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Consistent Perturbative Fixed Point Calculations in QCD and Supersymmetric QCD

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We suggest how to consistently calculate the anomalous dimension γ_* of the $\bar{\psi}\psi$ operator in finite order perturbation theory at an infrared fixed point for asymptotically free theories. If the $n + 1$ loop beta function and n loop anomalous dimension are known, then γ_* can be calculated exactly and fully scheme independently in a Banks-Zaks expansion through $O(\Delta_f^n)$, where $\Delta_f = \bar{N}_f - N_f$, N_f is the number of flavors, and \bar{N}_f is the number of flavors above which asymptotic freedom is lost. For a supersymmetric theory, the calculation preserves supersymmetry order by order in Δ_f . We then compute γ_* through $O(\Delta_f^2)$ for supersymmetric QCD in the dimensional reduction scheme and find that it matches the exact known result. We find that γ_* is astonishingly well described in perturbation theory already at the few loops level throughout the entire conformal window. We finally compute γ_* through $O(\Delta_f^3)$ for QCD and a variety of other nonsupersymmetric fermionic gauge theories. Small values of γ_* are observed for a large range of flavors.

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Introduction.—Our poor understanding of strongly interacting dynamics limits our progress on at least two frontiers of particle physics. First, we cannot, from fundamental principles, deduce that quantum chromodynamics (QCD) confines the quarks into hadrons and, simultaneously, exhibits chiral symmetry breaking explaining the lightness of the pions and providing the bulk of the mass of the nucleons. Second, a solution to the naturalness problem of the Higgs sector in the standard model begs for new physics to be discovered in the near future. Several new scenarios involve strongly coupled physics, but without a better understanding of such dynamics, it is difficult to pin down the viable theories able to correctly break the electroweak symmetry. One obstacle which needs to be overcome is to deduce the exact value of certain physical quantities (mass anomalous dimension, baryon anomalous dimension, etc.) at or near an interacting infrared fixed point.

Of our interest, therefore, is the calculation of the anomalous dimension of the bilinear operator $\bar{\psi}\psi$ of some gauge theory with gauge coupling $\alpha = [g^2/(4\pi)]$ and a single set of massless fermion representations ψ_f , $f = 1, \dots, N_f$ at an interacting fixed point. The anomalous dimension $\gamma(\alpha_*) \equiv \gamma_*$ evaluated at a fixed point α_* is a physical scheme independent quantity.

Unfortunately, however, in most cases, we are limited by knowing only the beta function and anomalous dimension to finite order in perturbation theory. For gauge theories with a single fermion representation, they are known to four loop order in the modified minimal subtraction scheme ($\overline{\text{MS}}$) [1], while for their supersymmetric cousin theories, they are known to three loop order in the dimensional reduction ($\overline{\text{DR}}$) scheme [2]. This limitation is the source of an induced problem we wish to solve.

Suppose that we expand the beta function and anomalous dimension in the gauge coupling as

$$\frac{2\pi}{\alpha^2}\beta(\alpha) = -\sum_{i=0}^{\infty}\beta_i\left(\frac{\alpha}{2\pi}\right)^i, \quad (1)$$

$$\gamma(\alpha) = \sum_{i=1}^{\infty}\gamma_i\left(\frac{\alpha}{2\pi}\right)^i. \quad (2)$$

We shall take the trivial fixed point to be in the ultraviolet such that the theory is asymptotically free. This is achieved provided $\beta_0 > 0$, i.e., for a sufficiently small number of flavors [3]. For a number of flavors just below the critical value where β_0 turns positive, the second beta function coefficient $\beta_1 < 0$ [4,5]. Suppose that we only have available the first two coefficients of the beta function and anomalous dimension. Then, the theory has a nontrivial positive fixed point located at

$$\frac{\alpha_*}{2\pi} = -\frac{\beta_0}{\beta_1}. \quad (3)$$

Evaluating the anomalous dimension at this fixed point yields

$$\gamma_* = -\frac{\gamma_1\beta_0}{\beta_1} + \frac{\gamma_2\beta_0^2}{\beta_1^2}. \quad (4)$$

Now, since β_0 , β_1 , and γ_1 are all scheme independent, whereas γ_2 is scheme dependent, γ_* must also be scheme dependent. This seems to be a contradiction. The scheme dependent contamination of γ_* is, of course, due to our artificial truncation of the beta function and anomalous

dimension. Also, it should be clear that this problem persists to any finite order in perturbation theory.

The value of γ_* obtained in his way at three and four loops in the $\overline{\text{MS}}$ scheme has been studied in [6] and in various other schemes in [7]. Higher order perturbation theory has also been used in [8] to study the phase transition separating the conformal from the chirally broken phase. However, no method for disentangling the induced scheme dependent error from the perturbative approximation was given. Also, in [9], an all orders beta function was conjectured which allowed for a determination of γ_* in principle through all orders. As we shall see, this result agrees with ours only through first order while it disagrees through higher orders. Therefore, our results also imply that it cannot be exact.

Finally, we note that, if the theory is supersymmetric, then the above calculation of γ_* also generically breaks supersymmetry which is again due to the artificial truncation of the perturbative expansion. In [10], the two and three loop evaluation of γ_* was studied with the unsatisfactory conclusion that γ_* turned negative at three loops for a number of flavors not very much below the critical value where asymptotic freedom is lost. As we will see below, the culprit can now be identified to be induced scheme dependence and explicit supersymmetry breaking. Also, see [11,12] for alternative approaches. The latter work compared gap equation predictions with the exact results. Considerable disagreement was observed.

Consistently computing the anomalous dimension.— Instead, we follow [5] and suggest calculating γ_* as a series expansion in $\Delta_f \equiv \bar{N}_f - N_f$

$$\gamma_* = \sum_{i=1}^{\infty} c_i \Delta_f^i, \quad (5)$$

where \bar{N}_f is the fixed number of flavors above which asymptotic freedom is lost. For illustrative purposes, we shall view the number of flavors as a continuous parameter, although ultimately, only integer values are of physical interest. Studies of other physical quantities can be found in [13–16].

Such an expansion has several attractive features. Of course, if we know the beta function and anomalous dimension to all orders, then we can, in principle, exactly compute each coefficient c_i . Each coefficient must then also be scheme independent: Imagine that we have computed $\tilde{\gamma}_*$ in another scheme, then we can similarly expand it in Δ_f (which is scheme independent) but with coefficients \tilde{c}_i . Forming the difference $\gamma_* - \tilde{\gamma}_* = 0$, we find that this can only be satisfied provided $c_i - \tilde{c}_i = 0$; i.e., the coefficients c_i are scheme independent.

Fortunately, as we shall see, each c_i will not depend on the full beta function and anomalous dimension. In fact, c_i will only depend on information stored in the $i + 1$ loop beta function and i loop anomalous dimension. It will not

receive contributions from higher loop orders. Hence, if we only know the beta function through $n + 1$ loop order and the anomalous dimension through n loop order [we will refer to this as an $(n + 1, n)$ loop order computation] this will still enable us to compute γ_* in an exact and scheme independent manner through $O(\Delta_f^n)$. We will now show this by providing a method for how, in principle, to compute the coefficients c_i to any desired order.

We will formally expand the fixed point solution in a Banks-Zaks expansion as

$$\frac{\alpha_*}{2\pi} = \sum_{i=1}^{\infty} a_i \Delta_f^i. \quad (6)$$

Our first job is to find the coefficients a_i . Evaluating the beta function at this fixed point and expanding in Δ_f gives

$$\begin{aligned} 0 = & (a_1 \bar{\beta}_1^{(0)} - \bar{\beta}_0^{(1)}) \Delta_f + (a_2 \bar{\beta}_1^{(0)} + a_1^2 \bar{\beta}_2^{(0)} - a_1 \bar{\beta}_1^{(1)}) \Delta_f^2 \\ & + (a_3 \bar{\beta}_1^{(0)} + 2a_1 a_2 \bar{\beta}_2^{(0)} - a_2 \bar{\beta}_1^{(1)} \\ & - a_1^2 \bar{\beta}_2^{(1)} + a_1^3 \bar{\beta}_3^{(0)}) \Delta_f^3 + O(\Delta_f^4), \end{aligned} \quad (7)$$

where

$$\bar{\beta}_i^{(n)} = \frac{\partial^n \beta_i}{\partial N_f^n |_{N_f = \bar{N}_f}}. \quad (8)$$

Each coefficient must vanish since Δ_f is taken to be arbitrary and positive. This allows us to, first, solve for a_1 , then for a_2 , then for a_3 , etc. For the first three, this gives

$$a_1 = \frac{\bar{\beta}_0^{(1)}}{\bar{\beta}_1^{(0)}}, \quad a_2 = (\bar{\beta}_1^{(0)} \bar{\beta}_1^{(1)} - \bar{\beta}_0^{(1)} \bar{\beta}_2^{(0)}) \frac{\bar{\beta}_0^{(1)}}{\bar{\beta}_1^{(0)3}}, \quad (9)$$

$$\begin{aligned} a_3 = & (\bar{\beta}_1^{(0)2} \bar{\beta}_1^{(1)2} - 3\bar{\beta}_0^{(1)} \bar{\beta}_1^{(0)} \bar{\beta}_1^{(1)} \bar{\beta}_2^{(0)} + 2\bar{\beta}_0^{(1)2} \bar{\beta}_2^{(0)2} \\ & + \bar{\beta}_0^{(1)} \bar{\beta}_1^{(0)2} \bar{\beta}_2^{(1)} - \bar{\beta}_1^{(0)} \bar{\beta}_0^{(1)2} \bar{\beta}_3^{(0)}) \frac{\bar{\beta}_0^{(1)}}{\bar{\beta}_1^{(0)5}}. \end{aligned} \quad (10)$$

The pattern for solving for the coefficients a_i one at a time continues and, in principle, allows us to solve for the fixed point to any desired order in Δ_f in a quite straight forward manner. In general, a_i will only depend on the first $i + 1$ loop coefficients. Finally, evaluating the anomalous dimension at the fixed point and expanding in Δ_f , we find for the first three coefficients

$$\begin{aligned} c_1 = & a_1 \bar{\gamma}_1^{(0)}, \quad c_2 = a_2 \bar{\gamma}_1^{(0)} + a_1^2 \bar{\gamma}_2^{(0)} - a_1 \bar{\gamma}_1^{(1)}, \\ c_3 = & a_3 \bar{\gamma}_1^{(0)} + 2a_1 a_2 \bar{\gamma}_2^{(0)} + a_1^3 \bar{\gamma}_3^{(0)} - a_1^2 \bar{\gamma}_2^{(1)}, \end{aligned} \quad (11)$$

with

$$\bar{\gamma}_i^{(n)} = \frac{\partial^n \gamma_i}{\partial N_f^n |_{N_f = \bar{N}_f}}. \quad (12)$$

In general, c_i will only depend on the $(i + 1, i)$ loop coefficients of the beta function and anomalous dimension. They will not receive any further corrections from higher loops. Therefore, if we have available the $n + 1$ loop beta function and the n loop anomalous dimension, we can compute γ_* through $O(\Delta_f^n)$ in an exact and fully scheme independent manner. This is precisely what we wanted.

Supersymmetric QCD.—We are now at a point where we can put our explicit formula and results for γ_* to work. Before discussing QCD we will take a small departure and test our investigations against exact known results in supersymmetric QCD (SQCD). Here, N_f now counts the number of superflavors and $\bar{N}_f = \frac{3}{2}(C_A/T_r)$. The group factors C_r and T_r are, respectively, the quadratic Casimir and trace normalization factor for the representation r , and A denotes the adjoint representation. Using the (3,2) loop coefficients calculated in the $\overline{\text{DR}}$ scheme in [2], the coefficients c_1 and c_2 can be found, and therefore, for the value of γ_* , we arrive at

$$\gamma_* = \frac{2T_r}{3C_A} \Delta_f + \left(\frac{2T_r}{3C_A}\right)^2 \Delta_f^2 + O(\Delta_f^3). \quad (13)$$

This result is directly comparable to the exact result (computed through a different scheme) already known to exist [17] and which we write in the following suggestive form:

$$\gamma_* = \frac{\frac{2T_r}{3C_A} \Delta_f}{1 - \frac{2T_r}{3C_A} \Delta_f}. \quad (14)$$

Through $O(\Delta_f^2)$, we see there is complete agreement with the result obtained in the $\overline{\text{DR}}$ scheme, Eq. (13), as there should be.

It is instructive to plot γ_* as a function of the number of superflavors. This we do in Fig. 1 for supersymmetric QCD with an $SU(3)$ gauge group and fundamental matter. It is plotted through $O(\Delta_f)$ (green), through $O(\Delta_f^2)$ (red), and exactly (black). The blue curve is γ_* through $O(\Delta_f^3)$ obtained from the exact result and which we now know must correspond to a (4,3) loop computation. In fact, knowing the exact result, our investigations allow us to predict that an $(n + 1, n)$ loop computation must yield

$$c_n = \left(\frac{2T_r}{3C_A}\right)^n. \quad (15)$$

It is clear that perturbation theory provides a remarkably accurate estimate already at $O(\Delta_f^3)$ (blue curve) as

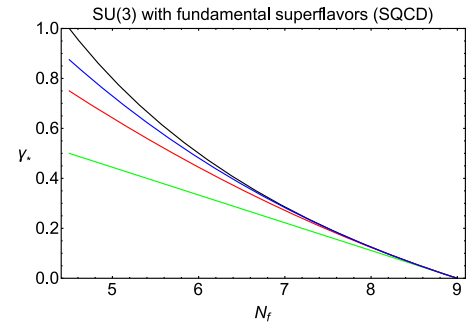


FIG. 1. The value of γ_* for supersymmetric QCD with gauge group $SU(3)$ and N_f fundamental superfields.

compared to the exact result (black curve). To our surprise, the physics at the fixed point seems to be very well described by higher order perturbation theory throughout the entire conformal window, $0 < \gamma_* < 1$.

Initially, when we set out to perform the computation of γ_* , one might have feared that potential problems of convergence of the perturbative expansion would appear since Δ_f is, typically, not a small number. However, there is no need for such worries. In fact, by inspecting the exact result, we see that γ_* has a series expansion in Δ_f provided $[2T_r/(3C_A)]\Delta_f < 1$, which corresponds to $N_f > 0$. Therefore, the series expansion formally exists for all asymptotically free theories. Of course, it only makes sense to calculate γ_* for the theories that actually reach the fixed point and preserve unitarity [18] $0 < \gamma_* < 1$, corresponding to $[3C_A/(4T_r)] < N_f < [3C_A/(2T_r)]$ [17].

If we only had perturbation theory available, how well would we have predicted this range for the conformal window? Remember that, even though we assumed N_f to be continuous, only integer values correspond to real theories. We would still require $\gamma_* < 1$ due to unitarity. Using the value of γ_* as computed through $O(\Delta_f^3)$, we find that the theories with $N_f = 5, \dots, 9$ superflavors are infrared conformal, precisely in agreement with the exact result.

QCD.—Having seen how accurately γ_* is described by a (4,3) loop computation in supersymmetric QCD, we turn our attention to QCD for which the similar computation can be directly done. Here, the critical number of flavors below which the theory is asymptotically free is $\bar{N}_f = [11C_A/(4T_r)]$. Using the (4,3) loop coefficients of the beta function and anomalous dimension in the $\overline{\text{MS}}$ scheme found in [1], we can calculate γ_* through $O(\Delta_f^3)$ as done above. Direct evaluation gives

$$c_1 = \frac{8T_r C_r}{C_A(7C_A + 11C_r)}, \quad c_2 = \frac{4T_r^2 C_r (35C_A^2 + 636C_A C_r + 352C_r^2)}{3C_A^2 (7C_A + 11C_r)^3}, \quad (16)$$

$$\begin{aligned}
c_3 = & \frac{4T_r C_r}{81C_A^4(7C_A + 11C_r)^5} \left[-55419T_r^2 C_A^5 + 432012T^2 C_A^4 C_r + 5632T_r^2 C_r \frac{d_F^{abcd} d_A^{abcd}}{d_A} (-5 + 132\zeta_3) \right. \\
& + 16C_A^3 \left(122043T_r^2 C_r^2 + 6776 \frac{d_F^{abcd} d_F^{abcd}}{d_A} (-11 + 24\zeta_3) \right) \\
& + 704C_A^2 \left(1521T_r^2 C_r^3 + 112T_r \frac{d_F^{abcd} d_A^{abcd}}{d_A} (4 - 39\zeta_3) + 242C_r \frac{d_F^{abcd} d_F^{abcd}}{d_A} (-11 + 24\zeta_3) \right) \\
& \left. + 32T_r C_A \left(53361T_r C_r^4 - 3872C_r \frac{d_F^{abcd} d_A^{abcd}}{d_A} (-4 + 39\zeta_3) + 112T_r \frac{d_A^{abcd} d_A^{abcd}}{d_A} (-5 + 132\zeta_3) \right) \right], \quad (17)
\end{aligned}$$

where d_F^{abcd} and d_A^{abcd} are a set of fully symmetrical tensors that can be found in [1]. [As an independent check, we have also performed the calculation of c_1 and c_2 in the $\overline{\text{DR}}$ scheme (for which evanescent and ϵ -scalar self-couplings have to be studied as well) for $SU(3)$ QCD and found exact agreement with the results given here. These results will appear in a future publication.] The dimension of the adjoint representation is d_A while ζ_3 is the Riemann zeta function evaluated at three. With these coefficients in hand, we have, at our disposal, the exact scheme independent value of γ_* through $O(\Delta_f^3)$.

Again, it is instructive to plot γ_* as a function of the number of flavors. We do this for an $SU(2)$ and $SU(3)$ gauge group with fundamental fermions in Fig. 2 and with two index symmetric fermions in Fig. 3. The green curve is $O(\Delta_f)$, the red curve is $O(\Delta_f^2)$, and the blue curve is $O(\Delta_f^3)$.

In Tables I and II, we provide the value of γ_* for a variety of theories. This includes $SU(3)$ QCD with fundamental fermions. If the conformal window is bounded by $\gamma_* < 1$, then the theories with $N_f = 8, \dots, 16$ flavors are infrared conformal. Remember that, even though we have taken N_f to be continuous, only a discrete value corresponds to a real theory. For the $SU(2)$ case, the theory is infrared conformal for a number of $N_f = 5, \dots, 11$ flavors. Although we expect that higher order corrections will push up the value of γ_* slightly, the conformal window extends quite far down in the number of flavors.

For higher dimensional representations, the conformal window is much more narrow. If we again assume that $\gamma_* < 1$ marks the lower boundary, only the two theories with $N_f = 2$ and $N_f = 2.5$ Dirac flavors, corresponding to four and five Weyl fermions, respectively, are infrared conformal for fermions in the adjoint representation of

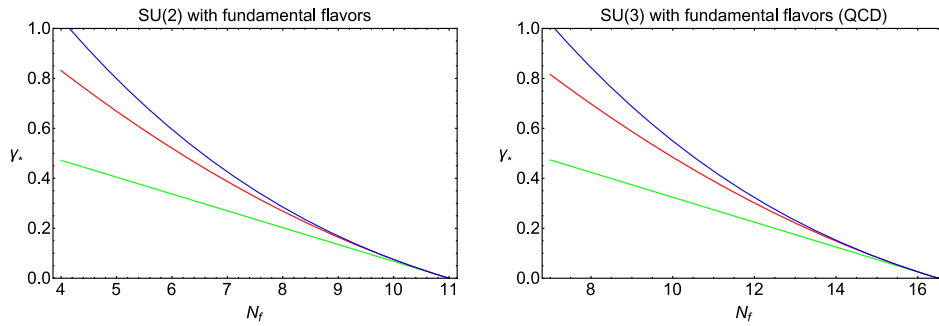


FIG. 2. The value of γ_* for a fermionic gauge theory with gauge group $SU(2)$ (left) and $SU(3)$ (right) and N_f fundamental flavors.

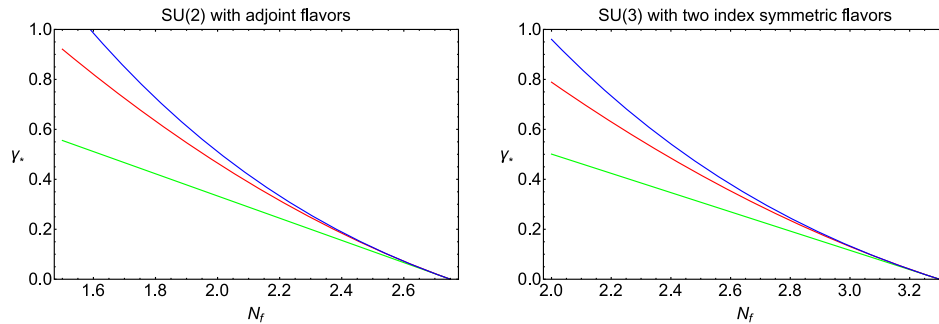


FIG. 3. The value of γ_* for a fermionic gauge theory with gauge group $SU(2)$ (left) and $SU(3)$ (right) and N_f two index symmetric flavors. For two colors the two index symmetric representation is equivalent to the adjoint representation.

TABLE I. Values of γ_* calculated through $O(\Delta_f^3)$ for N_f fermions in the fundamental representation of an $SU(2)$ or $SU(3)$ gauge group. The critical value where asymptotic freedom is lost is in the two cases marked by $\bar{N}_f = 11$ and $\bar{N}_f = 16.5$, respectively.

$SU(2)$										
N_f	4	5	6	7	8	9	10			
γ_*	1.04	0.799	0.596	0.426	0.285	0.169	0.0754			
$SU(3)$										
N_f	7	8	9	10	11	12	13	14	15	16
γ_*	1.02	0.844	0.687	0.549	0.428	0.323	0.231	0.152	0.0841	0.0259

$SU(2)$. For fermions in the two index symmetric representation of $SU(3)$, only the $N_f = 2$ and $N_f = 3$ theories are infrared conformal.

There are two theories with higher dimensional representations that have received much attention as potential strongly interacting theories able to break the electroweak symmetry [19]. The first is $SU(2)$ with $N_f = 2$ adjoint flavors. This theory has $\gamma_* \sim 0.511$ and must be assumed to lie within the conformal window. The second is $SU(3)$ with $N_f = 2$ two index symmetric flavors for which $\gamma_* \sim 0.960$. This theory seems to lie just around the boundary of the conformal window and could potentially exhibit walking dynamics. It provides a value of γ_* in the desired range of the order unity and is minimal with only two Dirac flavors, not to be in conflict with precision measurements.

We also would like to make a comment on the convergence and accuracy of our result. In the supersymmetric case, the ratio of two consecutive expansion coefficients is $[(c_{n+1})/c_n] = [2T_r/(4C_A)]$. For supersymmetric QCD with $SU(3)$ gauge group and fundamental superflavors, this is $[(c_{n+1})/c_n] \sim 0.11$. In the nonsupersymmetric case, on the other hand, we find $(c_2/c_1) \sim 0.076$ and $(c_3/c_2) \sim 0.063$. Hence, the first expansion coefficients decrease more rapidly in the nonsupersymmetric case compared to the supersymmetric case. If this continues to higher orders, the radius of convergence and the accuracy

TABLE II. Values of γ_* calculated through $O(\Delta_f^3)$ for N_f fermions in the two index symmetric representation of an $SU(2)$ or $SU(3)$ gauge group. The critical value where asymptotic freedom is lost is in the two cases marked by $\bar{N}_f = 2.75$ and $\bar{N}_f = 3.3$, respectively. For $SU(2)$ the two index symmetric representation is equivalent to the adjoint representation.

$SU(2)$		
N_f	2	$\frac{5}{2}$
γ_*	0.511	0.127
$SU(3)$		
N_f	2	3
γ_*	0.960	0.133

of our results will be even more favorable than in the supersymmetric case. The same pattern holds for the other gauge groups and representations discussed here.

We stress that the above theories have been subject to thorough studies by the lattice community in recent years. For a recent review, see [20]. Our investigations should serve as an analytic background against which the lattice simulations should be compared. Here, however, there seems to be some conflict with the existing lattice literature, since some (but not all) investigations predict a value of γ_* slightly smaller than the one predicted here. We imagine that such discrepancies will be resolved in the foreseeable future by the lattice groups such that a direct comparison with our results will rest on a firmer basis.

Having in mind the high level of accuracy we have seen to exist for supersymmetric theories, we cannot help but speculate that the computation of γ_* through $O(\Delta_f^3)$ is, at least, equally precise for QCD and similar nonsupersymmetric fermionic gauge theories.

Finally, we make a brief comment on the adjoint theory. Occasionally, it has been speculated that the physics of a $SU(N)$ gauge theory with a set of adjoint flavors inside the conformal window should be independent of N . We can, finally, show that this is not the case. Although $c_1 = \frac{4}{9}$ and $c_2 = \frac{341}{1458}$ are both independent on N , actually $c_3 = \{[61873 - (42624/N^2)]/(472392)\}$ has a mild dependence on N . Note that ζ_3 drops out. Since these coefficients are exact, γ_* is bound to have at least some minor N dependence. Potentially, this could change the boundary of the conformal window as a function of N .

Summary.—Predicting the value of critical exponents in a reliable way is of vital importance in both condensed matter and particle physics. For instance, in condensed matter physics, they are needed in order to understand certain phase transitions. In particle physics, such physical quantities are often difficult to calculate due to the strong interactions that are involved between the elementary particles. Here, we have proposed a way of calculating such critical exponents. The method makes use of only standard perturbation theory but in a quite novel way. It matches earlier known results very accurately.

Last, in the literature, there are many proposals where physics that might appear at distances currently probed by the Large Hadron Collider is described by composite states. Our results not only give us a better understanding of strongly interacting physical systems in general, but also shed light on the possibility of observing the Higgs boson and/or dark matter particles to be composite.

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