected, \delta \geq 1)$ -,  $(\delta \geq 1, \delta \geq 1)$ -partition problems are all  $NP$ -complete.*

**Proof.** Let  $\mathcal{E} = \{Connected, B^+, B^-, \delta \geq 1\}$ , let  $(G = (U_1, U_2, E), k)$  be an instance of Hitting Set, and let  $n_i = |U_i|$ ,  $i = 1, 2$ .

**Construction:**

Let  $x = |U_1| + |U_2| = n_1 + n_2$  and let  $Z_a$  be the digraph we obtain from the star  $S_{a, x+1}$  on  $x + 1$  vertices with center  $a$  by replacing each edge by a directed 2-cycle. Construct  $D = (V, A)$  as follows. Start by setting  $V = U_1 \cup U_2$  and  $A = \emptyset$ . Now add arcs and new vertices as follows: For each edge  $(u, v) \in E$  add a directed 2-cycle between  $u$  and  $v$ . Furthermore, for every pair of vertices  $u, v \in U_2$  add a directed 2-cycle between  $u$  and  $v$ . Finally each  $u \in U_2$  identify  $u$  with the vertex  $a$  in a private copy of  $Z_a$ . Let  $k_1 = |U_1| + k \cdot (x + 1)$  and  $k_2 = (|U_2| - k)(x + 1)$ , that is,  $k_1 + k_2 = |V(D)|$ .

**Proof of correctness:**

We now argue that  $(G, k)$  is a yes-instance of Hitting Set if and only if  $(D, k_1, k_2)$  is a yes-instance of  $(\varepsilon_1, \varepsilon_2)$ -partition for every choice of  $\varepsilon_1, \varepsilon_2 \in \mathcal{E}$ .

Suppose first that  $(G, k)$  is a yes-instance of Hitting Set. It means that there exists a solution  $H \subseteq U_2$  such that all the vertices in  $U_1$  are covered by  $H$ . Partition the vertices of  $D$  into  $V_1$  and  $V_2 = V(D) \setminus V_1$ , such that,  $V_1$  contains exactly the vertices  $U_1$  and for each  $u \in H$  all the vertices of  $Z_u$ . Both  $D[V_1]$  and  $D[V_2]$  are strongly connected since every pair of vertices in  $U_2$  is connected with

a directed 2-cycle and for every  $(u, v) \in E$  there is also a directed 2-cycle between  $u$  and  $v$ . Since  $D[V_1]$  and  $D[V_2]$  are strongly connected they also have every property in  $\mathcal{E}$ . Furthermore,  $|V_1| = k_1$  and  $|V_2| = k_2$ .

To prove the other direction, assume that  $(D, k_1, k_2)$  is a yes-instance of  $(\varepsilon_1, \varepsilon_2)$ -partition for  $\varepsilon_1, \varepsilon_2 \in \mathcal{E}$ , that is, there exists a partition  $V_1$  and  $V_2$  of the vertices  $D$  such that  $|V_i| \geq k_i$  and  $D[V_i]$  satisfies  $\varepsilon_i$  for  $i = 1, 2$ . Note that this implies that  $|V_i| = k_i$  for  $i = 1, 2$  as  $k_1 + k_2 = |V(D)|$ . For any vertex  $u \in U_2$  the vertices of  $Z_u - u$  only has arcs going to and coming from  $u$ . It means that, for each  $u \in U_2$  all vertices of  $Z_u$  must be in the same partition  $V_i$  for  $i \in [1, 2]$  since otherwise there would be an isolated vertex in one of the partitions  $V_i$  but then  $D[V_i]$  would not have any of the properties in  $\mathcal{E}$ .

Recall that  $V_1$  is of size  $|U_1| + k \cdot (x + 1)$ . It means that  $V_1$  must contain exactly  $k$  vertices from  $U_2$ , because if it contained  $r > k$  or  $t < k$  then the size of  $V_1$  would be at least  $r(x + 1) \geq x + k(x + 1) = |U_1| + |U_2| + k(x + 1)$  or at most  $|U_1| + t(x + 1) < |U_1| + x + t(x + 1) \leq |U_1| + k(x + 1)$  but this would contradict that  $|V_1| = |U_1| + k \cdot (x + 1)$ . From this it can also be concluded that  $V_1$  must contain all the vertices in  $U_1$ . Moreover,  $U_1$  in itself forms an independent set in  $D$ . So for  $D[V_1]$  to have the property  $\varepsilon_1$  each  $v \in U_1$  must be incident with an arc in  $D[V_1]$ . Hence  $H = V_1 \cap U_2$  is a hitting set in  $G$ .  $\square$

**Theorem 5.** *The (Connected, Connected)-,  $(B^+, B^+)$ -,  $(B^-, B^-)$ -, (Connected,  $\delta \geq 1$ )-,  $(\delta \geq 1, \delta \geq 1)$ -partition problems are all FPT.*

As we remarked in the introduction  $(D, k_1, k_2)$  is a yes-instance of  $(B^-, B^-)$ -partition if and only if  $(\overleftarrow{D}, k_1, k_2)$  is a yes-instance of  $(B^+, B^+)$ -partition, where  $\overleftarrow{D}$  is the digraph (called the **converse** of  $D$ ) which we obtain from  $D$  by reversing all arcs. Furthermore  $(D, k_1, k_2)$  is a yes-instance of (Connected, Connected)-partition if and only if  $(S(D), k_1, k_2)$  is a yes-instance of  $(B^+, B^+)$ -partition, where  $S(D)$  is the symmetric digraph on the same vertex set as  $D$  and with a 2-cycle between  $x, y \in V(D)$  precisely when there is at least one arc between  $x$  and  $y$  in  $D$ . Hence it suffices to show that the  $(B^+, B^+)$ -partition problem, the (Connected,  $\delta \geq 1$ )-partition problem and the  $(\delta \geq 1, \delta \geq 1)$ -partition problem are all FPT. The following three lemmas cover these cases. The three proofs follow the same scheme. For each of the problems we may assume that  $n = |V(D)| \geq 2(k_1 + k_2)$  since otherwise we can just try all possible 2-partitions  $V_1, V_2$  with  $|V_i| \geq k_i$  for  $i = 1, 2$ .

**Lemma 6.** *The  $(B^+, B^+)$ -partition problem is FPT.*

**Proof.**

Let  $(D, k_1, k_2)$  be an instance of the  $(B^+, B^+)$ -partition problem with  $n = |V(D)| \geq 2(k_1 + k_2)$ . There are only  $O(n^2)$  possible choices for the roots of the two out-branchings in a  $(B^+, B^+)$ -partition so clearly it suffices to show that for a given choice  $s_1, s_2$  of such vertices we can check in FPT-time whether there is a solution where they are the roots of the two out-branchings.

First we fix two vertices  $s_1, s_2 \in V$  and obtain an initial partition  $(V_1^0, V_2^0)$  by letting  $V_1^0$  consist of those vertices that can be reached from  $s_1$  in  $D - s_2$  and  $V_2^0 = V - V_1^0$ . If  $|V_1^0| < k_1$  or  $s_2$  cannot reach all vertices in  $V_2^0$ , there is no solution for the pair  $s_1, s_2$  and we may proceed to the next pair. So assume below that  $|V_1^0| \geq k_1$  and that  $s_2$  can reach all vertices in  $V_2^0$ . If we also have  $|V_2^0| \geq k_2$  we are done so we may also assume that this is not the case.

Below we denote by  $V_1^i, V_2^i$  the current partition. The algorithm will obtain the next partition  $V_1^{i+1}, V_2^{i+1}$  by moving one or more vertices from  $V_1^i$  to  $V_2^i$ . For each vertex  $u \in V_1^i - s_1$  we denote by  $X_u^i$  the set of vertices that are not reachable from  $s_1$  in  $D[V_1^i - u]$ . Thus if we move  $u$  to  $V_2^i$  we must also move all of  $X_u^i$  to  $V_2^i$  to maintain the invariant that  $s_1$  can reach all vertices in its part of the partition.

Now the algorithm can easily be stated: as long as we have  $|V_2^i| < k_2$  try to find an arc  $vu$  from  $V_2^i$  to  $V_1^i - s_1$  so that we can move  $X_u^i \cup \{u\}$  from  $V_1^i$  to  $V_2^i$  and still have at least  $k_1$  vertices in  $V_1^i - (X_u^i \cup \{u\})$ . If we find such an arc, the new set  $V_2^{i+1}$  will have at least one more vertex than  $V_2^i$  so after at most  $k_2 - 1$  steps we have either obtained a solution or we have reached a partition  $V_1^j, V_2^j$  for which one of the following holds:

1. There is no arc from  $V_2^j$  to  $V_1^j - s_1$  or

2. for every vertex  $u \in V_1^j - s_1$  such that there is an arc  $vu$  from  $V_2^j$  to  $u$  we have that  $|V_1^j - (X_u^j \cup \{u\})| < k_1$

In the first case it is easy to see that there is no solution  $V_1, V_2$  with  $s_i \in V_i$  so assume that we are in the second case. Choose  $u \in V_1^j - s_1$  such that there is an arc from  $V_2^j$  to  $u$  but there is no arc from  $V_2^j$  to  $X_u^j$ . This is easy to do in polynomial time as  $X_w^j \subset X_u^j$  for every  $w \in X_u^j$ . Since 2. holds we have  $|V_1^j - (X_u^j \cup \{u\})| + |V_2^j| \leq k_1 + k_2 - 2$  and since all paths from  $s_1, s_2$  to a vertex in  $X_u^j$  must go through  $u$ , all of these vertices must belong to the same set in every good 2-partition. This means that we can check for a solution by putting  $s_i$  in  $V_i$ ,  $i = 1, 2$ , putting  $X_u^j \cup \{u\}$  together with either  $s_1$  or  $s_2$  and then trying all possible distributions of the remaining at most  $k_1 + k_2 - 2$  vertices in the two sets. This is clearly doable in time  $O(2^{k_1+k_2}n^2)$ .  $\square$

In the two remaining problems the direction of the arcs play no role, so we consider the underlying undirected graph  $G = (V, E)$  of the input digraph  $D = (V, A)$ , that is,  $G$  has the same vertex set as  $D$  and  $xy$  is an edge in  $E$  if and only if there is at least one arc between  $x$  and  $y$  in  $D$ . Clearly we can assume that  $G$  has no isolated vertices as otherwise  $(D, k_1, k_2)$  is a no-instance.

**Lemma 7.** *The (Connected,  $\delta \geq 1$ )-partition problem is FPT.*

**Proof.** Let  $(D, k_1, k_2)$  be an instance of the (Connected,  $\delta \geq 1$ )-partition problem. It is easy to see that we may check for a solution by at most  $|E|$  applications of a procedure which fixes one edge  $xy$  and then checks in FPT-time whether there is a 2-partition  $V_1, V_2$  where  $x, y$  are both in  $V_2$ ,  $G[V_1]$  is connected and  $G[V_2]$  has no isolated vertex.

**Remark:** Note that if  $G$  contains an induced star  $S_{v,p}$  such that  $|V| - p < k_1 + k_2$  then all vertices of  $S_{v,p}$  must belong to the same part of the partition and hence if we have found such a star, we can search for a solution as we did above by trying all possible distributions of the vertices of  $V - V(S_{v,p})$  to the two sets  $V_1, V_2$ . Note that we are not going to search for such a star directly as that is not FPT.

We first construct  $V_2^0$  starting with the vertices  $x, y$  and then adding all vertices that must be added to  $V_2^0$  because all their neighbours are already added. If every connected component of  $G - V_2^0$  has size less than  $k_1$ , then there is no solution  $V_1, V_2$  with  $x, y \in V_2$  so assume below that at least one connected component  $C$  of  $G - V_2^0$  has size at least  $k_1$ . If there exists a component  $C$  where  $|C| \geq k_1$  and  $|V - C| \geq k_2$ , then  $V_1 = C$  and  $V_2 = V - C$  is a solution. Hence we can assume that for every connected component  $C$  of  $G - V_2^0$  with  $|C| \geq k_1$  we have  $|V - C| < k_2$ . Let  $C$  be an arbitrary component of  $G - V_2^0$  such that  $|C| \geq k_1$  and let  $V_1^1 = C$  and  $V_2^1 = V - C$ . Below we denote by  $V_1^i, V_2^i$  the current partition which will always satisfy that  $G[V_1^i]$  is connected,  $|V_1^i| \geq k_1$  and  $G[V_2^i]$  has no isolated vertices.

Note that if  $u$  and  $v$  are joined by an edge in  $G[V_1^i]$  and  $G[V_1^i - \{u, v\}]$  is connected, then we may move them to  $V_2^i$  (recall that  $n \geq 2(k_1 + k_2)$ ). Furthermore if there exists a  $u \in V_1^i$  and  $v \in V_2^i$  where  $u$  and  $v$  are joined by an edge in  $G$  and  $G[V_1^i - \{u\}]$  is connected, then we may move  $u$  to  $V_2^i$ . Performing such moves as long as we can, we either obtain a good 2-partition  $V_1^j, V_2^j$ , in which case we are done, or at some iteration  $j$  there are no such edges or vertices remaining and we still have  $|V_2^j| < k_2$ .

Now consider a spanning tree  $T$  of  $G[V_1^j]$  and root it in some vertex  $s$ . Since the process above stopped, the leaves of  $T$  form an independent set. Thus if the induced star  $S_{v,q}$  rooted at any vertex  $v$  at height 1 in  $T$  has more than  $n - k_1 - k_2$  vertices we can solve the problem as described in the remark above. So we may assume that  $|V_1^j| - q \geq k_1$  and hence we can move all vertices of  $S_{v,q}$  to  $V_2^j$ .

By the assumption that  $n \geq 2(k_1 + k_2)$ , this process of moving edges, vertices, and stars will either end in a solution or by identifying an induced star  $S_{v,q}$  with more than  $n - k_1 - k_2$  vertices. In the latter case we can solve the problem as described in the remark.  $\square$

**Lemma 8.** *The ( $\delta \geq 1, \delta \geq 1$ )-partition problem is FPT.*



**Proof.**

Let  $(D, k_1, k_2)$  be an instance of the  $(\delta \geq 1, \delta \geq 1)$ -partition problem. The algorithm uses the algorithm above for the  $(Connected, \delta \geq 1)$ -partition problem. Follow that algorithm until one of the following situations occur:

- (a) The algorithm stops because all connected components of  $G[V - V_2^0]$  have size less than  $k_1$ .
- (b) The algorithm has identified an induced star  $S_{v,q}$  with  $q > n - k_1 - k_2$ .
- (c) The algorithm has found a partition  $V_1, V_2$  such that  $|V_i| \geq k_i$ ,  $G[V_1]$  is connected and  $G[V_2]$  has no isolated vertices.

If (c) occurs we are done as  $G[V_i]$  has no isolated vertices for  $i = 1, 2$ . If (b) occurs we can solve the problem in time  $O(2^{k_1+k_2}n^2)$  since all vertices of  $S_v$  must belong to the same set  $V_i$  in any solution. Hence we may assume that we are in situation (a). Let  $C_1, C_2, \dots, C_r$  be the connected components of  $G - V_2^0$  ordered so that  $|C_i| \geq |C_j|$  when  $i < j$ . Let  $1 < t$  be the smallest index so that  $|C_1| + \dots + |C_t| \geq k_1$ . Since  $n \geq 2(k_1 + k_2)$  it is easy to see that  $V_1 = C_1 \cup \dots \cup C_t$  and  $V_2 = V - V_1$  is a solution showing that  $(D, k_1, k_2)$  is a yes-instance. Here we used that  $G - V_2^0$  has no trivial connected component. □

## 5 The $W[1]$ -hard partition problems

In this subsection, we give a set of reductions to show that the remaining 2-partition problems from Table 1 are  $W[1]$ -hard. All the reductions are from INDEPENDENT SET which is the problem of deciding, for a given graph  $G = (V, E)$  and a natural number  $k$ , whether  $G$  has an independent set  $S \subseteq V$  such that  $|S| \geq k$ . This problem is  $W[1]$ -hard [5].

**Lemma 9.** *The  $(B^+, \delta \geq 1)$ -,  $(B^+, Connected)$ -,  $(B^-, \delta \geq 1)$ -, and  $(B^-, Connected)$ -partition problems are all  $W[1]$ -hard when  $k_1 > 3$ .*

**Proof.** It suffices to show that  $(B^+, Connected)$ - and  $(B^+, \delta \geq 1)$ -partition are  $W[1]$ -hard since the fact that  $(B^-, Connected)$ - and  $(B^-, \delta \geq 1)$ -partition are  $W[1]$ -hard follows by considering the converse digraph. Let  $\mathcal{E} = \{Connected, \delta \geq 1\}$ .

Given an instance  $(G = (V, E), k)$  of INDEPENDENT SET with  $k \geq 3$  we make a digraph  $D$  as follows: Set  $V(D) = V(G) \cup \{v_e | e \in E\} \cup \{v_{u,y} | u \in V\} \cup \{x, y\}$  and let  $A(D) = \{xv | v \in V\} \cup \bigcup_{uw \in E} \{v_{uw}u, v_{uw}w\} \cup \bigcup_{u \in V} \{v_{u,y}y, v_{u,y}u\}$ . It is easy to check that every out-tree on at least 4 vertices in  $D$  must be rooted at  $x$  ( $D$  contains no directed path of length 2 and all vertices of  $V(D) - x$  have out-degree at most 2). Let  $k_1 = k + 1$  and  $k_2 = 1$ .

We now argue that  $(G, k)$  is a yes-instance of INDEPENDENT SET if and only if  $(D, k_1, k_2)$  is a yes-instance of  $(B^+, \varepsilon)$ -partition for all  $\varepsilon \in \mathcal{E}$ .

Suppose first that there exists an independent set  $S \subseteq V(G)$  of size at least  $k$  in  $G$ . Partition the vertices of  $D$  into  $V_1 = S \cup \{x\}$  and  $V_2 = V(D) \setminus V_1$ . Then  $V_1$  is of size  $k + 1$ , and  $D[V_1]$  contains an out-branching rooted in  $x$  since for every vertex  $v \in S$  there is an arc  $(x, v) \in A$ . Observe that  $V_2$  contains all the vertices  $\{v_e | e \in E\}$  and  $\{v_{u,y} | u \in V\}$ , together with  $V \setminus S$  and  $y$ . Each  $v_{u,y}$  has an arc to  $y$  and an arc to the vertex  $u \in V$ . Hence  $D[(V \setminus S) \cup \{v_{u,y} | u \in V\} \cup \{y\}]$  is connected. Moreover, since  $S$  is an independent set, for every vertex  $v_{uw}$  at least one of  $u, w$  will be in  $V_2$ . Hence  $D[V_2]$  is connected and has  $\delta \geq 1$ .

In the converse direction, suppose that  $(D, k_1, k_2)$  is a yes-instance of  $(B^+, \varepsilon)$ -partition for  $\varepsilon \in \mathcal{E}$ . Then there exists a 2-partition  $V_1, V_2$  of  $V(D)$ , satisfying that  $|V_1| > 3$ ,  $D[V_1]$  contains an out-branching, and  $D[V_2]$  has the property  $\varepsilon$ . By the remark above,  $x$  must be the root of the out-branching in  $D[V_1]$ . Furthermore, all vertices of  $\{v_e | e \in E\} \cup \{v_{u,y} | u \in V\} \cup \{y\}$  must belong to  $V_2$  as these cannot be reached from  $x$ . Furthermore, as no vertex in  $V_2$  is isolated, for every edge  $uw \in E$  at least one of  $u, w$  must belong to  $V_2$  (to give  $v_{uw}$  a neighbour). Thus  $V_1 \setminus \{x\}$  is an independent set of size at least  $k$  in  $G$ . □

**Lemma 10.** For every  $\epsilon \in \mathcal{E} = \{\text{Strong, Connected, } B^+, B^-, \delta \geq 1, \delta^0 \geq 1\}$  the (Acyclic,  $\epsilon$ )-partition problem is  $W[1]$ -hard when  $k_1 \geq 2$ .

**Proof.** Let  $(G = (V, E), k)$  be an instance of INDEPENDENT SET. Construct a digraph  $D = (V, A)$  by adding two new vertices  $x, x'$ , replacing each edge  $uv \in E$  by a 2-cycle, adding the 2-cycles  $xvx, x'vx'$  for each  $v \in V$  and finally adding the 2-cycle  $xx'x$ . Clearly an induced acyclic subdigraph of  $D$  of size at least 2 corresponds to an independent set of  $G$  and conversely. Moreover, no acyclic subdigraph  $H$  of  $D$  with at least 2 vertices can contain  $x$  or  $x'$  since they form 2-cycles with all other vertices. Hence, for every acyclic subdigraph  $H$  of  $D$  the digraph  $D' = D - V(H)$  is strongly connected since it contains  $x$  which has directed 2-cycles to every  $u \in V(D)$ . Furthermore,  $|D'| \geq 2$  since  $x, x' \in V(D')$  and therefore  $D'$  will have each of the properties  $\epsilon \in \mathcal{E}$ . It means that  $(G, k)$  is a yes-instance of INDEPENDENT SET if and only if  $(D, k_1 = k, 2)$  is a yes-instance of (Acyclic,  $\epsilon$ )-partition for  $\epsilon \in \mathcal{E}$ .  $\square$

**Lemma 11.** For every  $\epsilon \in \mathcal{E} = \{\text{Strong, Connected, } B^+, B^-, \delta \geq 1, \delta^0 \geq 1\}$  the (Complete,  $\epsilon$ )-partition problem is  $W[1]$ -hard when  $k_1 \geq 3$ .

**Proof.** Let  $(G = (V, E), k)$  be an instance of Independent Set with  $k \geq 3$ . Construct a digraph  $D = (V, A)$  as follows. Set  $V(D) = V \cup V' \cup \{x\}$ , where  $V' = \{v' | v \in V\}$  is a copy of  $V$ , and let  $A(D)$  consist of the following arcs: a 2-cycle between  $v$  and its copy  $v'$  and a 2-cycle between  $v'$  and  $x$  for every  $v \in V$  and for each pair  $u, v \in V$  such that  $(u, v) \notin E$  insert a directed 2-cycle between  $u$  and  $v$ .

We now argue that  $(G, k)$  is a yes-instance of Independent Set if and only if  $(D, k_1 = k, 1)$  is a yes-instance of (Complete,  $\epsilon$ )-partition for  $\epsilon \in \mathcal{E}$ . Suppose first that  $(G, k)$  is a yes-instance of INDEPENDENT SET, and let  $S$  be an independent set of size  $k$ . Partition the vertices of  $D$  into  $V_1 = S$  and  $V_2 = V(D) \setminus V_1$ . Observe that  $D[V_1]$  is a complete digraph since, by construction, there are directed 2-cycles between any pair of vertices  $u, v \in V(G)$  where  $(u, v) \notin E$ . Furthermore, for every vertex in  $V_2 = V(D) \setminus V_1$  there is an path to and from  $x$ , implying that  $D[V_2]$  is strongly connected. Hence  $D[V_2]$  has every property in  $\mathcal{E}$ . For the converse direction, suppose that  $(D, k_1, 1)$  is a yes-instance of (Complete,  $\epsilon$ )-partition for  $\epsilon \in \mathcal{E}$ . Let  $V_1$  and  $V_2 = V(D) \setminus V_1$  form a 2-partition fulfilling the properties for (Complete,  $\epsilon$ )-partition. Each vertex of  $V'$  has exactly one directed 2-cycle to  $V$  and one directed 2-cycle to  $x$ . Furthermore,  $x$  is not adjacent to any vertex in  $V$ . It means that  $x$  and every vertex of  $V'$  must be in  $V_2$  since  $|V_1| \geq k_1 > 2$ .  $V_1$  can not contain two vertices  $u, v \in V(G)$  which are adjacent in  $G$  since they do not have a directed 2-cycle between them in  $D$ . It means that  $V_1$  contains vertices which forms an independent set in  $G$ . Hence  $S = V_1$  is an independent set of size at least  $k$  in  $G$ .  $\square$

## 6 Conclusion

In this paper, we have further refined the results obtained in the papers [2, 3] by studying the parameterized complexity of 2-partition problems of digraphs where each of the two partitions had one of two structural properties. The problems have been parameterized by the size of the partitions, and out of the 23 problems studied, we found 2 were polynomial time solvable, 5 were fixed-parameter tractable, and 16 were  $W[1]$ -hard.

The parameterized complexity status of some of the other problems that were studied in [2, 3] remain open. The most important among these are the following.

**Problem 12.** What is the parameterized complexity of  $(B^+, \delta^- \geq 1)$ -partition and  $(\delta^- \geq 1, \delta^- \geq 1)$ -partition?

Both problems are in XP as shown in [2].

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