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ON ENDOMORPHISMS OF THE CUNTZ ALGEBRA WHICH PRESERVE THE CANONICAL UHF-SUBALGEBRA, II

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ABSTRACT. It was shown recently by Conti, Rørdam and Szymański that there exist endomorphisms λ_u of the Cuntz algebra $\mathcal{O}_n$ such that $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ but $u \notin \mathcal{F}_n$, and a question was raised if for such a u there must always exist a unitary $v \in \mathcal{F}_n$ with $\lambda_u|_{\mathcal{F}_n} = \lambda_v|_{\mathcal{F}_n}$. In the present paper, we answer this question to the negative. To this end, we analyze the structure of such endomorphisms λ_u for which the relative commutant $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$ is finite dimensional.

1. Introduction and preliminaries

This paper is devoted to continuation of the line of investigation of exotic endomorphisms of the Cuntz algebras initiated in [4]. Our main result is solution of a question raised therein, see below for details. Our startegy is based on a detailed analysis of such endomorphisms λ_u of $\mathcal{O}_n$ that globally preserve the core UHF subagebra $\mathcal{F}_n$ and have finite dimensional relative commutant $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$, and builds on the earlier results in this direction obtained in [10].

The Cuntz algebra $\mathcal{O}_n$, $n \geq 2$, is the C^*-algebra generated by isometries $S_1, \ldots, S_n$ satisfying $\sum_{i=1}^n S_i S_i{}^* = 1$. It is a purely infinite, simple C^*-algebra, independent of the choice of generating isometries, [7]. We denote by W_n^k the set of k-tuples $\mu = (\mu_1, \ldots, \mu_k)$ with $\mu_m \in \{1, \ldots, n\}$, and by W_n the union $\cup_{k=0}^\infty W_n^k$, where $W_n^0 = \{0\}$. If $\mu \in W_n^k$ then $|\mu| = k$ is the length of μ. If $\mu = (\mu_1, \ldots, \mu_k) \in W_n$ then $S_\mu = S_{\mu_1} \ldots S_{\mu_k}$ ($S_0 = 1$ by convention) is an isometry in $\mathcal{O}_n$. Every word in $\{S_i, S_i^* \mid i = 1, \ldots, n\}$ can be uniquely expressed as $S_\mu S_\nu^*$, for $\mu, \nu \in W_n$ [7, Lemma 1.3].

The gauge action γ of the circle group $\mathbb{T}$ on $\mathcal{O}_n$ is defined by $\gamma_z(S_i) = zS_i$, $z \in \mathbb{T}$. Let $\mathcal{F}_n$ be the fixed point algebra of γ. Denote $\mathcal{F}_n^{(k)} := \text{span}\{S_\mu S_\nu^* \mid \mu, \nu \in W_n^k\}$. Then $\mathcal{F}_n$ is generated by $\mathcal{F}_n^{(k)}$, $k = 1, 2, \ldots$, and each $\mathcal{F}_n^{(k)}$ is isomorphic to the matrix algebra $M_{n^k}(\mathbb{C})$. Thus $\mathcal{F}_n$ is isomorphic to the UHF-algebra of type n^∞, and hence it has a

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unique tracial state τ. There exists a faithful conditional expectation $E : \mathcal{O}_n \to \mathcal{F}_n$, defined by integration with respect to the Haar measure on $\mathbb{T}$ as

$$E(x) = \int_{\mathbb{T}} \gamma_z(x) dz.$$

For each $k \in \mathbb{Z}$ we denote by $\mathcal{O}_n^{(k)}$ the corresponding spectral subspace for γ in $\mathcal{O}_n$,

$$\mathcal{O}_n^{(k)} := \{x \in \mathcal{O}_n \mid \gamma_z(x) = z^k,\ \forall z \in \mathbb{T}\}.$$

Thus, in particular, $\mathcal{O}_n^{(0)} = \mathcal{F}_n$.

The C^*-subalgebra of $\mathcal{O}_n$ generated by projections $P_\mu := S_\mu S_\mu^*$, $\mu \in W_n$, is a MASA (maximal abelian subalgebra) in $\mathcal{O}_n$. We call it the diagonal and denote $\mathcal{D}_n$, also writing $\mathcal{D}_n^k$ for $\mathcal{D}_n \cap \mathcal{F}_n^{(k)}$.

The canonical shift endomorphism $\varphi : \mathcal{O}_n \to \mathcal{O}_n$ is defined by

$$\varphi(x) = \sum_{i=1}^{n} S_i x S_i^*.$$

It is easy to see that $S_i x = \varphi(x) S_i$ and $x S_i{}^* = S_i{}^* \varphi(x)$ for all $x \in \mathcal{O}_n$.

As shown by Cuntz in [8], there exists a bijective correspondence between unitaries in $\mathcal{O}_n$ (whose collection is denoted $\mathcal{U}(\mathcal{O}_n)$) and unital $*$-endomorphisms of $\mathcal{O}_n$, determined by

$$\lambda_u(S_i) = uS_i, \quad i = 1, \ldots, n.$$

We have $\mathrm{Ad}(u) = \lambda_{u\varphi(u^*)}$ for all $u \in \mathcal{U}(\mathcal{O}_n)$. If $u \in \mathcal{U}(\mathcal{O}_n)$ then for each positive integer k we denote

$$u_k = u\varphi(u) \cdots \varphi^{k-1}(u). \tag{1}$$

Here $\varphi^0 = \mathrm{id}$, and we agree that u_k^* stands for $(u_k)^*$. If α and β are multi-indices of length k and m, respectively, then $\lambda_u(S_\alpha S_\beta^*) = u_k S_\alpha S_\beta^* u_m^*$.

The Cuntz correspondence between unitaries and endomorphisms of $\mathcal{O}_n$ provides a very efficient tool for investigations of the latter. In this note, we continue the study (by several authors) of those unital endomorphisms which globally preserve the UHF-subalgebra $\mathcal{F}_n$. For example, such endomorphisms were analyzed from the point of view of the Jones-Kosaki-Watatani index theory in [12] and [3], and in connection with Hopf algebra actions in [9] and [13]. More recently, interesting combinatorial approaches to the study of permutative endomorphisms of this type have been found (e.g. see [6], [2], and a survey article [1]).

It was observed by Cuntz in his groundbreaking paper [8] that an *automorphism* λ_u globally preserves $\mathcal{F}_n$ if and only if $u \in \mathcal{F}_n$. The situation is more complex with *proper endomorphisms*. Clearly, $u \in \mathcal{F}_n$ implies $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$, [8], but the question if the converse is true remained open until very recently. Indeed, it was shown in [4] that there exist unitaries u in $\mathcal{O}_n \setminus \mathcal{F}_n$ such that $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$. All such examples found therein were of the form $u = wv$ with $w \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ and $v \in \mathcal{F}_n$. In such a case, we also have $\lambda_u(x) = \lambda_v(x)$ for all $x \in \mathcal{F}_n$. Thus a natural question arises if such a factorization of u is always possible whenever $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ (cf. [1, Problem 5.3]).

Some progress towards answering this question has been made recently in [10] and [11]. The main purpose of the present paper is to develop definite methods for analyzing endomorphisms λ_u of $\mathcal{O}_n$ satifying $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ and an additional condition that the relative commutant $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$ be finite dimensional. In particular, we give a verifiable criterion for determining if the aforementioned decomposition is possible, Corollary 3.4. Based on this criterion, in Section 3 we give an explicit example of a unitary $u \in \mathcal{O}_2$ such that $\lambda_u(\mathcal{F}_2) \subseteq \mathcal{F}_2$ and $\dim \lambda_u(\mathcal{F}_2)' \cap \mathcal{F}_2 < \infty$ but there is no unitary $v \in \mathcal{F}_2$ such that $\lambda_u|_{\mathcal{F}_2} = \lambda_v|_{\mathcal{F}_2}$, see Example 3.6. In this way, we answer to the negative the question raised in [4] and [1].

2. The relative commutants

We begin by recording for future references a few simple facts, essentially contained in [4] and [10].

Proposition 2.1. *Let $u \in \mathcal{U}(\mathcal{O}_n)$. Then the following conditions are equivalent.*

(1) $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$,
(2) $\lambda_{\gamma_z(u)}|_{\mathcal{F}_n} = \lambda_u|_{\mathcal{F}_n}$ *for all* $z \in \mathbb{T}$,
(3) $u\gamma_z(u^*) \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ *for all* $z \in \mathbb{T}$.

Proof. Clearly, $\gamma_z \lambda_u \gamma_z^{-1} = \lambda_{\gamma_z(u)}$ for all $z \in \mathbb{T}$. Thus condition (2) above is equivalent to $\gamma_z \lambda_u|_{\mathcal{F}_n} = \lambda_u|_{\mathcal{F}_n}$ for all $z \in \mathbb{T}$. Obviously, this holds if and only if $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$. That is, (1) is equivalent to (2).

It is an immediate consequence of Proposition 2.1 and Proposition 4.7 from [4] that $\lambda_u|_{\mathcal{F}_n} = \lambda_v|_{\mathcal{F}_n}$ if and only if $vu^* \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$. This gives (2) is equivalent to (3). □

Proposition 2.2. *If $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ and $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n = \mathbb{C}1$, then $u \in \mathcal{F}_n$.*

Proof. If $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ and $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n = \mathbb{C}1$, then $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n = \mathbb{C}1$ as well, [10, Theorem 1.1]. As shown in [4], this implies that $u \in \mathcal{F}_n$. □

Proposition 2.3. *Let u be a unitary in $\mathcal{O}_n$. Then $u = wv$ for some $w \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ and a unitary $v \in \mathcal{F}_n$ if and only if there exists a unitary $y \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ such that $u\gamma_z(u^*) = y\gamma_z(y^*)$ for all $z \in \mathbb{T}$.*

Proof. If $u = wv$ for some $w \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ and $v \in \mathcal{U}(\mathcal{F}_n)$, then $u\gamma_z(u^*) = w\gamma_z(w^*)$, and it suffices to put $y = w$.

Conversely, if there exists a unitary $y \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ such that $u\gamma_z(u^*) = y\gamma_z(y^*)$ for all $z \in \mathbb{T}$ then y^*u is fixed by all γ_z. Thus $y^*u \in \mathcal{F}_n$ and it suffices to put $w = y$ and $v = y^*u$. □

From now on, we make **a standing assumption** that $u \in \mathcal{U}(\mathcal{O}_n)$ is such that

$$\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n \text{ and } \dim\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n < \infty. \tag{2}$$

As shown in [10], assumption (2) above entails a number of important consequences, which we summarize as follows.

- We also have $\dim\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n < \infty$.
- There exists a unitary group $\{u_z\}_{z\in\mathbb{T}}$ in the center of $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$ such that $\mathrm{Ad}u_z(x) = \gamma_z(x)$ for all $x \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$.
- Minimal projections in $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$ are minimal in $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ as well. Thus $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ contains a MASA consisting of projections in $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$.

The proof of the following theorem is modelled after that of [10, Lemma 1.11].

Theorem 2.4. *Let $u \in \mathcal{U}(\mathcal{O}_n)$ be such that $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ and $\dim\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n < \infty$. Then there exist unitaries $w \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ and $v \in \mathcal{O}_n$, and a unitary group $\{v_z\}_{z\in\mathbb{T}} \subseteq \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$ satisfying $u = wv$ and $\gamma_z(v) = v_z v$ for all $z \in \mathbb{T}$.*

Proof. At first we note that $u\gamma_z(u^*)u_z$ is a unitary group in $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$. Indeed,

$$\begin{aligned}(u\gamma_{z_1}(u^*)u_{z_1})(u\gamma_{z_2}(u^*)u_{z_2}) &= u\gamma_{z_1}(u^*)(\mathrm{Ad}\, u_{z_1})(u)u_{z_1}\gamma_{z_2}(u^*)u_{z_2} \\ &= u\gamma_{z_1}(u^*)\gamma_{z_1}(u)\gamma_{z_1}(\gamma_{z_2}(u^*))u_{z_1}u_{z_2} = u\gamma_{z_1z_2}(u^*)u_{z_1z_2}.\end{aligned}$$

Since $\dim\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n < \infty$, this unitary group may be diagonalized. On the other hand, $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ contains a MASA composed of projections in $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$. Thus, there exists a unitary $w \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ such that $y_z := w^*(u\gamma_z(u^*)u_z)w$ is a unitary group in $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$. Since each u_z is in the center of $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$, the unitary groups $\{y_z\}_{z\in\mathbb{T}}$ and $\{u_z\}_{z\in\mathbb{T}}$ commute.

Set $v_z := u_z y_z^*$, $z \in \mathbb{T}$, and $v := w^*u$. Then v_z is a unitary group in $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$ and

$$\gamma_z(v) = \gamma_z(w^*u) = u_z(u_z^*\gamma_z(w^*u)u^*w)w^*u = u_z y_z^* w^*u = v_z w^*u = v_z v.$$

for all $z \in \mathbb{T}$. This completes the proof. □

We keep the notation from Theorem 2.4, assuming that unitaries w, v and v_z have the properties described therein. Thus, in particular, $\lambda_u|_{\mathcal{F}_n} = \lambda_v|_{\mathcal{F}_n}$ by [4, Proposition 2.1]. Consequently, $\mathrm{Ad}v \circ \varphi$ is an automorphism of $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$, by [4, Proposition 2.3 and Lemma 2.4].

Lemma 2.5. *With unitaries u, v, v_z and u_z as above, put*

$$X_z := (\mathrm{Ad}v \circ \varphi)(u_z)u_z^* v_z.$$

Then $\{X_z\}_{z\in\mathbb{T}}$ *is a unitary group in the center of* $\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n$*, and we have*

$$\gamma_z(v) = X_z u_z(\mathrm{Ad}v\circ\varphi)(u_z^*)v.$$

Proof. For each $x\in\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n$, we see that

$$\begin{aligned}u_z(\mathrm{Ad}v\circ\varphi)(x)u_z^* &= \gamma_z(v\varphi(x)v^*) = \gamma_z(v)\gamma_z(\varphi(x))\gamma_z(v^*) = \gamma_z(v)\varphi(\gamma_z(x))\gamma_z(v)^* \\ &= v_z v\varphi(u_z x u_z^*)v^*v_z^* = v_z v\varphi(u_z)v^*v\varphi(x)v^*v\varphi(u_z^*)v^*v_z^* \\ &= v_z\,\mathrm{Ad}(v\varphi(u_z)v^*)((\mathrm{Ad}\,v\circ\varphi)(x))v_z^*.\end{aligned}$$

Hence, we have

$$\mathrm{Ad}(v_z^*u_z)((\mathrm{Ad}\,v\circ\varphi(x)) = \mathrm{Ad}((\mathrm{Ad}\,v\circ\varphi)(u_z))((\mathrm{Ad}\,v\circ\varphi)(x)).$$

Since $\mathrm{Ad}v\circ\varphi$ is an automorphism of $\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n$, this shows that

$$\mathrm{Ad}(v_z^*u_z) = \mathrm{Ad}((\mathrm{Ad}\,v\circ\varphi)(u_z)) \ \text{ on } \ \lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n. \tag{3}$$

Consequently, X_z belongs to the center of $\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n$.

Now, $\{u_z\}_{z\in\mathbb{T}}$ and $\{v_z\}_{z\in\mathbb{T}}$ are commuting unitary groups, and both commute with X_z, by the above argument. Therefore the unitary group $(\mathrm{Ad}\,v\circ\varphi)(u_z) = X_z u_z v_z^*$ commutes with both of them. Consequently, X_z being a product of three mutually commuting unitary groups itself is a unitary group.

The final claim of the lemma now follows from the fact that $\gamma_z(v) = v_z v$. □

Before proceeding further, we introduce the following notation. For $x\in\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n$ and $k\in\mathbb{N}$, we set

$$x^{(k)} := x(\mathrm{Ad}v\circ\varphi)(x)(\mathrm{Ad}v\circ\varphi)^2(x)\cdots(\mathrm{Ad}v\circ\varphi)^{k-1}(x). \tag{4}$$

Lemma 2.6. *With unitaries* u*,* v*,* v_z *and* u_z *as above, and* $x\in\lambda_u(\mathcal{F}_n)'\cap\mathcal{O}_n$*, for all* $g\in\mathcal{U}(\mathcal{O}_n)$*,* $z\in\mathbb{T}$ *and* $k\in\mathbb{N}$ *we have the following identities.*

(i) $\gamma_z(v_k) = v_z^{(k)}v_k$,
(ii) $(\mathrm{Ad}g\circ\varphi)^k(x) = g_k\varphi^k(x)g_k^*$,
(iii) $v_z^{(k)} = X_z^{(k)}u_z(\mathrm{Ad}v\circ\varphi)^k(u_z^*)$,
(iv) $(gv)_k = g^{(k)}v_k$.

Proof. In all three cases, we proceed by induction on k.

Ad (i). Case $k=1$ is the identity $\gamma_z(v_1) = \gamma_z(v) = v_z v = v_z^{(1)}v_1$ from Theorem 2.4. For the inductive step, we calculate

$$\gamma_z(v_{k+1}) = \gamma_z(v_k\varphi^k(v)) = \gamma_z(v_k)\varphi^k(\gamma_z(v)) = v_z^{(k)}v_k\varphi^k(v_z)v_k^*v_k\varphi^k(v) = v_z^{(k+1)}v_{k+1}.$$

In this calculation we used identity (ii) of the present lemma, whose proof does not depend on (i).

Ad (ii). Case $k = 1$ is clear. For the inductive step, we have

$$(\mathrm{Ad}\, g \circ \varphi)^{k+1} = (\mathrm{Ad}\, g \circ \varphi)(g_k \varphi^k(x) g_k^*) = g\varphi(g_k)\varphi^{k+1}(x)\varphi(g_k^*)g^* = g_{k+1}\varphi^{k+1}(x)g_{k+1}^*.$$

Ad (iii). Case $k = 1$ is clear. For the inductive step, we see that

$$\begin{aligned} v_z^{(k+1)} &= v_z^{(k)}(\mathrm{Ad}\, v \circ \varphi)^k(v_z) = X_z^{(k)} u_z (\mathrm{Ad} v \circ \varphi)^k(u_z^*)(\mathrm{Ad}\, v \circ \varphi)^k(v_z) \\ &= X_z^{(k)} u_z (\mathrm{Ad} v \circ \varphi)^k(u_z^* v_z) = X_z^{(k)} u_z (\mathrm{Ad} v \circ \varphi)^k(X_z(\mathrm{Ad}\, v \circ \varphi)(u_z^*)) \\ &= X_z^{(k)} u_z (\mathrm{Ad} v \circ \varphi)^k(X_z)(\mathrm{Ad}\, v \circ \varphi)^{k+1}(u_z^*) = X_z^{(k+1)} u_z (\mathrm{Ad} v \circ \varphi)^{k+1}(u_z^*). \end{aligned}$$

Ad (iv). Case $k = 1$ is clear. For the inductive step, we calculate using part (ii) above,

$$(gv)_{k+1} = (gv)_k \varphi^k(gv) = g^{(k)} v_k \varphi^k(gv) = g^{(k)}(v_k \varphi^k(g) v_k^*) v_k \varphi^k(v) = g^{(k+1)} v_{k+1},$$

and this completes the proof. □

The following lemma provides a key step in the proof of our second main result, Theorem 2.8, below. We continue keeping the notation of Theorem 2.4. Here we remark that since $v = w^* u$, $w \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$, and $\mathrm{Ad}\, u \circ \varphi$ is an automorphism of $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$, we see that $\mathrm{Ad}\, v \circ \varphi = \mathrm{Ad}\, w^* \circ (\mathrm{Ad}\, u \circ \varphi)$ is an automorphism of $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ as well. We also note that for each positive integer k, $\{X_z^{(k)}\}_{z \in \mathbb{T}}$ is a unitary group in the center of $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$.

Lemma 2.7. *With unitaries u, v, v_z and u_z as above, there exist a positive integer k and a unitary $U \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ such that*

$$(\mathrm{Ad}\, v \circ \varphi)^k(x) = \mathrm{Ad}\, U(x) \quad \textit{for all } x \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n.$$

Then $X_z^{(k)} = 1$. Furthermore, for such U and k, we have $U^* v_k \in \mathcal{F}_n$.

Proof. Since $\mathrm{Ad}\, v \circ \varphi$ is an automorphism of a finite dimensional C^*-algebra $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$, its restricts to the center has finite order. Thus there exists a positive integer k and a unitary $U \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ such that $(\mathrm{Ad}\, v \circ \varphi)^k = \mathrm{Ad}\, U$ on $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$. We claim that $U^* v_k \in \mathcal{F}_n$.

Indeed, by Lemma 2.6, for all $z \in \mathbb{T}$ we have

$$\gamma_z(v_k) = v_z^{(k)} v_k = X_z^{(k)} u_z (\mathrm{Ad} v \circ \varphi)^k(u_z^*) v_k = X_z^{(k)} u_z U u_z^* U^* v_k = X_z^{(k)} \gamma_z(U) U^* v_k,$$

and this yields

$$\gamma_z(U^* v_k) = X_z^{(k)} U^* v_k. \tag{5}$$

Since $\{X_z^{(k)}\}_{z \in \mathbb{T}}$ is a unitary group in the center of $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$, there exists a partition of unity $1 = \sum_i p_i$ in $\mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n)$ and integers k_i such that

$$X_z^{(k)} = \sum_i z^{k_i} p_i.$$

We have $(\mathrm{Ad}v \circ \varphi)^k(p_i) = Up_iU^* = p_i$ for all i. Combining this with part (ii) of Lemma 2.6, we get

$$v_k^* p_i v_k = \varphi^k(p_i). \tag{6}$$

We want to show that $k_i = 0$ for all i. Suppose for a moment this is not the case and let $k_i > 0$ for some i. We set $K := p_i U^* v_k (S_1^*)^{k_i}$. Since p_i being in $\mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n)$ belongs to $\mathcal{F}_n$ as well, it follows from identity (5) above that

$$\gamma_z(K) = \gamma_z(p_i U^* v_k (S_1^*)^{k_i}) = p_i X_z^{(k)} U^* v_k \gamma_z((S_1^*)^{k_i}) = z^{k_i} p_i U^* v_k (z^{-k_i}(S_1^*)^{k_i}) = K.$$

Hence K belongs to $\mathcal{F}_n$. We have $KK^* = p_i$. On the other hand, using identity (6) we get

$$K^*K = S_1^{k_i} v_k^* p_i v_k (S_1^*)^{k_i} = S_1^{k_i} \varphi^k(p_i)(S_1^*)^{k_i} = \varphi^{k+k_i}(p_i) S_1^{k_i}(S_1^*)^{k_i}.$$

It easily follows that $\tau(KK^*) > \tau(K^*K)$, which is a contradiction. A similar argument applies in the case $k_i < 0$. Hence $k_i = 0$ for all i and thus $X_z^{(k)} = 1$. Now, identity (5) implies that $U^* v_k$ is fixed by the gauge action and hence belongs to $\mathcal{F}_n$. $\square$

Now, we are ready to prove the second main result of this paper.

Theorem 2.8. *Let $u \in \mathcal{U}(\mathcal{O}_n)$ be such that $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ and $\dim \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n < \infty$. Then there exist a positive integer k and unitaries $W \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ and $V \in \mathcal{F}_n$ such that $u_k = WV$.*

Proof. By Theorem 2.4 and Lemma 2.7, there exist unitaries $w, U \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$, a unitary group $\{v_z\}_{z \in \mathbb{T}}$ in $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$ and a positive integer k satisfying $u = wv$, $\gamma_z(v) = v_z v$, $U^* v_k \in \mathcal{F}_n$. By part (iv) of Lemma 2.6, we have $w^{(k)} v_k = u_k$. Thus to complete the proof, it suffices to put $W := w^{(k)} U$ and $V := U^* v_k$. $\square$

It was observed in [4] (just above Remark 4.4) that if $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ and $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n = \mathbb{C}1$ then $u \in \mathcal{F}_n$. The following corollary gives a sharp strengthening of that result.

Corollary 2.9. *Let u be a unitary in $\mathcal{O}_n$. If $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$, $\dim \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n < \infty$ and the automorphism $\mathrm{Ad}\, u \circ \varphi$ of $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ is inner, then there exist a unitary $w \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ and a unitary $v \in \mathcal{F}_n$ such that $u = wv$, and hence also $\lambda_u|_{\mathcal{F}_n} = \lambda_v|_{\mathcal{F}_n}$. In particular, this is the case whenever $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ is a factor.*

Remark 2.10. The assumption in Corollary 2.9 above that the automorphism $\mathrm{Ad}\, u \circ \varphi$ of $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ be inner, is equivalent to demanding existence of a unitary g in the relative commutant $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ such that

$$\lambda_{gu}(\mathcal{F}_n)' \cap \mathcal{O}_n = \lambda_{gu}(\mathcal{O}_n)' \cap \mathcal{O}_n.$$

Indeed, if $\mathrm{Ad}\, u \circ \varphi$ is inner then $\mathrm{Ad}\, gu \circ \varphi = \mathrm{id}$ for a suitable unitary g in $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$. Hence $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n = \lambda_{gu}(\mathcal{F}_n)' \cap \mathcal{O}_n = \lambda_{gu}(\mathcal{O}_n)' \cap \mathcal{O}_n$. Conversely, if $\lambda_{gu}(\mathcal{F}_n)' \cap \mathcal{O}_n =$

$\lambda_{gu}(\mathcal{O}_n)' \cap \mathcal{O}_n$ then $\mathrm{Ad}\, gu \circ \varphi = \mathrm{id}$ on $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n = \lambda_{gu}(\mathcal{F}_n)' \cap \mathcal{O}_n$, and hence $\mathrm{Ad}\, u \circ \varphi$ is inner. □

Remark 2.11. We remark that the implication in Corollary 2.9 above cannot be reversed. In fact, there exist unitaries $u \in \mathcal{F}_n$ such that $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ is finite dimensional and the automorphism $\mathrm{Ad}\, u \circ \varphi$ is outer on $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$. For example, take

$$u = S_{22}S_{11}^* + S_{12}S_{22}^* + S_{11}S_{12}^* + P_{21},$$

a permutative unitary in $\mathcal{F}_2$. Then $\mathrm{Ad}\, u \circ \varphi$ is outer on $\lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2$. For otherwise let h be a unitary in $\lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2$ such that $\mathrm{Ad}\, u \circ \varphi = \mathrm{Ad}\, h$ on $\lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2$. Then $\mathrm{Ad}\, u \circ \varphi(h) = h$ and thus $h \in \lambda_u(\mathcal{O}_2)' \cap \mathcal{O}_2$. But it can be shown that λ_u is irreducible on $\mathcal{O}_2$ (e.g., see [5], where this endomorphism is denoted ρ_{142}), and hence $\lambda_u(\mathcal{O}_2)' \cap \mathcal{O}_2 = \mathbb{C}1$. Thus h is a scalar and consequently $\mathrm{Ad}\, u \circ \varphi$ is identity on $\lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2$. This however is not the case, since one can calculate directly that $\mathrm{Ad}\, u \circ \varphi$ permutes P_1 and P_2, and both these projections are in $\lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2$. □

We want to elaborate a little bit the statement of Theorem 2.8 above. We continue keeping our standing assumption (2).

Lemma 2.12. *Let α be an automorphism of $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ and let $k \in N$ be such that α^k acts trivially on $\mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n)$. Then there exists a MASA D of $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$ and a unitary g in $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ such that*

(i) $(\mathrm{Ad}\, g \circ \alpha)^k = \mathrm{id}$, *and*

(ii) $(\mathrm{Ad}\, g \circ \alpha)(D) = D$.

Proof. Automorphism α permutes the finitely many minimal central projections of $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$. Write this permutation as a product of disjoint cycles. Clearly, it suffices to prove the lemma for each cycle separately. Thus we may simply assume that α acts transitively on minimal projections $p_1, p_2, \ldots, p_l$ in $\mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n)$, so that $\alpha(p_i) = p_{i+1}$, with $p_{l+1} = p_1$. Let $\{e_{r,s}^{(i)}\}$ be matrix units of the full matrix algebra $p_i(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n)$, such that all $e_{r,r}^{(i)}$ are in $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$. Then $D := \mathrm{span}\{e_{r,r}^{(i)}\}$ is a MASA in $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$. Since $p_i(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n) \cong p_{i+1}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n)$, we can find a unitary $g_i \in p_{i+1}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n)$ such that $(\mathrm{Ad}\, g_i \circ \alpha)(e_{r,s}^{(i)}) = e_{r,s}^{(i+1)}$. Setting $g := \sum_{i=1}^{l} g_i$ we obtain the desired result. □

Lemma 2.13. *Let $u \in \mathcal{U}(\mathcal{O}_n)$ be such that $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ and $\dim \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n < \infty$. Then there exist a positive integer k, a unitary $g \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$, and a unitary group $\{d_z\}_{z \in \mathbb{T}} \subseteq \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$ such that $(gv)_k \in \mathcal{F}_n$ and $\gamma_z(gv) = d_z gv$.*

Proof. Put $\alpha := \mathrm{Ad}\, v \circ \varphi$, and let g and k be as in Lemma 2.12. Then we have

$$(\mathrm{Ad}\, gv \circ \varphi)^k = \mathrm{id},$$

and thus

$$\operatorname{Ad} v_k \circ \varphi^k = (\operatorname{Ad} v \circ \varphi)^k = \operatorname{Ad}(g^{(k)})^*$$

by parts (ii) and (iv) of Lemma 2.6. Then arguing as in the proof of Lemma 2.7 (with ${g^{(k)}}^*$ playing the role of U), we get

$$(gv)_k = g^{(k)} v_k \in \mathcal{F}_n.$$

Now, let D be a MASA as in Lemma 2.12. For all $x \in D$ and $z \in \mathbb{T}$, we see that

$$\begin{aligned} gv\varphi(x)v^*g^* &= \gamma_z(gv\varphi(x)v^*g^*) = \gamma_z(g)v_z v\varphi(x)v^*v_z^*\gamma_z(g^*) \\ &= (\gamma_z(g)v_z g^*)(gv\varphi(x)v^*g^*)(\gamma_z(g)v_z g)^*, \end{aligned}$$

which implies that $\gamma_z(g)v_z g^*$ is in the commutant of MASA D, and hence in D itself. Set $d_z = \gamma_z(g)v_z g^*$, a unitary in D. Now, $d_z = u_z g u_z^* v_z g^*$ implies $u_z^* d_z = g(u_z^* v_z)g^*$. Since $\{u_z\}_{z\in\mathbb{T}}$ and $\{v_z\}_{z\in\mathbb{T}}$ are commuting unitary groups, so is $\{u_z^* d_z\}_{z\in\mathbb{T}}$, and consequently also is $\{d_z\}_{z\in\mathbb{T}}$. Finally, we see that $\gamma_z(gv) = \gamma_z(g)v_z v = d_z g v$. □

Now, we are ready to prove the following result.

Theorem 2.14. *Let $u \in \mathcal{U}(\mathcal{O}_n)$. If $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ and $\dim \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n < \infty$, then there exists a unitary $W \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ satisfying the following.*

(i) *There exists a unitary group $\{d_z\}_{z\in\mathbb{T}} \subseteq \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$ such that $\gamma_z(Wu) = d_z W u$ for all $z \in \mathbb{T}$.*

(ii) *There exists a positive integer k such that $(Wu)_k \in \mathcal{F}_n$.*

Proof. Let $u = wv$ be a factorization as in Theorem 2.4, and let $k \in \mathbb{N}$ and $g \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ be as in Lemma 2.13 above. Then setting $W := gw^*$ gives the claim. □

3. The criterion and examples

In this section, we give a dynamic characterization of those unitaries $u \in \mathcal{O}_n$ satisfying our standing assumptions which either belong to $\mathcal{F}_n$ (Theorem 3.2) or admit a unitary $v \in \mathcal{F}_n$ such that $\lambda_u|_{\mathcal{F}_n} = \lambda_v|_{\mathcal{F}_n}$ (Corollary 3.4). Before proving these results, we still need one technical lemma about the structure of the relative commutants. We keep our standing assumptions (2).

Lemma 3.1. *There exist a unitary group $\{q_z\}_{z\in\mathbb{T}}$ in $\mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n)$ such that*

$$X_z = q_z(\operatorname{Ad} v \circ \varphi)(q_z^*).$$

Proof. Since $\operatorname{Ad} v \circ \varphi$ restricts to an automorphism of $\mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n)$, there exist minimal projections $p_i^{(j)}$, $j = 1, \dots, N$, $i = 1, \dots, n_j$, in $\mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n)$ such that

$$\mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n) = \bigoplus_{j=1}^{N} \bigoplus_{i=1}^{n_j} p_i^{(j)}$$

and

$$(\mathrm{Ad}\, v \circ \varphi)(p_i^{(j)}) = p_{i+1}^{(j)} \ \text{ for } i < n_j, \ \text{ and } (\mathrm{Ad}\, v \circ \varphi)(p_{n_i}^{(j)}) = p_1^{(j)}.$$

Then X_z from Lemma 2.5 can be written as

$$X_z = \sum_{j=1}^{N} \sum_{i=1}^{n_j} z^{m_i^{(j)}} p_i^{(j)},$$

for some $m_i^{(j)} \in \mathbb{N}$. Now, let $k \in \mathbb{N}$ be such that $\mathrm{Ad}\, v \circ \varphi$ is an inner automorphism of $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$. Then

$$X_z^{(k)} = X_z (\mathrm{Ad}\, v \circ \varphi)(X_z)(\mathrm{Ad}\, v \circ \varphi)^2(X_z) \dots (\mathrm{Ad}\, v \circ \varphi)^{k-1}(X_z) = 1$$

by Lemma 2.7. Since each n_j divides k, this implies that

$$\sum_{i=1}^{n_j} m_i^{(j)} = 0$$

for each $j = 1, \dots, N$. Now, we want to define q_z as follows,

$$q_z = \sum_{j=1}^{N} \sum_{i=1}^{n_j} z^{r_i^{(j)}} p_i^{(j)},$$

for suitable chosen integers $r_i^{(j)}$, so that $X_z = q_z (\mathrm{Ad}\, v \circ \varphi)(q_z^*)$. To this end, it suffices to put

$$r_1^{(j)} ;= 0, \quad j = 1, \dots, N,$$
$$r_k^{(j)} := \sum_{r=2}^{k} m_r^{(j)}, \quad j = 1, \dots, N, \ k = 2, \dots, n_j.$$

□

Theorem 3.2. *Let $u \in \mathcal{U}(\mathcal{O}_n)$ be such that $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ and $\dim \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n < \infty$. Put $\alpha := \mathrm{Ad}\, u \circ \varphi$. If α satisfies the following two conditions:*

(i) *$\alpha(\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n) = \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$, and*

(ii) *$\alpha|_{\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n}$ preserves the τ-trace,*

then $u \in \mathcal{F}_n$.

Proof. At first, we observe that there exists a unitary group $\{u'_z\}_{z \in \mathbb{T}}$ in $\mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n)$ such that $\mathrm{Ad}\, u'_z(x) = \gamma_z(x)$ for all $x \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ and $\gamma_z(u) = u'_z \alpha({u'_z}^*) u$. Indeed, it suffices to put $u'_z := q_z u_z$, with q_z as in Lemma 3.1 above. Then $\alpha(u'_z) \in \mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n)$ by condition (i) of the theorem, and hence $\{u'_z \alpha({u'_z}^*)\}_{z \in \mathbb{T}}$ is a unitary group. Thus, $u'_z \alpha({u'_z}^*) = \sum z^{k_j} p_j$ for some integers k_j and a partition of unity by projections p_j from $\mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n)$.

Now, we claim that $p_j = 0$ whenever $k_j \neq 0$. To this end, suppose first that $k_j > 0$ for some index j, and put $R := p_{k_j} u (S_1^*)^{k_j}$. We have $\gamma_z(R) = R$ for all

$z \in \mathbb{T}$, and thus $R \in \mathcal{F}_n$. However, an easy calculation shows that $RR^* = p_{k_j}$ and $R^*R = \varphi^{k+1}(\alpha^{-1}(p_{k_j}))S_1^k(S_1^*)^k$. In view of condition (ii) of the theorem, this would imply $\tau(RR^*) \neq \tau(R^*R)$ if $p_j \neq 0$, a contradiction. Therefore $p_j = 0$ for all $k_j > 0$. A similar argument shows that $p_j = 0$ if $k_j < 0$.

Consequently, $u'_z\alpha({u'_z}^*) = 1$. But this gives $\gamma_z(u) = u$ for all $z \in \mathbb{T}$. Hence $u \in \mathcal{F}_n$ and the theorem is proved. □

We note that Theorem 3.2 gives a necessary and sufficient condition for $u \in \mathcal{F}_n$, since the reverse implication is trivial. Likewise, Corollary 3.3 below, gives a necessary and sufficient condition for $u_k \in \mathcal{F}_n$.

Corollary 3.3. *Let $u \in \mathcal{U}(\mathcal{O}_n)$ be such that $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ and $\dim\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n < \infty$. Put $\alpha := (\mathrm{Ad}\, u \circ \varphi)^k$, for some positive integer k. If α satisfies the following two conditions:*

(i) *$\alpha(\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n) = \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$, and*
(ii) *$\alpha|_{\lambda_u(\mathcal{F}_n)'\cap\mathcal{F}_n}$ preserves the τ-trace,*

then $u_k \in \mathcal{F}_n$.

Now, we are ready to give the following decomposability criterion.

Corollary 3.4. *Let $u \in \mathcal{U}(\mathcal{O}_n)$ be such that $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ and $\dim\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n < \infty$. Put $\alpha := \mathrm{Ad}\, u \circ \varphi$. Then the following two conditions are equivalent:*

(1) *There exist unitaries $w \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ and $v \in \mathcal{F}_n$ such that $u = wv$.*
(2) *For each minimal projection $p \in \mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n)$ there exists a τ-preserving isomorphism*

$$p(\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n) \cong \alpha(p)(\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n).$$

Now, we show how to construct examples of endomorphisms λ_u of $\mathcal{O}_n$ globally preserving the core UHF-subalgebra $\mathcal{F}_n$ but such that no unitary $v \in \mathcal{F}_n$ exists for which $\lambda_u|_{\mathcal{F}_n} = \lambda_v|_{\mathcal{F}_n}$.

To begin with, take two non-zero, orthogonal projections q_1, q_2 in $\mathcal{F}_n$ such that $\tau(q_2)/\tau(q_1) = n^r$ for some non-zero integer r. Let A_1 be a partial isometry in $\mathcal{O}_n^{(-r)}$ with domain projection $\varphi(q_1)$ and range projection q_2. Likewise, let A_2 be a partial isometry in $\mathcal{O}_n^{(r)}$ with domain projection $\varphi(q_2)$ and range projection q_1. Finally, let A_3 be a partial isometry in $\mathcal{F}_n$ with domain projection $1-\varphi(q_1)-\varphi(q_2)$ and range projection $1 - q_1 - q_2$. Put $u := A_1 + A_2 + A_3$. Then u is a unitary in $\mathcal{O}_n$ such that

$$\mathrm{Ad}\, u \circ \varphi(q_1) = q_2 \quad \text{and} \quad \mathrm{Ad}\, u \circ \varphi(q_2) = q_1. \tag{7}$$

Now, $u\gamma_z(u^*) = z^r q_1 + z^{-r}q_2 + 1 - q_1 - q_2$ belongs to $\mathrm{span}\{1, q_1, q_2\}$, and $\mathrm{span}\{1, q_1, q_2\} \subseteq \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ by [4, Proposition 2.3] and (7) above. Thus $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ by Proposition 2.1 above.

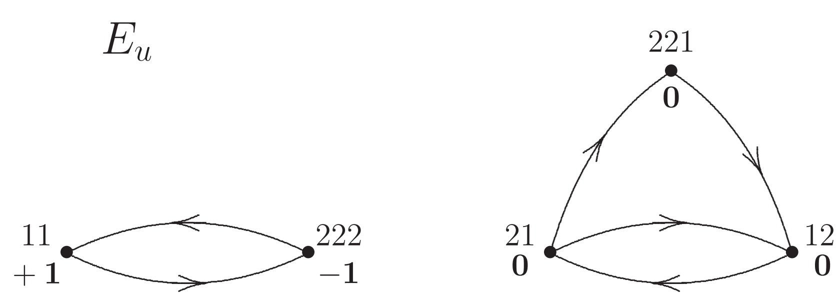
More generally, Let $1 = \sum q_j$ be a partition of unity by projections in $\mathcal{O}_n$. Let u be any unitary in $\mathcal{O}_n$ such that $\mathrm{Ad}\, u \circ \varphi$ permutes projections $\{q_j\}$ and for each j there is a $k_j \in \mathbb{Z}$ such that $q_j u \in \mathcal{O}_n^{(k_j)}$. Then $u\gamma_z(u^*) \in \mathrm{span}\{q_j\} \subseteq \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ for all $z \in \mathbb{T}$. This simple construction gives a large class of examples of unitaries $u \in \mathcal{O}_n \setminus \mathcal{F}_n$ such that $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$. However, to verify the conditions of Corollary 3.4 one needs more detailed information on the relative commutants $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n \subseteq \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$. Exact determination of these relative commutants is rather difficult and does not seem possible in general, despite the identity from [4, Proposition 2.3]. However, it is quite doable in concrete cases.

Now, we illustrate the above discussion with two concrete examples in $\mathcal{O}_2$. In these examples, along with the main algebra $C^*(S_1, S_2) \cong \mathcal{O}_2$, we consider its other subalgebras, also isomorphic to $\mathcal{O}_2$. For example, if T_1, T_2 are isometries in $C^*(S_1, S_2)$ generating a copy of $\mathcal{O}_2$, then we use subscript T along with the standard notation to indicate that the object comes from $C^*(T_1, T_2)$ and its generators. Thus φ_T denotes the usual shift on $C^(T_1, T_2)$, that is a map $\varphi : C^*(T_1, T_2) \to C^*(T_1, T_2)$ such that $\varphi(x) = T_1 x T_1^* + T_2 x T_2^*$. Similarly, $\mathcal{D}_T$ denotes the diagonal MASA of $C^*(T_1, T_2)$, and so on. The proof of one technical lemma needed in Example 3.6 is given afterwards.

Example 3.5. Take $q_1 = P_{11}$, $q_2 = P_{222}$, and set

$$A_1 = S_{2221} S_{111}^* + S_{2222} S_{211}^*$$
$$A_2 = S_{111} S_{1222}^* + S_{112} S_{2222}^*$$
$$A_3 = S_{1222} S_{2221}^* + S_{211} S_{112}^* + P_{121} + P_{1221} + P_{212} + P_{221}$$

We note that unitary $u := A_1 + A_2 + A_3$ falls within the class of polynomial unitaries considered in [4, Section 5]. In particular, its graph E_u, as defined therein, admits the $\{-1, 0, +1\}$ labelling:



This labelled graph satisfies the path condition defined in [4, p. 616], and this is an alternative way of showing that $\lambda_u(\mathcal{F}_2) \subseteq \mathcal{F}_2$.

Now, we have $P_{11}\mathcal{O}_2P_{11} \cong \mathcal{O}_2 = C^*(T_1,T_2)$, under the isomorphism sending T_1 to $S_{111}S_{11}^*$ and T_2 to $S_{112}S_{11}^*$. Similarly, $P_{222}\mathcal{O}_2P_{222} \cong \mathcal{O}_2 = C^*(R_1,R_2)$, under the isomorphism sending R_1 to $S_{2221}S_{222}^*$ and R_2 to $S_{2222}S_{222}^*$. Then an easy calculation shows that

$$\mathrm{Ad}\, u\circ\varphi(T_j) = \varphi_R(R_j),$$
$$\mathrm{Ad}\, u\circ\varphi(R_j) = \varphi_T(T_j),$$

for $j=1,2$. Consequently, the restriction of $(\mathrm{Ad}\, u\circ\varphi)^2$ to $P_{11}\mathcal{O}_2P_{11}$ is conjugate to φ_R^2. Likewise, the restriction of $(\mathrm{Ad}\, u\circ\varphi)^2$ to $P_{222}\mathcal{O}_2P_{222}$ is conjugate to φ_T^2. This immediately implies

$$\lambda_u(\mathcal{F}_2)'\cap P_{11}\mathcal{O}_2P_{11} \subseteq \bigcap_{k=1}^{\infty}(\mathrm{Ad}\, u\circ\varphi)^{2k}(P_{11}\mathcal{O}_2P_{11}) = \mathbb{C}P_{11},$$
$$\lambda_u(\mathcal{F}_2)'\cap P_{222}\mathcal{O}_2P_{222} \subseteq \bigcap_{k=1}^{\infty}(\mathrm{Ad}\, u\circ\varphi)^{2k}(P_{222}\mathcal{O}_2P_{222}) = \mathbb{C}P_{222}.$$

That is, both P_{11} and P_{222} are minimal projections in $\lambda_u(\mathcal{F}_2)'\cap\mathcal{O}_2$. One easily checks that $\mathrm{Ad}\, u\circ\varphi(S_{11}S_{222}^*) = S_{222}S_{11}^*$. Thus $S_{11}S_{222}^*$ is in $\lambda_u(\mathcal{F}_2)'\cap\mathcal{O}_2$, and we see that $(P_{11}+P_{222})\lambda_u(\mathcal{F}_2)'\cap\mathcal{O}_2(P_{11}+P_{222}) \cong \mathbb{M}_2(\mathbb{C})$. We remark that the restriction of $\mathrm{Ad}\, u\circ\varphi$ to $(P_{11}+P_{222})\mathcal{O}_2(P_{11}+P_{222})$ is conjugate to endomorphism ρ_{1342} from [5]. Let

$$w := S_{11}S_{222}^* + S_{222}S_{11}^* + 1 - P_{11} - P_{222}.$$

Then w is a unitary in $\lambda_u(\mathcal{F}_2)'\cap\mathcal{O}_2$ such that $w^*u\in\mathcal{F}_2$. □

Example 3.6. Take $q_1 = P_1$, $q_2 = P_{21}$, and set

$$A_1 = S_{211}S_{21}^* + S_{2121}S_{112}^* + S_{2122}S_{111}^*,$$
$$A_2 = S_{12}S_{121}^* + S_{11}S_{221}^*,$$
$$A_3 = S_{221}S_{122}^* + P_{222}.$$

We put $u := A_1+A_2+A_3$. By construction, $\mathrm{Ad}\, u\circ\varphi(P_1) = P_{21}$ and also $\mathrm{Ad}\, u\circ\varphi(P_{21}) = P_1$. Hence $\mathrm{Ad}\, u\circ\varphi(P_{22}) = P_{22}$ as well.

We have $P_{22}C^*(S_1,S_2)P_{22} \cong \mathcal{O}_2 = C^*(R_1,R_2)$, under the identification of $S_{221}S_{22}^*$ with R_1 and $S_{222}S_{22}^*$ with R_2. This isomorphism yields a conjugation between the restriction of $\mathrm{Ad}\, u\circ\varphi$ to $P_{22}C^*(S_1,S_2)P_{22}$ and the shift φ_R. Consequently,

$$\lambda_u(\mathcal{F}_2)'\cap P_{22}C^*(S_1,S_2)P_{22} = \bigcap_{k=1}^{\infty}(\mathrm{Ad}\, u\circ\varphi)^{k}(P_{22}C^*(S_1,S_2)P_{22}) = \mathbb{C}P_{22}.$$

We have $P_1C^*(S_1,S_2)P_1 \cong \mathcal{O}_2 = C^*(T_1,T_2)$, under the identification of $S_{11}S_1^*$ with T_1 and $S_{12}S_1^*$ with T_2. This isomorphism carries the restriction of $(\mathrm{Ad}\, u\circ\varphi)^2$ to $P_1C^*(S_1,S_2)P_1$ to the endomorphism of $C^*(T_1,T_2)$ given as composition $\varphi_T\circ\psi_T$, where ψ_T is an endomorphism of $C^*(T_1,T_2)$ such that

$$\psi_T(x) = T_1xT_1^* + T_2(\mathrm{Ad}\, F_T(x))T_2^*,$$

where $F_T := T_2T_1^* + T_1T_2^*$. By Lemma 3.7, we have

$$\lambda_u(\mathcal{F}_2)' \cap P_1C^*(S_1,S_2)P_1 \subseteq \bigcap_{k=1}^{\infty}(\mathrm{Ad}\, u \circ \varphi)^{2k}(P_1C^*(S_1,S_2)P_1) = \mathbb{C}P_1.$$

We have $P_{21}C^*(S_1,S_2)P_{21} \cong \mathcal{O}_2 = C^*(V_1,V_2)$, under the identification of $S_{211}S_{21}^*$ with V_1 and $S_{212}S_{21}^*$ with V_2. This isomorphism carries the restriction of $(\mathrm{Ad}\, u \circ \varphi)^2$ to $P_{21}C^*(S_1,S_2)P_{21}$ to $\psi_V \circ \varphi_V$. An argument similar to that from Lemma 3.7 shows that $\lambda_u(\mathcal{F}_2)' \cap P_{21}C^*(S_1,S_2)P_{21} = \mathbb{C}P_{21}$. Alternatively, this also easily follows from the preceding argument and the fact that $\mathrm{Ad}\, u \circ \varphi(P_{21}) = P_1$.

In view of the above, either $\lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2 = \mathrm{span}\{P_1, P_{21}, P_{22}\} \cong \mathbb{C}^3$, or P_1 and P_{21} are equivalent in $\lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2$. In the latter case, $\lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2$ contains a subalgebra isomorphic to $M_2(\mathbb{C})$ which is invariant under $\mathrm{Ad}\, u \circ \varphi$ and has P_1 and P_{21} as its minimal projections. Suppose for a moment that this is the case. Then $\mathrm{Ad}\, u \circ \varphi$ restricts to a non-trivial automorphism of $M_2(\mathbb{C})$, by necessity inner. The implementing unitary matrix g is fixed by $\mathrm{Ad}\, u \circ \varphi$ and thus belongs to $\lambda_u(\mathcal{O}_2)' \cap \mathcal{O}_2$. Matrix g has both diagonal entries equal to 0. Multiplying g by a suitable scalar of modulus 1, we can find such g that is self-adjoint. Now we see that there is a unitary element x of $\mathcal{O}_2$ such that

$$g = S_{21}x^*S_1^* + S_1xS_{21}^* \in \lambda_u(\mathcal{O}_2)' \cap \mathcal{O}_2.$$

Now, writing $F := S_1S_2^* + S_2S_1^*$, we compute

$$\begin{aligned} \mathrm{Ad} u \circ \varphi(g) &= u(S_{11}xS_{121}^* + S_{121}x^*S_{11}^* + S_{21}xS_{221}^* + S_{221}x^*S_{21}^*)u^* \\ &= S_{212}FxS_{12}^* + S_{12}x^*FS_{212}^* + S_{211}xS_{11}^* + S_{11}x^*S_{211}^*, \end{aligned}$$

and hence we get

$$S_1xS_{21}^* + S_{21}x^*S_1^* = S_{212}FxS_{12}^* + S_{12}x^*F^*S_{212}^* + S_{211}xS_{11}^* + S_{11}x^*S_{211}^*.$$

Multiplying by S_1^* from the left-side and by S_{21} from the right-side, we obtain

$$x = S_2x^*FS_2^* + S_1x^*S_1^*. \tag{8}$$

Equation (8) implies $xS_1 = S_1x^*$ and $S_1^*x = x^*S_1^*$. These two combined then yield $(x + x^*)S_1 = S_1(x + x^*)$ and $(x - x^*)S_1 = -S_1(x - x^*)$. By [14, Proposition 4], both $x + x^*$ and $x - x^*$ are scalars, and thus so is x. This however contradicts (8).

Thus $\lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2 = \mathrm{span}\{P_1, P_{21}, P_{22}\}$ and since $\tau(P_1) \neq \tau(P_{21})$, we conclude from Corollary 3.4 that there are no unitaries $w \in \lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2$ and $v \in \mathcal{F}_2$ such that $u = wv$. □

Lemma 3.7. *Let ψ_T be an endomorphism of $C^*(T_1,T_2) \cong \mathcal{O}_2$ such that*

$$\psi_T(x) = T_1xT_1^* + T_2(\mathrm{Ad}\, F_T(x))T_2^*,$$

where $F_T := T_2T_1^* + T_1T_2^*$. *Then we have*

$$\bigcap_{k=1}^{\infty}(\varphi_T\psi_T)^k(C^*(T_1,T_2)) = \mathbb{C}1.$$

Proof. We note that

$$\varphi_T\psi_T(x) = T_{11}xT_{11}^* + T_{21}xT_{21}^* + T_{12}(\mathrm{Ad}\,F_T(x))T_{12}^* + T_{22}(\mathrm{Ad}\,F_T(x))T_{22}^*.$$

Also, we clearly have $F_TT_1 = T_2$ and $F_TT_2 = T_1$. Thus $(\varphi_T\psi_T)^k(x)$ may be written as a finite sum of elements of the form $T_\mu XT_\mu^*$ with $|\mu| = 2k$. This gives

$$\bigcap_{k=1}^{\infty}(\varphi_T\psi_T)^k(C^*(T_1,T_2)) \subseteq \mathcal{D}_T' \cap C^*(T_1,T_2) = \mathcal{D}_T.$$

For a positive ineger k, let

$$Q_k := \sum_{|\mu|=k-1} T_{\mu 1}T_{\mu 1}^*.$$

Then a straightforward induction on k shows that

$$Q_{2k}(\varphi_T\psi_T)^k(x) = Q_{2k}\varphi_T^{2k}(x) \tag{9}$$

for all $x \in C^*(T_1,T_2)$. Take a $d = d^* \in \mathcal{D}_T$ that belongs to $\bigcap_{k=1}^{\infty}(\varphi_T\psi_T)^k(C^*(T_1,T_2))$. Suppose d is not a constant multiple of 1. Then there exist $k \in \mathbb{N}$, $t \in \mathbb{R}$, $\epsilon > 0$ and $\mu,\nu \in W_2^{2k-1}$ such that

$$T_{\mu 1}T_{\mu 1}^*d \geq (t+\epsilon)T_{\mu 1}T_{\mu 1}^* \quad \text{and} \quad T_{\nu 1}T_{\nu 1}^*d \leq (t-\epsilon)T_{\mu 1}T_{\mu 1}^*.$$

Let $x = x^* \in \mathcal{D}_2$ be such that $d = (\varphi_T\psi_T)^k(x)$. Then $Q_{2k}d = Q_{2k}\varphi_T^{2k}(x)$. Since $T_{\mu 1}T_{\mu 1}^* \leq Q_{2k}$ and $T_{\nu 1}T_{\nu 1}^* \leq Q_{2k}$, we get

$$T_{\mu 1}xT_{\mu 1}^* = T_{\mu 1}T_{\mu 1}^*Q_{2k}\varphi_T^{2k}(x) \geq (t+\epsilon)T_{\mu 1}T_{\mu 1}^*, \quad \text{and}$$
$$T_{\nu 1}xT_{\nu 1}^* = T_{\nu 1}T_{\nu 1}^*Q_{2k}\varphi_T^{2k}(x) \leq (t-\epsilon)T_{\nu 1}T_{\nu 1}^*.$$

This, however, is a contradiction. Indeed, since $T_{\mu 1}$ and $T_{\nu 1}$ are isometries, the above two inequalities would imply that both $x \geq (t+\epsilon)$ and $x \leq (t-\epsilon)$. Consequently,

$$\bigcap_{k=1}^{\infty}(\varphi_T\psi_T)^k(C^*(T_1,T_2)) = \mathbb{C}1,$$

as required. □

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