Review of Fernandez-Duque "Dynamic Topological Logic of Metric Spaces"

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The article is a contribution to dynamic topological logic $\mathcal{DTL}$, which is a modal logic for reasoning about dynamical systems. A dynamic system is considered to be a structure $\mathfrak{X} = \langle |\mathfrak{X}|, \mathcal{T}_\mathfrak{X}, f_\mathfrak{X} \rangle$ in which $\langle |\mathfrak{X}|, \mathcal{T}_\mathfrak{X} \rangle$ is a topological space and $f_\mathfrak{X}$ a continuous function whose domain and range is $|\mathfrak{X}|$. Such structures are called “dynamic topological systems” ($\mathcal{DTS}$). In order to reason about such systems a language of modal logic is used which extends classical propositional logic by three modalities: $\Box$ (‘interior’), $(f)$ (‘next’), and $[f]$ (‘henceforth’). An $X$-assignment assigns subsets of $|\mathfrak{X}|$ to the propositional variables of the language, and a dynamic topological model ($\mathcal{DTM}$) is the extension of a dynamic topological system by an assignment. $X$-assignments are extended to all formulas by interpreting the standard connectives by their corresponding Boolean operations and the modal operators according to the following rules:

$$
\begin{align*}
[\Box \varphi]_X &= [\varphi]_X^X, \\
[\neg f] \varphi}_X &= f_X^{-1} [\varphi]_X \\
[[f] \varphi]_X &= \bigcap_{n \geq 0} f_X^{-n} [\varphi]_X
\end{align*}
$$

(Here $X^0$, for $X \subseteq |\mathfrak{X}|$, is the interior of $X$.) Satisfaction within a $\mathcal{DTM}$ as well as satisfiability and validity in a single $\mathcal{DTS}$ and in a class of $\mathcal{DTS}$s is defined in the standard way. If $X \subseteq \mathcal{DTS}$, then $\mathcal{DTS}_X$ is the set of formulas valid in the elements of $X$. The set $\mathcal{DTS}$ of all valid formula is, of course, $\mathcal{DTS}_\mathcal{DTS}$. It is proved that any formula $\varphi$ which is satisfiable at all is already satisfied in a $\mathcal{DTM}$ which is based on a perfect countable metric space (i.e., a countable metric space without isolated points). Since any such space is homeomorphic to $\mathbb{Q}$, it follows as a corrollary that $\mathcal{DTS}$ is complete with respect to $\mathbb{Q}$, i.e., that $\mathcal{DTS} = \mathcal{DTS}_{\mathbb{Q}}$.

The second part of the paper is devoted to the proof that a corresponding completeness result does not hold true for the class $\mathcal{CompM}$ of complete metric spaces. The proof makes use of the Bair characterization of complete metric spaces. A formula $\text{Baire}$ which “translates” the Baire Category Theorem (that in a complete metric space a countable collection of dense, open sets is dense itself) into the language of $\mathcal{DTS}$ is constructed (from two propositional variables $p$ and $s$) as follows:

$$
\begin{align*}
\varphi_0 &= \Diamond s \land \Diamond \neg s, \\
\varphi_1 &= s \leftrightarrow (f)s, \\
\varphi_2 &= p \rightarrow \Box p, \\
\varphi_3 &= s \land \neg p \rightarrow \Box (\neg s \rightarrow p), \\
\text{Baire} &= \Box [f] \bigwedge_{n \leq 3} \varphi_n \rightarrow \Diamond [f]p.
\end{align*}
$$

(Here, as is usual, $\Diamond \varphi$ is defined as $\neg \Box \neg \varphi$.) It is shown then that $\text{Baire} \in \mathcal{DTS}_{\mathcal{CompM}}$ (as one may expect from the Baire Category Theorem, which in
effect says that a complete metric space is a Baire space). On the other hand, however, the author specifies a model over \( \mathbb{Q} \) which does not satisfy Baire, hence \( \text{Baire} \notin \mathcal{D}L_{\mathbb{Q}} \). Actually, the latter is proved in two ways: first by constructing a DTM, and then by using “quasimodels”. Quasimodels resemble the more usual Kripkean models else used in modal logic (cf. §3 of the article) and hence render it possible to apply standard techniques of modal logic (like the use of bisimulations) to \( \mathcal{D}L \). DTMs can be constructed from quasimodels in such a way that formulas which are satisfiable at all are already satisfied in a DTM generated from a special quasimodel; cf. the article’s §4.

In the third part of the article it is proved that the Cantor set \( K \) plays a similar role for the class \( \text{CompM} \) as \( \mathbb{Q} \) does for the entire class \( \text{DTS} \): it is shown that \( \mathcal{D}L_{\text{CompM}} = \mathcal{D}L_{K} \). — Combining the results achieved in the present paper with already known results of the author [MR2289646] as well of other researchers [MR2097225, MR2191470, MR2460927, and MR2609948], the relationships between sets of formulas valid in different classes of topological spaces can be depicted in the following diagram.

\[
\begin{array}{ccc}
\mathcal{D}L_{R} & \mathcal{D}L_{C} & \mathcal{D}L_{A} \\
\mathcal{D}L_{C} & \mathcal{D}L_{K} & \mathcal{D}L_{\text{CompM}} \\
\mathcal{D}L_{Q} & \mathcal{D}L \\
\end{array}
\]

Here the arrows indicates proper inclusion, \( C \) is the complex plane, and \( A \) the class of systems based on Alexandrov spaces.

Reviewed by Klaus Robering