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Published in:

DOI:
10.1093/mnrasl/slaa146

Publication date:
2020

Document version:
Final published version

Citation for published version (APA):

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Observations in statistically homogeneous, locally inhomogeneous cosmological toy models without FLRW backgrounds

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Accepted 2020 August 14. Received 2020 August 14; in original form 2020 July 24

ABSTRACT

We study observations in toy models that constitute exact cosmological solutions to the Einstein equation. These models are statistically homogeneous but locally inhomogeneous, without an a priori introduced Friedmann–Lemaître–Roberston–Walker (FLRW) background and with ‘structures’ evolving fairly slowly. The mean redshift–distance relation and redshift drift along 500 light rays in each of two models are compared with relations based on spatial averages. The relations based on spatial averages give a good reproduction of the mean redshift–distance relation, although most convincingly in the model where the kinematical backreaction and average spatial curvature cancel each other to a subpercentage precision. In both models, the mean redshift drift clearly differs from the drift of the mean redshift. This indicates that redshift drift could be an important tool for computing the redshift drift is straightforward to generalize and can thus be utilized to fairly easily compute this quantity in a general space–time.

Key words: large-scale structure of Universe - cosmology: observations - cosmology: theory.

1 INTRODUCTION

Modern cosmology is based on the Friedmann–Lemaître–Roberston–Walker (FLRW) solution to the Einstein equation. The dynamics of an FLRW universe is given by the Friedmann equations (in the subscripts, commas followed by one or more coordinates or indices indicate partial derivatives, and \(a = 1\)):

\[
\begin{align*}
\frac{\dot{a}}{a} &= \frac{8\pi G}{3} \rho - \frac{\kappa}{R_0^2} a + \frac{\Lambda}{3}, \\
\frac{\ddot{a}}{a} &= -\frac{4\pi G}{3} (\rho + 3p) + \frac{\Lambda}{3}. \quad (2)
\end{align*}
\]

Here, \(\rho\) is the density, \(p\) is the pressure and \(a\) is the scalefactor appearing in the FLRW line element,

\[
ds^2 = -dt^2 + a^2 \left( \frac{dr^2}{1 - k/r^2} + r^2 d\Omega^2 \right),
\]

where \(k = \kappa r^2 / R_0^2\) is the curvature parameter (\(\kappa = \pm 1, 0\)).

Unlike an FLRW universe that is spatially exactly homogeneous and isotropic, the real Universe is at most spatially statistically homogeneous and isotropic. This difference is a potentially vital detail because the spatially averaged expansion of a generic inhomogeneous universe in general deviates from the FLRW evolution. This deviation is known as a cosmic backreaction (Clarkson et al. 2011; Rasanen 2011; Buchert & Rasanen 2012) and it can be explained by the fact that, in general relativity, spatial averages and time derivatives do not commute.

In the case where averages are computed on hypersurfaces orthogonal to the fluid velocity field and with a metric lapse function equal to 1, the equations describing the average evolution of the Universe, called the Buchert equations (Buchert 2001), are

\[
\begin{align*}
\frac{1}{3} (\theta^2) &= 3 \left( \frac{a_D}{a_D} \right)^2 = 8\pi G_N (\rho) - \frac{1}{2} (\theta^2) + \Lambda - \frac{1}{2} Q, \\
\iff \Omega_\theta + \Omega_R + \Omega_\mu + \Omega_\lambda &= 1, \quad (3)
\end{align*}
\]

Triangle brackets denote spatial averaging of a scalar, i.e. \((s) := \langle \int_D sg^{(3)} d^3x \rangle / \langle \int_D g^{(3)} d^3x \rangle\), where \(g^{(3)} d^3x\) is the infinitesimal spatial volume element, \(D\) is the spatial domain of averaging and \(s\) is some scalar. The volume-averaged scalefactor is defined through the proper volume of a spatial averaging domain such that

\[
a_D := \left[ \frac{\int_{D_0} g^{(3)} d^3x}{\int_{D_0} g^{(3)} d^3x} \right]^{1/3}
\]

where subscripted zeros indicate evaluation at the present time. As equation (3) shows, the evolution of \(a_D\) is determined by the spatially averaged local expansion rate, \(\theta\). Density parameters, \(\Omega_x\), are defined by dividing the respective terms in equation (3) by \(3H_D^2 := 3(a_D/a_D)^2\). Note that \(\Lambda\) is included in the above equations for completeness but is set to zero in the studied models.

By comparing with the Friedmann equations, we can see that there is an extra term in the Buchert equations (i.e. \(Q\), which is known as the kinematical backreaction). The kinematical backreaction is defined by \(Q := (2/3) (\langle \theta^2 \rangle - \langle \theta^2 \rangle ) - 2 \langle \sigma^2 \rangle\), where \(\sigma^2 := (1/2)\sigma_{\mu\nu}\sigma^{\mu\nu}\) is the shear scalar of the fluid. In addition to the kinematical
backreaction, the Buchet equations differ from the Friedmann equations by the curvature term, which in the Buchet equations is given by the spatial average of the hypersurface Ricci scalar, \( (3^2) R \). This may evolve differently from the curvature in the Friedmann equations (i.e. differently from simply \( \propto a_J^2 \)).

It is unknown how cosmic backreaction affects the large-scale/average evolution of the Universe. It may turn out to be negligible but, as the equations above show, the backreaction can, for example, lead to average accelerated expansion. It has also been suggested that the backreaction may be the true explanation for the apparent late-time accelerated expansion of the Universe. However, a realistic quantification of the backreaction is highly non-trivial as it requires a realistic, general relativistic description of the Universe, which is simply not feasible at this time. Another route to quantifying the importance of the backreaction in our Universe is through observations. Specifically, several relations have been identified that can test the FLRW assumption observationally (Clarkson et al. 2008; Rasanen 2014). If observations fail these tests, then the Universe cannot be described by an FLRW metric on large scales and the backreaction is likely to be important. If observations fo fulfils these tests, it is not guaranteed that the Universe is FLRW on large scales. So, there is currently no known way of unambiguously falsifying the idea that the backreaction has an important effect on the dynamics of the Universe.

A main obstacle is that it is not known how to relate spatially averaged quantities to observations. Several methods have been proposed in the literature (e.g. Paranjape & Singh, 2008; Rosenthal & Flanagan, 2008; Larena et al., 2009, 2010); see also, for example, the discussions in Koksbang (2019a, b). However, in order to determine if any of these are accurate, they must be tested using exact solutions to the Einstein equation, and these solutions are not a priori based on FLRW backgrounds. The usual Swiss-cheese construction and, for example, relativistic codes based on weak-field approximations do not fulfill this requirement as they are based on pre-specified FLRW backgrounds. In this letter, exact, inhomogeneous, statistically homogeneous cosmological models are constructed without introducing an FLRW ‘background’ and the mean redshift–distance relation and redshift drift along 500 light rays in each model are computed and compared with relations based on spatially averaged quantities.

### 2 MODEL CONSTRUCTION

The studied model is of the type introduced in Hellaby (2012), where space is tessellated by cubes of a specific type of the homogeneous Bianchi I models (generalized Kasner models; Kasner 1925), resulting in an inhomogeneous cosmological model.

The considered local line element of a cube is

\[
\text{ds}^2 = -\text{d}t^2 + \left( \frac{1}{t_0} \right)^{2\alpha} \text{d}x^2 + \left( \frac{1}{t_0} \right)^{2\beta} \text{d}y^2 + \left( \frac{1}{t_0} \right)^{2\gamma} \text{d}z^2,
\]

where \( \alpha, \beta, \gamma \) are constants and \( t_0 \) is present time.

This metric fulfills the Einstein equation for a comoving perfect fluid with homogeneous density and homogeneous, anisotropic pressure (\( p_x \neq p_y \neq p_z \), in general); see Hellaby (2012) for details. The local expansion rate of the fluid is \( \theta = (\alpha \beta + \beta \gamma + \gamma \alpha) / t \). The shear scalar is

\[
\sigma^2 = \frac{2}{3t^4} \left[ (\alpha^2 + \beta^2 + \gamma^2) - (\alpha \beta + \alpha \gamma + \gamma \beta) \right] = -{(3^2)} R.
\]

As in Hellaby (2012), eight cubes are arranged in a ‘fundamental’ block that is used to tessellate all of space in order to construct an inhomogeneous cosmological model, which is statistically homogeneous. To fulfills the Darmois junction conditions (Darmois 1927), those of the metric parameters \( \alpha, \beta, \gamma \) that correspond to a direction orthogonal to a junction must be constant across the junction, while there is no restriction on the parameter in the direction parallel to the junction. That is, for a junction with \( x = \text{const.} \), \( \alpha \) may change across the junction while \( \beta \) and \( \gamma \) must be constant. The arrangement of the eight cubes is illustrated in Fig. 1, with numerical values for two particular models given in Table 1.

The parameter values given in Table 1 do not correspond to realistic values of density and pressure. Indeed, because \( \rho \propto (\alpha \beta + \beta \gamma + \gamma \alpha) / t^2 \), some regions have negative density, and some regions will not have a big bang singularity in all spatial directions. This is of no issue here as the models are not meant to be realistic renderings of the Universe. The purpose is to study the principles of light propagation in an inhomogeneous universe that does not contain an FLRW background, preferably with non-negligible backreaction. The principles of light propagation do not depend on particular values of pressure and density, for example – not even the signs matter. Nonetheless, realistic values of \( \rho \) and \( p \) must generally be considered favourable. However, such a requirement was found to be difficult to fulfils while also obtaining average accelerated expansion and non-negligible kinematical backreaction without introducing large regions expanding or contracting very fast in one or more directions, making the models very impractical for a light propagation study. The question of whether \( \rho \) and \( p \) take on realistic values was therefore not considered when choosing parameter values. The parameter values of model 1 were chosen to lead to a late-time average accelerated expansion without local accelerated volume expansion, while keeping expansion rates and regions small enough for ‘structures’ not to evolve much during the time it takes light rays to traverse the homogeneity scale (a natural requirement for expecting a simple relation between observables and spatial averages; Rasanen 2009, 2010). Model 1 has \( \Omega_k + \Omega_\Lambda \approx 0 \), so even though \( \Omega_\Lambda \) is not numerically small, \( \Omega_k \approx 1 \) to subpercentage precision. Nonetheless, the model is interesting for light propagation studies as the model everywhere locally behaves quite differently from the spatial average. This is seen in Fig. 2, which shows the expansion rates of both models. Model 2 does not have average accelerated expansion but it has roughly \( \Omega_\Lambda \approx [0.89, 0.95] \) and \( \Omega_k \approx 0.11 \neq -\Omega_k \) in the total time interval along the studied light rays.

### Figure 1.

Two-dimensional rendering of the fundamental Bianchi I block with their values of \( \alpha, \beta \) and \( \gamma \) as well as their size and numbering (\( \alpha \) takes the value a or A, \( \beta \) takes the value b or B, and \( \gamma \) takes the value g or G). The cubes on the left side of the figure have a comoving height of \( dz_1 \), and those to the left \( dz_2 \). The two sets of cubes are stacked on top of each other such that cube 1 is below cube 5, etc.
Integrating this, we see that this implies that the effect by observing in many, random directions. spatial averages and mean observations, so it is important to remove of statistical isotropy is expected to impair any relationship between random because the space–time is not statistically isotropic. A lack I model at the point of observation. The lines of sight must be random because the mean spatial position of the observers must be random because the mean position of the present-time (computed along 500 light rays in each model. For each light ray, the results would otherwise be biased according to the local Bianchi spatial positions of the observers must be random because the mean of observation are random. As the model is inhomogeneous, the spatial position of the present-time (t = t0) observer and the direction of observation is random. As the model is inhomogeneous, the spatial position of the observers must be random because the mean results would otherwise be biased according to the local Bianchi I model at the point of observation. The lines of sight must be random because the space–time is not statistically isotropic. A lack of statistical isotropy is expected to impair any relationship between spatial averages and mean observations, so it is important to remove the effect by observing in many, random directions.

Light paths are computed from the geodesic equations

\[
\frac{\,d}{\,d\lambda} \left( g_{\alpha\beta} \dot{k}^\beta \right) = \frac{1}{2} g_{\mu\nu;\alpha} k^\mu k^\nu,
\]

where the Einstein summation convention is used and Greek indices are space–time indices while Latin indices are pure space indices. On the junctions between different Bianchi I cubes with the junction of the form \( x' = \) const., \( g_{\mu\nu;\alpha} \) contains a \( \delta \) (delta-Dirac) function describing the change in \( \alpha, \beta \) or \( \gamma \) across the junction. For instance, at a boundary at constant \( x = x_0 \) where \( \alpha \) changes by \( \Delta \alpha \), the equation for \( \frac{\,d}{\,d\lambda} \) contains the term

\[
- \frac{1}{2} \frac{\,g_{xx\alpha}}{\,g_{xx}} (k^\alpha)^2 = - (k^\alpha)^2 \log \left( \frac{\,t}{\,t_0} \right) \delta(x - x_0) \Delta \alpha.
\]

Integrating this, we see that this implies that \( k^\alpha \) is modified according to

\[
k^\alpha \rightarrow k^\alpha + k^\alpha \log \left( \frac{\,t}{\,t_0} \right) \Delta \alpha
\]

when the light ray crosses the boundary. Numerically, this is seen to correspond to a renormalization of \( k^\alpha \) so that the light ray remains null. Thus, in general, the effect of the \( \delta \) function is simply to renormalize \( k^\alpha \) so that the geodesic remains null. The \( \delta \) functions do not appear in the Riemann tensor and hence in the transport equation \( \frac{\,d^2 D^\mu}{\,d\lambda^2} = T^\mu_{\rho} D^\rho_{\nu} \), from which the angular diameter distance is obtained (through \( D_\lambda = \sqrt{|\det[D]|} \)). The tidal matrix has the components

\[
T_{\alpha\beta} = \begin{bmatrix} R - Re(F) & Im(F) \\ Im(F) & R + Re(F) \end{bmatrix},
\]

where \( R := -(1/2)R_{\alpha\mu\nu} k^\mu k^\nu \) and \( F := -(1/2)R_{\alpha\mu\nu\rho} (e^\nu)^{\rho\mu}(e^\nu)^{\rho\nu} k^\nu \). Here, \( R_{\alpha\mu\nu} \) is the Ricci tensor, \( R_{\alpha\mu\nu\rho} \) is the Riemann tensor and \( e^\mu := E^\mu_0 + i E^\mu_1 \) where \( E^\mu_0, E^\mu_1 \) are orthonormal vectors spanning the space orthogonal to the propagation of a light ray in the rest frame of the observer (taken to be comoving with the fluid with \( u^\nu = \frac{\partial}{\partial \lambda} \)).

The redshift drift, \( \delta z \), describes the change/drift in the redshift of a comoving source as measured by a comoving observer as a function of observer proper time (Sandage 1962; McVittie 1962); non-comoving effects have also been studied (Bolejko, Wang & Lewis 2019; Marcori et al. 2018) but are not considered here. In an FLRW universe, \( \delta z = \delta t_0 (1 + \dot{z}) (\alpha(t_0) - \alpha(t)) \), which means that \( \delta z \) measures space–time expansion, and late-time accelerated expansion will show as \( \delta z > 0 \) for small \( z \). For the model studied

<table>
<thead>
<tr>
<th>Model</th>
<th>(a, b, g)</th>
<th>(A, B, G)</th>
<th>(dx1, dy1, dz1)</th>
<th>(dx2, dy2, dz2)</th>
<th>t0 (Gyr)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1, 1, 1)</td>
<td>-0.05 · (1, 1, -1)</td>
<td>30 · (1, 1, 1)</td>
<td>10 · (1, 1, 1)</td>
<td>8.9</td>
</tr>
<tr>
<td>2</td>
<td>(1/5, 2/3, 1/4)</td>
<td>10 · (1, 1, 1)</td>
<td>30 · (1, 1, 1)</td>
<td>7</td>
<td></td>
</tr>
</tbody>
</table>

Table 1. Model parameters of Bianchi I cubes in the fundamental blocks. The parameters a and A, b and B, g and G refer to values of \( \alpha, \beta \) and \( \gamma \), respectively, in different cubes of the fundamental block, as illustrated in Fig. 1. Similarly, dx1, dx2, etc., refer to the comoving side lengths of the cubes of the fundamental block, according to the illustration in Fig. 1.
Figure 3. Average, mean and spread of redshift–distance relation and redshift drift along 500 light rays compared with predictions based on spatial averages. The average and mean redshift–distance relations are indistinguishable in the figure for model 1. The redshift drift was computed using \( \delta t_0 = 30 \text{ yr} \).

\[
\frac{\delta z}{\delta t_0} = \frac{dz}{dt_0} = \left[ \frac{(k^\mu u_\mu)_0}{(k^\alpha u_\alpha)_0} \right] = \left[ \frac{\partial}{\partial x_\nu} \frac{d k^\mu}{d \lambda} - k^\nu_k k^\mu_\beta \right]_{|_0}.
\]

where the subscript \( e \) denotes evaluation at the space–time point of emission. The top line in the equation is valid for any space–time so the above illustrates a fairly simple general method for computing the redshift drift. We used \( \frac{dt_0}{d \epsilon} = 1 + z \) and \( \frac{dk^\nu}{d \lambda} = k^\nu_k k^\mu_\beta \). The last line is included to emphasize that \( \delta z \), unlike \( z \), is not given solely by an integral along the light ray, but also depends on local qualities of space–time through boundary terms (i.e. through \( \frac{dk^\nu}{d \lambda} \)). For a homogeneous space–time, \( k^\nu_k = 0 \) and the above can be used to compute the redshift drift if one solves the geodesic equations. This is, for example, the case for FLRW and Bianchi I space–times, but for a space–time with several Bianchi I regions joined, \( \delta \) functions on boundaries between different regions lead to \( k^\nu_k \neq 0 \). In order to obtain an expression for \( k^\nu_k \) (or \( k^\nu_\beta \)), the geodesic equations are differentiated as in Nwankwo, Ishak & Thompson (2011), yielding

\[
\frac{d}{d \lambda} k^\mu_\nu = \frac{\partial}{\partial x^\nu} \frac{dk^\mu}{d \lambda} - k^\nu_k k^\mu_\beta,
\]

which are solved simultaneously with the geodesic equations and the transport equation. Most of the equations for \( k^\mu_\nu \) contain \( \delta \) functions related to the junctions between different regions. When crossing the boundary of \( x' = \text{const.} \), the \( \delta \) functions in \( \frac{dk^\nu}{d \lambda} \) and \( \frac{dk^\mu_\nu}{d \lambda} \) are non-zero. Their contributions are taken into account by renormalizing \( k^\nu_\nu \) with the partial derivatives of the null condition and by renormalizing \( k^\nu_\beta \) through the definition \( \frac{dk^\nu}{d \lambda} = k^\nu_k k^\mu_\beta \).

As each Bianchi I region is locally homogeneous, \( k^\nu_\beta = 0 \) can be used as initial conditions when solving \( k^\mu_\nu / d \lambda \). The initial conditions for \( k^\nu_\nu \) are then simply \( (1/k^\nu_\nu)(dk^\nu / d \lambda) \).

The results from applying the above set of equations to the models specified in Table 1 are shown in Fig. 3. Specifically, the figure shows the mean and spread of the redshift–distance relation and redshift drift along the 500 light rays in each model. These exact results are compared with the average redshift–distance relation, \( (D^D_{\Lambda, z_{\Lambda}})^2 \), proposed in Rasanen (2009, 2010) to be given by

\[
H_D \frac{d}{dz_D} \left[ (1 + z_D)^2 H_D \frac{d D^D_{\Lambda}}{dz_D} \right] = -4\pi G (\rho + p) D^D_{\Lambda},
\]
where \( z_D := (1/a_D) - 1 \) and the appropriate average of the anisotropic pressure is set as \( \langle p \rangle := (1/3) \left( p_x + p_y + p_z \right) \), which is assessed by considering spatial directions of light rays along coordinate axes and comparing with Rasanen (2010). Note that a term representing the average null-shear has been dropped in the equation because the random directions of the sampled light rays should effectively lead this term to be negligible.

Fig. 3 shows a good agreement between the mean and average redshift–distance relation in both models with the notable difference that, in model 1, the agreement is nearly exact while it is clearly only approximate for model 2. In both models, there is a significant difference between the mean redshift drift and the drift of the mean redshift, \( \delta z_D := \delta t_0 \left( 1 + z_D \right) H_0^D - H_D \) (see Koksbang & Hannestad 2016). This result is in agreement with that found in Koksbang (2019b), which was, however, based on a toy model of disjoint FLRW regions and not an exact solution to the Einstein equation. Note also that the spread around the mean of \( \delta z \) is very large. This is presumably because of the anisotropy of the model and the large local effects, which are not expected to be seen with the real Universe.

4 SUMMARY

By studying light propagation in two exact cosmological solutions to the Einstein equation, we have shown that spatial averages can be used to describe the mean redshift–distance relation through an FLRW-like ‘average’ redshift–distance relation. We have also shown that the drift of the average redshift entering this average redshift–distance relation is not equal to the mean redshift drift. This difference in the average-versus-mean relation for the two types of observations implies that redshift drift can be useful for quantifying the importance of the backreaction; the results found here indicate that a non-negligible backreaction should be expected to lead to a clear disagreement between observations based on the redshift–distance relation and redshift drift. It is especially interesting that the mean redshift drift is negative while the drift of the mean redshift is positive, in agreement with what was found in Koksbang (2019b). However, the anisotropy of the models makes it unclear if the mean redshift drift will necessarily be negative if there is no local accelerated expansion.

ACKNOWLEDGEMENTS

The author thanks Syksy Rasanen for comments on the manuscript. Part of the numerical work was done using computer resources from the Finnish Grid and Cloud Infrastructure urn:nbn:fi:research-infras-2016072533. During the review process, the author transitioned from being supported by the Independent Research Fund Denmark under grant number 7027-00019B to being supported by the Carlsberg Foundation.

DATA AVAILABILITY STATEMENT

The data and/or code used to generate the data and results presented here will be shared on reasonable request to the author.

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