Notes for X-MEM 1
Industrial Membrane Technology

2nd Edition
by Knud Christensen

UNIVERSITY OF SOUTHERN DENMARK
Institute of Chemical Engineering, Biotechnology
and Environmental Technology

2007
Preface

The Boy stood on the burning deck,
Whence all but him had fled;

Felicia Hermans

To determine the size, strength and ability of a specific piece of equipment to accomplish a given task has always been the job of the engineer and for a chemical engineer this has always meant working with molar, energy and momentum balances, difficult to setup and often impossible to solve.

These notes have been prepared on the request of the chemical engineering students that participated in the course K-MEM1 Industrial Membrane Technology at Odense University College of Engineering (OUCE) in the years 2003 and 2004. During this period it became apparent that most textbooks concerning membrane technology what ever their many other merits did not contain a systematic approach to mass or molar balances for membrane systems nor design equations beyond the very basic level. During the lectures this approach was therefore developed and have now precipitated as the notes presented here.

As the notes presented here are only concerned with the molar and energy balances for membrane systems it is vital that they are used only as a supplement to a text book on membrane technology. Though the notes are not specifically adjusted to R. Bakers book Membrane Technology and Applications, it is my recommendation as a main text book.

It follows from this introduction that the notes intended audience is undergraduate students in their final year or graduate students taking an elective course in industrial membrane technology.

Since the notes primary targets are chemical engineering students at University of Southern Denmark (SDU) I have decided to use the same nomenclatur as S. Fogler in his book Elements of Chemical Reaction Engineering [1] as this text is part of the chemical engineering curriculum at SDU and the nomenclatur therefore familiar to most students.

In the hope that this book will help students to better cope with the often painful task of setting up basic design equations from scratch

Knud Christensen
Institute of Chemical Engineering, Biotechnology and Environmental Technology
University of Southern Denmark
Odense
Denmark
2006
# Table of Contents

**Preface** ................................................................. 1  

1. General Mole Balance Equation over a Membrane System ............................................. 1  

2. Mole Balance Equations for Specific Membrane Systems ............................................. 5  
   2.1 Mole Balance and Design Equation for Dead-end filtration ........................................ 5  
   2.1.1 Dead-end Filtration With Reaction ................................................................. 5  
   2.1.2 Dead-end Filtration Without Reaction ......................................................... 11  
   2.2 Mole Balance and Design Equation for Perfectly Mixed-Mixed Flow Membrane Module (M&M-MM) ............................................................... 16  
   2.2.1 Perfectly Mixed-Mixed Flow Membrane Module with Reaction ......................... 16  
   2.2.2 Perfectly Mixed-Mixed Flow Membrane Module without Reaction ..................... 19  
   2.3 Mole Balance and Design Equation for Plug flow-Perfectly Mixed Flow Membrane Module (PF&M-MM) ................................................................. 26  
   2.3.1 Plug Flow-Perfectly Mixed Flow Membrane Module with Reaction .................... 26  
   2.3.2 Plug Flow-Perfectly Mixed Flow Membrane Module without Reaction ............... 33  
   2.4 Mole Balance and Design Equation for Cocurrent Plug Flow Membrane Module (coPF-MM) ................................................................. 40  
   2.4.1 Cocurrent Plug Flow Membrane Module with Reaction ................................... 40  
   2.4.2 Cocurrent Plug Flow Membrane Module without Reaction ................................ 42  
   2.5 Mole Balance and Design Equation for Countercurrent Plug Flow Membrane Module (counterPF-MM) ................................................................. 46  
   2.5.1 Countercurrent Plug Flow Membrane Module with Reaction ............................. 46  
   2.4.2 Countercurrent Plug Flow Membrane Module without Reaction ....................... 48  
   2.6 Mole Balance and Design Equation for Plug Flow - Differential Cross Flow Membrane Module (PF&DCF-MM) ................................................................. 52  
   2.6.1 Plug Flow - Differential Cross Flow Membrane Module (PF&DCF-MM) with Reaction ................................................................. 52  
   2.6.2 Plug Flow - Differential Cross Flow Membrane Module (PF&DCF-MM) without Reaction ................................................................. 55  
   2.7 Mole Balance and Design Equation for Cross Flow Membrane Module (CF-MM) ....... 63  
   2.7.1 Cross Flow Membrane Module (CF-MM) with Reaction .................................. 63  
   2.7.2 Cross Flow Membrane Module (CF-MM) without Reaction ............................... 66  

3. General Energy Balance for a Membrane System .......................................................... 67  

4. Symbols ................................................................. 73  

5. Literature ............................................................... 78  

Postscript ................................................................. 78
1. General Mole Balance Equation over a Membrane System

The general mole balances for a generalized membrane module consisting of a membrane, M, system I and system II as shown in figure 1.1 is:

$$\begin{align*}
\text{System: } & \quad \text{In} + \text{Generated} = \text{Out} + \text{Accumulated} \\
\text{I: } & \quad F_{jI}^\text{In} \cdot \Delta t + G_{jI} \cdot \Delta t = F_{jI}^\text{M} \cdot \Delta t + F_{jI}^\text{Out} \cdot \Delta t + \Delta N_{jI} \\
\text{II: } & \quad F_{jII}^\text{In} \cdot \Delta t + G_{jII} \cdot \Delta t = F_{jII}^\text{M} \cdot \Delta t + F_{jII}^\text{Out} \cdot \Delta t + \Delta N_{jII} \\
\text{M: } & \quad F_{jM}^\text{In} \cdot \Delta t + F_{jM}^\text{Out} \cdot \Delta t + G_{jM} \cdot \Delta t = 0 + \Delta N_{JM}
\end{align*}$$

(1.1)

Where

- $F_{jI}^\text{In}$ is the molecular flow rate of component j to system I [mole j s$^{-1}$]
- $F_{jII}^\text{In}$ is the molecular flow rate of component j to system II [mole j s$^{-1}$]
- $F_{jM}^\text{In}$ is the molecular flow rate of component j from system I to the membrane [mole j s$^{-1}$]
- $F_{jM}^\text{Out}$ is the molecular flow rate of component j from system I to the membrane [mole j s$^{-1}$]
- $F_{jII}^\text{Out}$ is the molecular flow rate of component j from system II [mole j s$^{-1}$]
- $G_{jM}$ is mole j generated in membrane [mole j s$^{-1}$]
- $G_{jI}$ is mole j generated in system I [mole j s$^{-1}$]
- $G_{jII}$ is mole j generated in system II [mole j s$^{-1}$]
- $N_{jM}$ is number of moles j in membrane [mole j]
- $N_{jI}$ is number of moles j i system I [mole j]
- $N_{jII}$ is number of moles j i system II [mole j]
- j denotes component number j
- $\Delta$ denotes a finite interval
Concentrating first on system I the mole balance can be rewritten into an apparent ordinary first order differential equation:

\[
\frac{\Delta N_{jI}}{\Delta t} = \frac{F_{jI}|_{\text{in}} - F_{jI}|_{\text{out}} - F_{jI}|_{M} + G_{jI}}{G_{jI}}
\]

\[\Delta t \to 0\]

\[
\frac{dN_{jI}}{dt} = \frac{F_{jI}|_{\text{in}} - F_{jI}|_{\text{out}} - F_{jI}|_{M} + G_{jI}}{G_{jI}}
\]

(1.2)

The same procedure applies for system II:

\[
\frac{\Delta N_{jII}}{\Delta t} = \frac{F_{jII}|_{\text{in}} - F_{jII}|_{\text{out}} - F_{jII}|_{M} + G_{jII}}{G_{jII}}
\]

\[\Delta t \to 0\]

\[
\frac{dN_{jII}}{dt} = \frac{F_{jII}|_{\text{in}} - F_{jII}|_{\text{out}} - F_{jII}|_{M} + G_{jII}}{G_{jII}}
\]

(1.3)

while the mole balance for the membrane becomes

\[
\frac{\Delta N_{jM}}{\Delta t} = \frac{F_{jM}|_{M} + F_{jM}|_{M} + G_{jM}}{G_{jM}}
\]

\[\Delta t \to 0\]

\[
\frac{dN_{jM}}{dt} = \frac{F_{jM}|_{M} + F_{jM}|_{M} + G_{jM}}{G_{jM}}
\]

(1.4)
If an expression for the rate of reaction of component \( j \) in the membrane, system I and system II is known the amount of component \( j \) generated in the membrane, system I and system II can be expressed as [1]:

\[
\begin{align*}
\text{I: } G_{ji} &= \int_r r_{ji} \, dV_i \\
\text{II: } G_{ji} &= \int_r r_{ji} \, dV_{ii} \\
\text{M: } G_{ji} &= \int_r r_{ji} \, dV_M
\end{align*}
\]

(1.5)

where

\( r_{jm} \) is rate of formation of component \( j \) in the membrane \( [\text{mole j} \cdot \text{m}^{-3} \cdot \text{s}^{-1}] \)

\( r_{ji} \) is rate of formation of component \( j \) in system I \( [\text{mole j} \cdot \text{m}^{-3} \cdot \text{s}^{-1}] \)

\( r_{ji} \) is rate of formation of component \( j \) in system II \( [\text{mole j} \cdot \text{m}^{-3} \cdot \text{s}^{-1}] \)

\( V_M \) is the volume of the membrane \( [\text{m}^3] \)

\( V_i \) is the volume of system I \( [\text{m}^3] \)

\( V_{ii} \) is the volume of system II \( [\text{m}^3] \)

This gives the following mole balances over membrane, system I and II:

\[
\begin{align*}
\text{System: } & \text{ Accumulated } = \text{ In } - \text{ Out } + \text{ Generated} \\
\text{I: } & \frac{dN_{ji}}{dt} = F_{ji}|_{\text{in}} - F_{ji}|_{\text{out}} - F_{ji}|_{\text{M}} + \int_r r_{ji} \, dV_i \\
\text{II: } & \frac{dN_{ji}}{dt} = F_{ji}|_{\text{in}} - F_{ji}|_{\text{out}} - F_{ji}|_{\text{M}} + \int_r r_{ji} \, dV_{ii} \\
\text{M: } & \frac{dN_{jm}}{dt} = F_{ji}|_{\text{M}} + F_{ji}|_{\text{M}} - 0 + \int_r r_{jm} \, dV_M
\end{align*}
\]

(1.6)
If an expression for the mole or mass transport per unit area, the mole or mass flux, is known the mole or mass transport to the membrane can be described by the mole or mass flux $J_j$ to the membrane:

$$I: \quad F_{jI|M} = \int J_{jI} \, dA_I$$

$$II: \quad F_{jII|M} = \int J_{jII} \, dA_{II}$$

where $A_I$ is the contact area between system I and the membrane [m$^2$]

$A_{II}$ is the contact area between system II and the membrane [m$^2$]

$J_{jI}$ is the flux of component $j$ from system I to the membrane [mole j m$^{-2}$ s$^{-1}$]

$J_{jII}$ is the flux of component $j$ from system II to the membrane [mole j m$^{-2}$ s$^{-1}$]}

This finally yields the general mole balance equations for membrane, system I and system II:

$$\text{System: } \quad \frac{dN_{jI}}{dt} = F_{jI|\text{In}} - F_{jI|\text{Out}} - \int J_{jI} \, dA_I + \int r_{jI} \, dV_I$$

$$\text{II: } \quad \frac{dN_{jII}}{dt} = F_{jII|\text{In}} - F_{jII|\text{Out}} - \int J_{jII} \, dA_{II} + \int r_{jII} \, dV_{II}$$

$$\text{M: } \quad \frac{dN_{jM}}{dt} = \int J_{jI} \, dA_I + \int J_{jII} \, dA_{II} - 0 + \int r_{jM} \, dV_M$$

These rather cumbersome equations can thereafter be simplified depending on the membrane configuration and membrane process.
2. Mole Balance Equations for Specific Membrane Systems

Most industrial and laboratory membrane systems can be categorized into a few though idealized membrane system configurations. For the moment disregarding the actual separation process of the membrane these configurations primarily differs in the way the fluid, gas or liquid, flow passed or to the membrane. These idealized membrane configurations tremendously simplifies the mole balances for the membrane system and thereby simplifies the design equations to be solved when scaling industrial membranes or the mole balances to be solved when measuring fluxes in the laboratory. This section will therefore dwell in some detail on these specific membrane configurations, their matching mole balances and design equations.

Based on flow pattern the membrane configurations can be divided into

- Dead-end filtration
- Perfectly Mixed-Mixed Flow Membrane Module (M&M-MM)
- Plug Flow - Perfectly Mixed Flow Membrane Module (PF&M-MM)
- Cocurrent Plug Flow Membrane Module (coPF-MM)
- Countercurrent Plug Flow Membrane Module (counterPF-MM)
- Plug Flow - Differential Cross Flow Membrane Module (PF&DCF-MM)
- Cross Flow Membrane Module (CF-MM)

A more detailed description with examples of simple applications will be given in the following paragraphs.

2.1 Mole Balance and Design Equation for Dead-end filtration

2.1.1 Dead-end Filtration With Reaction

The intuitively simplest membrane separation method is dead-end filtration as shown in figure 2.2. In this process all particles larger than the pore opening of the membrane is held back while the rest are forced through the membrane or filter. This is a flow configuration most coffee drinkers will recognize from daily life. On an industrial scale it is used for as diverse processes as screening pebbles, removing crystals from mother liquor by simple filtration and bacteria from drinking water using microfiltration.

In dead-end filtration ideally all particles larger than the pore diameter in the filter are held back thereby creating a layer of solids on the filter. This layer is traditionally termed the filter cake. Ideally all particles smaller than the pore diameter should pass through. In real life this is not so since many smaller particles are held back in the tortuous passages formed in the filter cake. At the beginning of any dead-end filtration there will therefore be some particle sizes that pass through to the product stream that as the filter cake builds up will be held back in this. As a result a range of particle sizes where a clean cut of pass/not pass cannot be achieved exists. Based on this a mole balance (or mass balance in most cases) has to be set up for each particle size. This can be done by taking the general mole balance and simplifying it to describe dead-end filtration.
The general mole balances for membrane, M, system I and system II is given by equation (1.8):

\[
\text{System: } \frac{dN_j}{dt} = \text{In} - \text{Out} + \text{Generated} \\
\text{I: } \frac{dN_{jI}}{dt} = F_{jI|\text{In}} - F_{jI|\text{Out}} - \int J_{jI} \, dA_1 + \int r_{jI} \, dV_1 \\
\text{II: } \frac{dN_{jII}}{dt} = F_{jII|\text{In}} - F_{jII|\text{Out}} - \int J_{jII} \, dA_{II} + \int r_{jII} \, dV_{II} \\
\text{M: } \frac{dN_{jM}}{dt} = \int J_{jI} \, dA_1 + \int J_{jII} \, dA_{II} + 0 + \int r_{jM} \, dV_M
\] (1.8)
The first simplification that can be made stems from the fact that no separate product flow leaves system I nor does any separate feed flow enter system II:

\[ \begin{align*}
F_{jI}^{\text{Out}} &= 0 \\
F_{jI}^{\text{In}} &= 0
\end{align*} \]  

(2.1)

In system I the liquid above the filter cake is assumed to be well mixed thereby having the same spatial composition in the system and at the liquid filter cake interface. For system II the liquid is also assumed well mixed leading to the same spatial composition of the liquid and the membrane liquid interface.

Taking the filter cake to be part of the membrane system now leads to the following simplifications:

\[ \begin{align*}
\int_{A_I} J_{ji} \, dA_I &= J_{ji} \cdot A_I \\
\int_{V_I} r_{ji} \, dV_I &= r_{ji} \cdot V_I \\
\int_{A_{II}} J_{ji} \, dA_{II} &= J_{ji} \cdot A_{II} \\
\int_{V_{II}} r_{ji} \, dV_{II} &= r_{ji} \cdot V_{II}
\end{align*} \]  

(2.2)

This leads to the following simplification of the general mole balance:

System:

\[ \begin{align*}
\text{I:} & \quad \frac{dN_{ji}}{dt} = F_{jI}^{\text{In}} - J_{ji} \cdot A_I + r_{ji} \cdot V_I \\
\text{II:} & \quad \frac{dN_{ji}}{dt} = -F_{jI}^{\text{Out}} - J_{ji} \cdot A_{II} + r_{ji} \cdot V_{II} \\
\text{M:} & \quad \frac{dN_{ji}}{dt} = J_{ji} \cdot A_I + J_{ji} \cdot A_{II} + \int r_{ji} \, dV_M
\end{align*} \]  

(2.3)
The flow through the membrane system \( M \) can be assumed to be plug flow. This can be used to simplify the general mole balance further. As the membrane system for dead-end filtration contains both the filter cake and the filter (membrane proper) the mole balance is logically divided into one equation for the filter cake and one for the filter:

\[
\text{System:}
\]

Filter cake:
\[
\frac{dN_j}{dt} = J_{ji} \cdot A_i - J_{jFC} \cdot A_{FC} + \int r_{FC} \, dV_{FC}
\]

Filter:
\[
\frac{dN_j}{dt} = J_{jFC} \cdot A_{FC} + J_{ji} \cdot A_i + \int r_{F} \, dV_{F}
\]

(2.4)

Where
- \( A_{FC} \) is the contact area between filter cake and filter [m²]
- \( J_{jFC} \) is flux of component \( j \) from filter cake to filter [mole \( j \cdot m^{-2} \cdot s^{-1} \)]
- \( N_j \) is the number of moles of component \( j \) in the system [mole]
- \( N_{jFC} \) is the number of moles of component \( j \) in the filter cake [mole]
- \( r_j \) is rate of formation of component \( j \) in the filter [mole \( j \cdot m^{-3} \cdot s^{-1} \)]
- \( r_{FC} \) is rate of formation of component \( j \) in the filter cake [mole \( j \cdot m^{-3} \cdot s^{-1} \)]
- \( V_f \) is the volume of filter [m³]
- \( V_{FC} \) is the volume of filter cake [m³]

These equations can be further simplified by rewriting the equation for number of moles in the filter cake and in the filter to an equation in concentration:

\[
\text{System:}
\]

Filter cake:
\[
\frac{d}{dt} \int_{V_{FC}} C_{jFC} \, dV_{FC} = J_{ji} \cdot A_i - J_{jFC} \cdot A_{FC} + \int r_{FC} \, dV_{FC}
\]

Filter:
\[
\frac{d}{dt} \int_{V_F} C_{jF} \, dV_F = J_{jFC} \cdot A_{FC} + J_{ji} \cdot A_i + \int r_{F} \, dV_{F}
\]

(2.5)

where
- \( C_{jF} \) is the concentration of component \( j \) in the filter [mole \( j \cdot m^{-3} \)]
- \( C_{jFC} \) is the concentration of component \( j \) in the filter cake [mole \( j \cdot m^{-3} \)]
Differentiation with the volume of filter cake and filter respectively gives

\[
\begin{align*}
\text{System:} & \quad \left( \frac{\partial}{\partial t} \int_{V_{FC}} C_{jFC} \, dV_{FC} \right) = \frac{\partial (J_{jFC} \cdot A_1 - J_{jFC} \cdot A_{FC})}{\partial V_{FC}} + r_{jFC}, \\
\text{Filter cake:} & \quad \left( \frac{\partial}{\partial t} \int_{V_F} C_{jF} \, dV_F \right) = \frac{\partial (J_{jFC} \cdot A_{FC} + J_{jF} \cdot A_{II})}{\partial V_F} + r_{jF} \\
\text{Filter:} & \quad \left( \frac{\partial}{\partial t} \int_{V_F} C_{jF} \, dV_F \right) = \frac{\partial (J_{jF} \cdot A_{II})}{\partial V_F} + r_{jF}
\end{align*}
\]

(2.6)

Using that the flux to the filter cake is independent of the volume of the filter cake, that the flux from the filter cake to the filter is independent of the filter volume and shifting the order of differentiation leads to

\[
\begin{align*}
\text{System:} & \quad \left( \frac{\partial}{\partial t} \int_{V_{FC}} C_{jFC} \, dV_{FC} \right) = \frac{\partial (J_{jFC} \cdot A_{FC})}{\partial V_{FC}} + r_{jFC}, \\
\text{Filter cake:} & \quad \left( \frac{\partial}{\partial t} \int_{V_F} C_{jF} \, dV_F \right) = \frac{\partial (J_{jF} \cdot A_{II})}{\partial V_F} + r_{jF}
\end{align*}
\]

(2.7)

so that

\[
\begin{align*}
\text{System:} & \quad \frac{\partial C_{jFC}}{\partial t} = - \frac{\partial (J_{jFC} \cdot A_{FC})}{\partial V_{FC}} + r_{jFC}, \\
\text{Filter cake:} & \quad \frac{\partial C_{jF}}{\partial t} = \frac{\partial (J_{jF} \cdot A_{II})}{\partial V_F} + r_{jF}
\end{align*}
\]

(2.8)
A final simplification is done by using that the filter cake volume can be described as a function of filter cake depth and filter cake cross area, and that the filter volume can be described as a function of filter depth and filter cross area as seen on figure 2.2.

Where \( y \) is the Cartesian coordinate vector in the direction through the filter [m]

The final general design equation for dead-end filtration therefore becomes
2.1.2 Dead-end Filtration Without Reaction

For most industrial applications in dead-end filtration no reactions occur to an appreciably extent during the filtration. In these cases the design equation (2.10) can be simplified further. As the rate of reaction \( r_i \) is zero the design equations becomes

\[
\text{System:}
\]

\[
\begin{align*}
I: \quad & \frac{dN_{jI}}{dt} = F_{jI,\text{in}} - J_{jI} \cdot A_I \\
II: \quad & \frac{dN_{jII}}{dt} = -F_{jII,\text{out}} - J_{jII} \cdot A_{II}
\end{align*}
\]

\[
\text{Filter cake:}
\]

\[
\frac{\partial C_{jFC}}{\partial t} = \frac{\partial J_{jFC}}{\partial y} - \frac{J_{jFC}}{A_{FC}} \frac{\partial A_{FC}}{\partial y}
\]

\[
\text{Filter:}
\]

\[
\frac{\partial C_{jF}}{\partial t} = \frac{\partial J_{jF}}{\partial y} + \frac{J_{jF}}{A_{F}} \frac{\partial A_{F}}{\partial y}
\]

At the same time during operation feed is added to the feed chamber, system I, as fast as it is removed leading to

\[
\begin{align*}
\frac{dN_{jI}}{dt} &= \frac{dN_{jI}}{dt} = \frac{dN_{jI}}{dt} = \frac{dN_{jI}}{dt} = 0 = F_{jI,\text{in}} - J_{jI} \cdot A_I \\
F_{jI,\text{in}} &= J_{jI} \cdot A_I
\end{align*}
\]

Equally product will be removed from the product chamber, system II, at the same speed as it is received from the filter so that

\[
\begin{align*}
\frac{dN_{jII}}{dt} &= \frac{dN_{jII}}{dt} = \frac{dN_{jII}}{dt} = \frac{dN_{jII}}{dt} = 0 = -F_{jII,\text{out}} - J_{jII} \cdot A_{II} \\
F_{jII,\text{out}} &= -J_{jII} \cdot A_{II}
\end{align*}
\]
which leads to the design equation

\[
\text{System:} \\
I: \quad F_{ji}|_{\text{In}} = J_{ji} \cdot A_i \\
II: \quad F_{ji}|_{\text{Out}} = - J_{ji} \cdot A_{ii} \\
\text{Filter cake:} \quad \frac{\partial C_{jFC}}{\partial t} = - \frac{\partial J_{jFC}}{\partial y} - \frac{J_{jFC} \cdot \partial A_{FC}}{A_{FC} \cdot \partial y} \tag{2.14}
\]

\[
\text{Filter:} \quad \frac{\partial C_{jF}}{\partial t} = \frac{\partial J_{jF}}{\partial y} + \frac{J_{jF} \cdot \partial A_{F}}{A_{F} \cdot \partial y}
\]

A further simplification is not possible unless an expression for the flux is known and the geometry of the filter and filter cake is given.

**Example 2.1 Dead-end filtration of binary mixture**

A simplified, but in simple filtration often valid, situation is that of a binary feed slurry containing a solid, 1, that is 100% rejected by the filter and a liquid phase, 2, that during filtration pass the filter unhindered. If such a slurry is treated on a Nutch filter as shown in figure E2.1 the design equation can be further simplified. Since the cross sectional area is constant and independent of y the design equation simplifies to

\[
\text{System:} \\
I: \quad F_{1i}|_{\text{In}} = v_{f1} \cdot C_{1f} = J_{1i} \cdot A_{F} \\
F_{2i}|_{\text{In}} = v_{f2} \cdot C_{2f} = J_{2i} \cdot A_{F} \\
II: \quad F_{1ii}|_{\text{Out}} = 0 = - J_{1ii} \cdot A_{F} \\
F_{2ii}|_{\text{Out}} = v_{p2} \cdot C_{2p} = - J_{2ii} \cdot A_{F} \\
\text{Filter cake:} \quad \frac{\partial C_{1FC}}{\partial t} = - \frac{\partial J_{1FC}}{\partial y} \\
\frac{\partial C_{2FC}}{\partial t} = - \frac{\partial J_{2FC}}{\partial y} \\
\text{Filter:} \quad \frac{\partial C_{1F}}{\partial t} = \frac{\partial J_{1F}}{\partial y} = 0 \\
\frac{\partial C_{2F}}{\partial t} = \frac{\partial J_{2F}}{\partial y}
\]

Where
- $C_{j}$ is the concentration of component $j$ in the feed [mole · m$^{-3}$]
- $C_{p}$ is the concentration of component $j$ in the product [mole · m$^{-3}$]
- $v_{f}$ is the volumetric feed flow rate [m$^3$ · s$^{-1}$]
- $v_{p}$ is the volumetric product flow rate [m$^3$ · s$^{-1}$]
Figure E2.1 Nutsch filter.

For most filtration operations the density of the filter and filter cake does not change during operation which directly leads to

Filter cake:
\[
\frac{\partial C_{1FC}}{\partial t} = 0 = -\frac{\partial J_{1FC}}{\partial y}
\]
\[
\frac{\partial C_{2FC}}{\partial t} = 0 = -\frac{\partial J_{2FC}}{\partial y}
\]

Filter:
\[
\frac{\partial C_{1F}}{\partial t} = 0 = \frac{\partial J_{1F}}{\partial y}
\]
\[
\frac{\partial C_{2F}}{\partial t} = 0 = \frac{\partial J_{2F}}{\partial y}
\]

The thickness of the filter cake increases with time. Since all solids in the feed are retained by the filter the volume of the filter cake can be found as

\[
C_{1FC} \cdot V_{FC}(t) = \int_{0}^{t} v_{f} \cdot C_{1f} \, dt
\]

or since the concentration in the feed slurry seldom change during a run

\[
V_{FC}(t) = \frac{C_{1F}}{C_{1FC}} \int_{0}^{t} v_{f} \, dt
\]
From this it follows that the filter cake thickness equals

\[ l_{FC}(t) = \frac{V_{FC}(t)}{A_F} = \frac{C_{1f}}{A_F \cdot C_{1FC}} \int_0^t \nu_4 \, dt \]

Where \( l_{FC} \) is the filter cake depth [m]

The amount of liquid retained in the filter cake is of course

\[ N_{2FC}(t) = l_{FC}(t) \cdot A_F \cdot C_{2FC} \]

The difference between the liquid flux from chamber I to the filter cake and from chamber II to the filter therefore becomes

\[ \frac{dN_{2FC}}{dt} = J_{2I} \cdot A_F + J_{2FC} \cdot A_F = J_{2I} \cdot A_F + J_{2II} \cdot A_F \]

\[ A_F \cdot C_{2FC} \frac{dl_{FC}(t)}{dt} = J_{2I} \cdot A_F + J_{2II} \cdot A_F \]

\[ C_{2FC} \frac{dl_{FC}(t)}{dt} = J_{2I} + J_{2II} \]

\[ C_{2FC} \frac{\nu_f}{A_F \cdot C_{1FC}} = J_{2I} + J_{2II} \]

\[ J_{2II} = -J_{2I} + \frac{C_{2FC} \cdot \nu_f}{C_{1FC} \cdot A_F} \]

To come any further the flux through the filter cake and filter need to be known.

For the liquid passing through the filter cake and the porous membrane D’Arcy’s law usually applies:

Filter cake: \[ J_{2FC} = \frac{C_{2p}}{R_{FC} \cdot \mu_{\text{liquid}}} (P_I - P_{IM}) \]

Filter: \[ J_{2F} = \frac{C_{2p}}{R_f \cdot \mu_{\text{liquid}}} (P_{IM} - P_{II}) \]

Where \( P_I \) is the pressure in system I [Pa]

\( P_{IM} \) is the pressure at the system I- membrane interface [Pa]

\( P_{II} \) is the pressure in system II [Pa]

\( R_f \) is the resistance to flow in the filter \([m^{-1}]\)

\( R_{FC} \) is resistance to flow in the filter cake \([m^{-1}]\)

\( \mu_{\text{liquid}} \) is the liquid viscosity \([\text{Pa} \cdot \text{s}]\)
The filter cake resistance, $R_{FC}$, is a function of cake depth and the specific cake resistance $\alpha$:

$$R_{FC}(t) = l_{FC}(t) \cdot \rho_{FC} \cdot \alpha_{spC} = \frac{C_{II} \cdot \int_{0}^{t} v_i \cdot dt}{A_F \cdot C_{1FC} \cdot M_{w1} \cdot \rho_{1FC} \cdot \alpha_{spC}}$$

$$= \alpha_{spC} \cdot M_{w1} \cdot C_{1f} \cdot \frac{\int_{0}^{t} v_i \cdot dt}{A_F}$$

Where $M_{wj}$ is the molecular weight of component $j$ [kg mol$^{-1}$]

$\alpha_{spC}$ is the specific cake resistance [m kg$^{-1}$]

$\rho_{FC}$ is bulk density of component $j$ in the filter cake [kg m$^{-3}$]

while the filter resistance, $R_F$, is an empirical constant to be determined by laboratory experiments.

Since

$$J_{2FC} = -J_{2F} = J_{2II}$$

the flux $J_{2II}$ is often described by an overall resistance defined by

$$J_{2II} = -\frac{C_{2p}}{R_O \cdot \mu_{\text{liquid}}} \cdot (P_I - P_{II})$$

which can be found from

$$\begin{align*}
(P_I - P_{II}) &= (P_I - P_{IM}) + (P_{IM} - P_{II}) \\
\downarrow \\
\frac{(P_I - P_{II})}{-J_{2II}} &= \frac{(P_I - P_{IM})}{-J_{2FC}} + \frac{(P_{IM} - P_{II})}{J_{2F}} \\
\downarrow \\
\frac{R_O \cdot \mu_{\text{liquid}}}{C_{2p}} &= \frac{R_{FC} \cdot \mu_{\text{liquid}}}{C_{2p}} + \frac{R_F \cdot \mu_{\text{liquid}}}{C_{2p}} \\
\downarrow \\
R_O &= R_{FC} + R_F
\end{align*}$$
For this specific case the design equations therefore becomes

\[ V_{FC}(t) = \frac{C_{1f}}{C_{IFC}} \int_0^1 v_f(t) \, dt \]
\[ v_p \cdot C_{2p} = -J_{2II} \cdot A_F \]
\[ J_{2II} = \frac{C_{2p}}{R_o \cdot \mu_{liquid}} (P_1 - P_{II}) \]
\[ J_{2II} = \frac{v_f}{A_F} \left( \frac{C_{2FC}}{C_{IFC}} \cdot C_{1f} - C_{2f} \right) \]
\[ l_{FC}(t) = \frac{V_{FC}(t)}{A_F} \]
\[ R_o = R_{FC} + R_F \]
\[ R_{FC}(t) = \alpha_{sp} \cdot \frac{M_{w1} \cdot C_{1f}' \cdot v_f}{A_F} \int_0^t \, dt \]

When the volumetric flow rate or the pressure drop is known as a function of time these equations can be solved to give the necessary filtration area for a given production rate and a maximum allowable pressure drop over the filter and filter cake.

2.2 Mole Balance and Design Equation for Perfectly Mixed-Mixed Flow Membrane Module (M&M-MM)

2.2.1 Perfectly Mixed-Mixed Flow Membrane Module with Reaction

For small size particles the resistance to flow as the filter cake builds up becomes excessive making dead-end filtration uneconomical or outright impossible. For membrane processes dealing with microfiltration size particles or smaller another approach has to be used. Instead of forcing the feed stream through the membrane leaving all particles above a certain size to pile up on the membrane surface the feed is passed along side the membrane whereby some feed will pass through the membrane while the retained feed stream will sweep the membrane surface for particles retained by the membrane. Though this does not totally eliminate the problem with cake built up in many cases it reduces the problem to an acceptable level. For these flow through membrane systems the solution to the general mole balance will depend both on the flux through the membrane and the flow pattern in the two membrane module chambers. The simplest flow pattern to treat from a mathematical view point is the case where both membrane module chambers are perfectly mixed as shown in figure 2.3.
This flow pattern is used frequently in the laboratory to test new membrane materials ability to separate gas or liquid components but is only seen on an industrial scale in some gas separation modules with low flux.

In the laboratory the Mixed-Mixed Membrane Module (M&M-MM) is typically configured as a flat-sheet module and used to determine the differential selectivity of the membrane.

As the two module chambers, system I and II, are well mixed the concentration is constant in both chambers. This means that the concentration inside chamber I is identical to the concentration in the stream leaving chamber I and that the concentration in chamber II equals the concentration in the stream leaving chamber II.

Since the chambers are well mixed pressure, concentration and temperatures in system I and II are independent of position and the general mole balance (1.8):

\[
\begin{align*}
\text{System: } & \\
I: & \quad \frac{dN_{ji}}{dt} = F_{ji|\text{in}} - F_{ji|\text{out}} - \int_{A_i} J_{ji} \, dA_i + \int_{V_i} r_{ji} \, dV_i \\
\text{II: } & \quad \frac{dN_{ji}}{dt} = F_{ji|\text{in}} - F_{ji|\text{out}} - \int_{A_{ii}} J_{ji} \, dA_{ii} + \int_{V_{ii}} r_{ji} \, dV_{ii} \\
\text{M: } & \quad \frac{dN_{jm}}{dt} = \int_{A_i} J_{ji} \, dA_i + \int_{A_{ii}} J_{ji} \, dA_{ii} - 0 + \int_{V_{mm}} r_{jm} \, dV_m
\end{align*}
\]

Figure 2.3 Mixed-Mixed Membrane Module.
can be simplified to

\[
\text{System:}
\begin{align*}
  \text{I:} & \quad \frac{dN_j}{dt} = F_{ji}\vert_{\text{in}} - F_{ji}\vert_{\text{out}} - J_{ji} \cdot A_I + r_{ji} \cdot V_I \\
  \text{II:} & \quad \frac{dN_{jl}}{dt} = F_{jl}\vert_{\text{in}} - F_{jl}\vert_{\text{out}} - J_{jl} \cdot A_{II} + r_{jl} \cdot V_{II} \\
  \text{M:} & \quad \frac{dN_{JM}}{dt} = J_{jI} \cdot A_I + J_{jII} \cdot A_{II} + \int_{0}^{l_M} r_{JM} \cdot A_M(y) \, dy
\end{align*}
\]

using equation (2.2). As described under dead-end filtration figure 2.2 assuming plug flow through the membrane leads to the further simplification

\[
\text{System:}
\begin{align*}
  \text{I:} & \quad \frac{dN_j}{dt} = F_{ji}\vert_{\text{in}} - F_{ji}\vert_{\text{out}} - J_{ji} \cdot A_I + r_{ji} \cdot V_I \\
  \text{II:} & \quad \frac{dN_{jl}}{dt} = F_{jl}\vert_{\text{in}} - F_{jl}\vert_{\text{out}} - J_{jl} \cdot A_{II} + r_{jl} \cdot V_{II} \\
  \text{M:} & \quad \frac{dN_{JM}}{dt} = J_{jI} \cdot A_I + J_{jII} \cdot A_{II} + \int_{0}^{l_M} r_{JM} \cdot A_M(y) \, dy
\end{align*}
\]

Where \( A_M(y) \) is cross sectional area of membrane in the y-direction [m²]

During operation it will normally be reasonable to assume steady state for shorter periods of time (typically hours or days of operation). This simplifies the equation further since

\[
\text{System:}
\begin{align*}
  \text{I:} & \quad \frac{dN_j}{dt} = 0 \\
  \text{II:} & \quad \frac{dN_{jl}}{dt} = 0 \\
  \text{M:} & \quad \frac{dN_{JM}}{dt} = 0
\end{align*}
\]
The design equations for an M&M-MM with reaction therefore becomes

\[
\begin{align*}
\text{System:} \\
\text{I: } & \quad 0 = F_{jI}^{\text{in}} - F_{jI}^{\text{out}} - J_{jI} \cdot A_l + r_{jI} \cdot V_l \\
\text{II: } & \quad 0 = F_{jII}^{\text{in}} - F_{jII}^{\text{out}} - J_{jII} \cdot A_{II} + r_{jII} \cdot V_{II} \\
\text{M: } & \quad 0 = J_{jI} \cdot A_l + J_{jII} \cdot A_{II} + \int_0^L r_{jM} \cdot A_M(y) \, dy
\end{align*}
\] (2.18)

To solve these design equations the geometry of the membrane, a description of the flux and the rate equations have to be known.

2.2.2 Perfectly Mixed-Mixed Flow Membrane Module without Reaction

For most applications no reactions occur in either the module chambers or the membrane. In this case the design equations (2.18) can be simplified further:

\[
\begin{align*}
\text{System:} \\
\text{I: } & \quad 0 = F_{jI}^{\text{in}} - F_{jI}^{\text{out}} - J_{jI} \cdot A_l \\
\text{II: } & \quad 0 = F_{jII}^{\text{in}} - F_{jII}^{\text{out}} - J_{jII} \cdot A_{II} \\
\text{M: } & \quad J_{jI} \cdot A_l = -J_{jII} \cdot A_{II}
\end{align*}
\] (2.19)

From an experimental point of view it follows that the flux can be found from

\[
\begin{align*}
J_{jI} & = \frac{F_{jI}^{\text{in}} - F_{jI}^{\text{out}}}{A_l} = \frac{v_l^{\text{in}} \cdot C_{jI}^{\text{in}} - v_l^{\text{out}} \cdot C_{jI}^{\text{out}}}{A_l} \\
J_{jII} & = \frac{F_{jII}^{\text{out}} - F_{jII}^{\text{in}}}{A_l} = \frac{v_{II}^{\text{out}} \cdot C_{jII}^{\text{out}} - v_{II}^{\text{in}} \cdot C_{jII}^{\text{in}}}{A_l}
\end{align*}
\] (2.20)
and that the differential or intrinsic selectivity as

$$\alpha_{ij,S&H}^* = \frac{C_{ii\mid \text{Out}}}{C_{ii\mid \text{In}}}$$

and

$$\alpha_{ij,Mulder}^* = \frac{C_{ii\mid \text{Out}}}{C_{ii\mid \text{In}}}$$

(2.21)

The results obtained using laboratory experiments and equations (2.20) and (2.21) can then be used to predict behavior of large scale equipment with often different flow patterns.

Example 2.2 Mixed-mixed flow membrane module for gas separation

Though not a common situation, in gas separation processes membrane modules with perfect mixed flow in both chambers can occur. This example based on a paper by Weller and Steiner [2] show how to use the general design equation (2.19) for an M&M-MM in gas separation. The gas separation module is shown in figure E2.2.
The following quantities are assumed to be specified in advance:

- $F_{totf}$ the total molecular feed flow $\left[ \text{mol} \, \text{s}^{-1} \right]$.
- $P_I$ the pressure in chamber I [Pa].
- $P_{II}$ the pressure in chamber II [Pa].
- $S$ the recovery or stage cut, defined as $S = \frac{F_{top}}{F_{totf}}$.
- $T_I$ the temperature in chamber I [K].
- $T_{II}$ the temperature in chamber II [K].
- $x_f^j$ the molecular fraction of component $j$ in the feed.

while the following quantities are to be determined:

- $A_I$ the membrane area facing chamber I $\left[ \text{m}^2 \right]$.
- $A_{II}$ the membrane area facing chamber II $\left[ \text{m}^2 \right]$.
- $F_{top}$ the total molecular permeate flow $\left[ \text{mol} \, \text{s}^{-1} \right]$.
- $F_{totr}$ the total molecular retentate flow $\left[ \text{mol} \, \text{s}^{-1} \right]$.
- $x_p^j$ the molecular fraction of component $j$ in the permeate.
- $x_r^j$ the molecular fraction of component $j$ in the retentate.

Furthermore the flux through the membrane of component $j$ is assumed to be described by an equation of the form

$$J_j = K_j \left( P_{jIM} - P_{jIIM} \right)$$

where

- $K_j$ is a mass transfer coefficient $\left[ \text{mol} \, \text{Pa}^{-1} \, \text{m}^2 \, \text{s}^{-1} \right]$.
- $P_{jIM}$ is the partial pressure of component $j$ on the membrane surface in module chamber I [Pa].
- $P_{jIIM}$ is the partial pressure of component $j$ on the membrane surface in module chamber II [Pa].

If steady state is assumed during the operation design equation (2.19) for an M&M-MM is the proper starting point:

System:

\[
\begin{align*}
\text{I: } & \quad 0 = F_{jI}^{\text{In}} - F_{jI}^{\text{Out}} - J_{jI}^I \cdot A_I \\
\text{II: } & \quad 0 = F_{jII}^{\text{In}} - F_{jII}^{\text{Out}} - J_{jII}^I \cdot A_{II} \\
\text{M: } & \quad J_{jI}^I \cdot A_I = -J_{jII}^I \cdot A_{II}
\end{align*}
\]
From this it is found that

\[
J_{jII} \cdot A_{II} = - J_{jI} \cdot A_I
\]

\[
J_{jI} \cdot A_I = F_{jI| \text{in}} \cdot F_{jI| \text{out}} = x_{jI} \cdot F_{\text{totf}} - x_{jI} \cdot F_{\text{totor}}
\]

\[
F_{jI| \text{out}} = x_{jP} \cdot F_{\text{top}} = - J_{jII} \cdot A_{II} = J_{jI} \cdot A_I = F_{jI| \text{in}} - F_{jI| \text{out}} = x_{jI} \cdot F_{\text{totf}} - x_{jI} \cdot F_{\text{totor}}
\]

An overall mass balance over the membrane chamber gives

\[
F_{\text{totf}} = F_{\text{top}} + F_{\text{totor}}
\]

From this and the definition of the recovery factor it follows that

\[
F_{\text{top}} = S \cdot F_{\text{totf}}
\]

\[
F_{\text{totor}} = F_{\text{totf}} - F_{\text{top}} = F_{\text{totf}} - S \cdot F_{\text{totf}} = (1 - S) \cdot F_{\text{totf}}
\]

Inserting this in the mass balance for membrane module chamber II lead to the following relationship between the mole fraction in the retentate and the permeate

\[
x_{jP} \cdot F_{\text{top}} = x_{jI} \cdot F_{\text{totf}} - x_{jI} \cdot F_{\text{totor}}
\]

\[
x_{jP} \cdot S \cdot F_{\text{totf}} = x_{jI} \cdot F_{\text{totf}} - x_{jI} \cdot (1 - S) \cdot F_{\text{totf}}
\]

\[
x_{jP} \cdot S = x_{jI} - x_{jI} \cdot (1 - S)
\]

\[
x_{jP} = \frac{x_{jI} - S \cdot x_{jP}}{1 - S}
\]

Inserting the flux equation into the mass balance for chamber II gives

\[
F_{jII| \text{out}} = J_{jI} \cdot A_I
\]

\[
x_{jP} \cdot F_{\text{top}} = A_{I} \cdot K_{j} \left( x_{jI} \cdot P_{\text{I}} - x_{jI} \cdot P_{\text{II}} \right) = A_{I} \cdot K_{j} \left( x_{jI} \cdot P_{\text{I}} - x_{jP} \cdot P_{\text{II}} \right)
\]

\[
x_{jP} \cdot S \cdot F_{\text{totf}} = A_{I} \cdot K_{j} \left( x_{jI} \cdot P_{\text{I}} - x_{jP} \cdot P_{\text{II}} \right)
\]
Choosing one component \( k \) as key component and dividing the flux equations for the remaining \( j \) components with the flux equation for the key component yields

\[
\frac{x_{jp}^* \cdot F_{totr}}{x_{kp} \cdot F_{totr}} = \frac{A_j \cdot K_j \cdot (x_{jp}^* P_I - x_{jp}^* P_{II})}{A_k \cdot K_k \cdot (x_{kp}^* P_I - x_{kp}^* P_{II})}
\]

\[
\frac{x_{jp}}{x_{kp}} = \frac{K_j \cdot (x_{jp}^* P_I - x_{jp}^* P_{II})}{K_k \cdot (x_{kp}^* P_I - x_{kp}^* P_{II})}
\]

\[
\frac{x_{jp}}{x_{kp}} = \frac{\alpha_{jk} \cdot x_{jp}^* - x_{jp}^* P_r}{x_{kp}^* - x_{kp}^* P_r}
\]

where \( \alpha_{jk} = \frac{K_j}{K_k} \) is the ideal gas separation factor

\( P_r = \frac{P_{II}}{P_I} \) is the pressure ratio

Inserting the equation for \( x_{jp}^* \) into the flux equation yield

\[
\frac{x_{jp}}{x_{kp}} = \alpha_{jk} \cdot \frac{x_{jp}^* - x_{kp}^* P_r}{x_{jp}^* - x_{kp}^* P_r}
\]

This together with the constraint on the permeate mole fractions

\[
1 = \sum_{j=1}^{n} x_{jp}
\]

gives the following \( n \) design equations with \( n \) unknowns

\[
\frac{x_{jp}}{x_{kp}} = \alpha_{jk} \cdot \frac{x_{jp}^* - P_r \cdot x_{kp}^* + S \cdot (P_r - 1) \cdot x_{kp}^*}{x_{kp}^* - P_r \cdot x_{kp}^* + S \cdot (P_r - 1) \cdot x_{kp}^*}
\]

\[
1 = \sum_{j=1}^{n} x_{jp}
\]

These equations will generally have to be solved by iteration.

The mole fraction in the retentate is then calculated from

\[
x_{jp}^* = \frac{x_{jp}^* - S \cdot x_{jp}}{1 - S}
\]
while the necessary membrane area is found from

\[
x_{jp} \cdot S \cdot F_{totf} = A_1 \cdot K_j \cdot (x_{jp} \cdot P_1 - x_{jp} \cdot P_{II})
\]

\[
A_1 = \frac{x_{jp} \cdot S \cdot F_{totf}}{K_j \cdot (x_{jp} \cdot P_1 - x_{jp} \cdot P_{II})}
\]

**Example 2.3 Mixed-mixed membrane module flow for binary feed mixture**

For binary mixtures the equations of example 2.2 have an analytical solution as derived by Weller and Steiner [2]. For the problem shown in figure E2.3 the design equations are reduced to

\[
x_{1p} = \frac{x_{1f} - \frac{p_t \cdot x_{1p}}{x_{2f} - \frac{p_t \cdot x_{2p}}{S \cdot (p_t - 1) \cdot x_{1p}}}}{1 - x_{1p} + x_{2p}}
\]

Figure E2.3 Mixed-mixed flow membrane module with binary feed mixture
Using that

\[ x_{2p} = 1 - x_{1p} \]
\[ x_{2f} = 1 - x_{1f} \]

leads to

\[ \frac{x_{1p}}{1 - x_{1p}} = \frac{x_{1f} - p_r \cdot x_{1p} + S \cdot (p_r - 1) \cdot x_{1p}}{(1 - x_{1p}) - p_r \cdot (1 - x_{1p}) + S \cdot (p_r - 1) \cdot (1 - x_{1p})} \]

which upon rearrangement yield a second order equation in the unknown \(x_{1p}\)

\[
0 = -x_{1p} \cdot (1 - x_{1f}) + p_r \cdot x_{1p} \cdot (1 - x_{1p}) - S \cdot (p_r - 1) \cdot (1 - x_{1p}) \cdot x_{1p} \\
+ a_{12} \cdot (1 - x_{1p}) \cdot x_{1f} - p_r \cdot x_{1p} \cdot a_{12} \cdot (1 - x_{1p}) + S \cdot x_{1p} \cdot a_{12} \cdot (1 - x_{1p}) \cdot (p_r - 1) \\
+ \left\{ a_{12} \cdot x_{1f} + (1 - a_{12}) \cdot (1 - S) \cdot p_r + (1 - a_{12}) \cdot S - 1 \right\} \cdot x_{1p} + a_{12} \cdot x_{1f}
\]

The solution to this second order equation is

\[ x_{1p} = \frac{-B \pm \sqrt{B^2 - 4 \cdot A \cdot C}}{2 \cdot A} \]

with

\[ A = (a_{12} - 1) \cdot S + (a_{12} - 1) \cdot (1 - S) \cdot p_r \]
\[ B = (1 - a_{12}) \cdot x_{1f} + (1 - a_{12}) \cdot (1 - S) \cdot p_r + (1 - a_{12}) \cdot S - 1 \]
\[ C = a_{12} \cdot x_{1f} \]

The necessary membrane area can then be determined from

\[ A_l = \frac{x_{1p} \cdot S \cdot F_{\text{mf}}}{K_1 \left( x_{1f} \cdot P_1 - x_{1p} \cdot P_{II} \right)} \]
2.3 Mole Balance and Design Equation for Plug flow-Perfectly Mixed Flow Membrane Module (PF&M-MM)

2.3.1 Plug Flow-Perfectly Mixed Flow Membrane Module with Reaction

\[ \frac{dN_{\text{I}}}{dt} = F_{\text{I, in}} - F_{\text{I, out}} - \int J_{\text{I, i}} dA_1 + \int r_{\text{I}, i} dV_1 \]
\[ \frac{dN_{\text{II}}}{dt} = F_{\text{II, in}} - F_{\text{II, out}} - \int J_{\text{II, i}} dA_2 + \int r_{\text{II}, i} dV_2 \]
\[ \frac{dN_{\text{M}}}{dt} = \int J_{\text{I}} dA_1 + \int J_{\text{II}} dA_2 - 0 + \int r_{\text{M}, i} dV_M \] (1.8)

Figure 2.4 Plug flow-mixed flow membrane module.

Situations where the flow in one membrane module chamber can be considered as plug flow while the flow in the other membrane chamber can be considered as perfectly mixed flow occurs for instance when an immersible membrane module is immersed in a well stirred tank. The membrane module is shown in figure 2.4.

Taking system I as the plug flow system and system II as the perfectly mixed flow system the general mass balance (1.8) can be simplified as follows.
For system I the mass balance can be rewritten as

\[
\frac{dN_{ji}}{dt} = \frac{d}{dt} \left( \int C_{ji} dV_i \right) = F_{ji,\text{in}} - F_{ji,\text{out}} - \int J_{ji} dA_i + \int r_{ji} dV_i \tag{2.21}
\]

where \( C_{ji} \) is the molar concentration of component \( j \) in chamber \( I \) [mol m\(^{-3}\)].

Differentiation with the volume \( V_i \) leads to

\[
\frac{\partial}{\partial V_i} \left( \frac{d}{dt} \left( \int C_{ji} dV_i \right) \right) = \frac{\partial}{\partial V_i} (F_{ji,\text{in}} - F_{ji,\text{out}}) - \frac{\partial}{\partial V_i} \left( \int J_{ji} dA_i \right) + \frac{\partial}{\partial V_i} \left( \int r_{ji} dV_i \right) \tag{2.23}
\]

A further simplification comes from considering the general geometry of a membrane chamber as shown in figure 2.5. As seen the differential volume can generally be expressed as

\[
dV_i = A_{TI}(x) \cdot dx \tag{2.24}
\]

where \( A_{TI}(x) \) is the cross sectional area of the membrane chamber perpendicular to the direction of flow [m\(^2\)].

\( dx \) is the differential length in the direction of flow [m].

![Figure 2.5 Geometry of a general membrane chamber](image-url)
The differential wetted membrane area in chamber I is expressed as

\[ dA_I = \Omega_I(x) \cdot dx \]  

(2.25)

where \( \Omega_I(x) \) is wetted periferi of the membrane in chamber I [m].

and the differential wetted membrane area in chamber II can be expressed as

\[ dA_{II} = \Omega_{II}(x) \cdot dx \]  

(2.26)

where \( \Omega_{II}(x) \) is wetted periferi of the membrane in chamber II [m].

Inserted in the mass balance for chamber I this gives

\[
\frac{\partial C_{jI}}{\partial t} = - \left( \frac{1}{A_{I}(x)} \frac{\partial F_{jI}}{\partial x} - \frac{1}{A_{I}(x)} \left( \int J_{jI} \cdot \Omega_{I}(x) \ dx \right) \right) + r_{jI}
\]  

(2.27)

For system II the concentration is the same everywhere in the chamber and therefore independent of \( dV_{II} \) so the mass balance can be written as

\[
\frac{dN_{jII}}{dt} = \frac{d(C_{jII} \cdot V_{II})}{dt} = F_{jII \mid in} - F_{jII \mid out} - \int J_{jII} \ dA_{II} + r_{jII} \cdot V_{II}
\]  

(2.28)

where \( C_{jII} \) is the molar concentration of component j in chamber II \( \left[ \frac{\text{mol}}{\text{m}^3} \right] \)

If the volume of chamber II furthermore is constant with time, the mass balance simplifies to

\[
V_{II} \cdot \frac{\partial C_{jII}}{\partial t} = F_{jII \mid in} - F_{jII \mid out} - \int J_{jII} \ dA_{II} + r_{jII} \cdot V_{II}
\]  

(2.29)
Dividing with $V_\text{II}$ and using the geometry of the membrane chamber leads to

$$\frac{\partial C_{\text{II}}}{\partial t} = \frac{F_{\text{jII,in}}}{V_\text{II}} - \frac{F_{\text{jII,out}}}{V_\text{II}} - \frac{\int_{x_0}^{x_\text{II}} J_{\text{II}} \cdot \Omega_\text{II}(x) \, dx}{V_\text{II}} + r_{\text{jII}} \quad (2.30)$$

For the membrane the mass balance is rewritten as

$$\frac{dN_{\text{JM}}}{dt} = \frac{d}{dt} \left( \int_{V_\text{M}} C_{\text{JM}} \, dV_\text{M} \right) = \int_{A_\text{I}} J_{\text{I}} \, dA_\text{I} + \int_{A_\text{II}} J_{\text{II}} \, dA_\text{II} + \int_{r_{\text{JM}}} r_{\text{JM}} \, dV_\text{M} \quad (2.31)$$

where $C_{\text{JM}}$ is the molar concentration of component $j$ in the membrane $\left[\text{mol} / \text{m}^3\right]$. Differentiation with the volume $V_\text{M}$ lead to

$$\frac{d}{dt} \left( \int_{V_\text{M}} C_{\text{JM}} \, dV_\text{M} \right) = \frac{\partial}{\partial V_\text{M}} \left( \int_{V_\text{M}} C_{\text{JM}} \, dV_\text{M} \right) = \frac{\partial}{\partial V_\text{M}} \left( \int_{A_\text{I}} J_{\text{I}} \, dA_\text{I} \right) + \frac{\partial}{\partial V_\text{M}} \left( \int_{A_\text{II}} J_{\text{II}} \, dA_\text{II} \right) + \frac{\partial}{\partial V_\text{M}} \left( \int_{r_{\text{JM}}} r_{\text{JM}} \, dV_\text{M} \right) \quad (2.32)$$

Using that

$$dV_\text{M} = dx \cdot dy \cdot dz \quad (2.33)$$
the equation for the membrane simplifies further

\[
\frac{\partial C_{jM}}{\partial t} = - \left( \frac{\partial }{\partial x} \left( \int \int J_{jM} \, dx \, dz \right) \right) + r_{jM} \\
\frac{\partial C_{jM}}{\partial t} = - \frac{\partial J_{jM}}{\partial y} + r_{jM}
\]

(2.34)

The general design equations for a plug flow - perfect mixed membrane module therefore becomes

**System:**

I: Plug flow chamber:

\[
\frac{\partial C_{jI}}{\partial t} = \frac{1}{A_{fI}(x)} \frac{\partial F_{jI}}{\partial x} - \frac{\Omega_{I}(x)}{A_{fI}(x)} \cdot J_{jI} + r_{jI}
\]

II: Perfect mixed chamber:

\[
\frac{\partial C_{jII}}{\partial t} = \frac{F_{jII|in} - F_{jII|out}}{V_{II}} - \frac{\int_{x_{II}}^{x_{II}} J_{jII} \cdot \Omega_{II}(x) \, dx}{V_{II}} + r_{jII}
\]

M: Membrane:

\[
\frac{\partial C_{jM}}{\partial t} = \frac{\partial J_{jM}}{\partial y} + r_{jM}
\]

(2.35)

Under steady-state conditions these design equations simplify to

**System:**

I: Plug flow chamber:

\[
\frac{dF_{jI}}{dx} = -\Omega_{I}(x) \cdot J_{jI} + r_{jI} \cdot A_{fI}(x)
\]

II: Perfect mixed chamber:

\[
0 = \frac{F_{jII|in} - F_{jII|out}}{V_{II}} - \frac{\int_{x_{II}}^{x_{II}} J_{jII} \cdot \Omega_{II}(x) \, dx}{V_{II}} + r_{jII}
\]

M: Membrane:

\[
\frac{\partial J_{jM}}{\partial y} = r_{jM}
\]

(2.36)
Example 2.4 How to calculate the cross flow area and the wetted membrane periferi in standard geometries

For the rectangular flatsheet module shown schematically in figure E2.4 the cross sectional flow area is

\[ A_{Tf}(x) = L_{yI} \cdot L_{zI} \]

Where \( L_{xI} \) is channel length in chamber I [m]

\( L_{yI} \) is the channel height in chamber I [m]

\( L_{zI} \) is the channel width in chamber I [m]

while the wetted membrane periferi is

\[ \Omega_I(x) = L_{zI} \]

This leads to

\[ dV_I = A_{Tf}(x) \cdot dx = L_{yI} \cdot L_{zI} \cdot dx \]

\[ dA_I = \Omega_I(x) \cdot dx = L_{zI} \cdot dx \]

\[ dA_{II} = \Omega_{II}(x) \cdot dx = L_{zI} \cdot dx \]
For the turbular module shown schematically in figure E2.5 the cross sectional flow area is

\[ A_{TI}(x) = \frac{\pi}{4} d_i^2 \]

Where \( d_i \) is the inner diameter of the turbular membrane [m].

while the wetted membrane periferi for system I is

\[ \Omega_I(x) = \pi \cdot d_i \]

and for system II

\[ \Omega_{II}(x) = \pi \cdot d_o \]

Where \( d_o \) is the outer diameter of the turbular membrane [m].

This leads to

\[ dV_I = A_{TI}(x) \cdot dx = \frac{\pi}{4} d_i^2 \cdot dx \]
\[ dA_I = \Omega_I(x) \cdot dx = \pi \cdot d_i \cdot dx \]
\[ dA_{II} = \Omega_{II}(x) \cdot dx = \pi \cdot d_o \cdot dx \]
2.3.2 Plug Flow-Perfectly Mixed Flow Membrane Module without Reaction

If no reaction occurs in the membrane module the steady state design equations reduce to

\[
\begin{align*}
\text{System:} & \\
\text{I: Plug flow chamber:} & \quad \frac{dF_j}{dx} = -\frac{\Omega(x) \cdot J_j}{\Omega(x)} \\
\text{II: Perfect mixed chamber:} & \quad \frac{F_{jm,\text{in}}}{V_{\text{II}}} - \frac{F_{jm,\text{out}}}{V_{\text{II}}} = \frac{\int_{x_{\text{II}}}^{x_{\text{II}}} \Omega_{\text{II}}(x) \, dx}{V_{\text{II}}} \\
\text{M: Membrane:} & \quad 0 = \frac{\partial J_{jm}}{\partial y} \Rightarrow J_{jm} = -J_{jm} \cdot \frac{\Omega_{\text{II}}(x)}{\Omega_{\text{I}}(x)} 
\end{align*}
\]

Example 2.5: Plug flow-mixed flow membrane module for gas separation

For a gas separation process where the feed membrane module chamber behaves as a plug flow chamber and the permeate membrane module chamber behaves as a perfect mixed chamber the following method can be used to solve the design equations. The gas separation module is shown in figure E2.6.

The following quantities are assumed to be specified in advance:

- \( F_{\text{totf}} \) the total molecular feed flow \( \text{[mol s]} \)
- \( F_k \) the molecular retentate flow for a key component \( k \) \( \text{[mol s]} \)
- \( P_\text{I} \) the total pressure in chamber I [Pa]
- \( P_\text{II} \) the total pressure in chamber II [Pa]
- \( T_\text{I} \) the temperature in chamber I [K]
- \( T_\text{II} \) the temperature in chamber II [K]
- \( x_j \) the molecular fraction of component j in the feed
while the following quantities are to be determined:

- $A_I$, the membrane area facing chamber I [m$^2$]
- $A_{II}$, the membrane area facing chamber II [m$^2$]
- $F_{tropol}$, the total molecular permeate flow [mol/s]
- $F_{totr}$, the total molecular retentate flow [mol/s]

$x_{j\text{p}}$, the molecular fraction of component j in the permeate

$x_{jr}$, the molecular fraction of component j in the retentate

Furthermore the flux through the membrane of component j is assumed to be described by an equation of the form

$$J_j = K_j \cdot \left( P_{j\text{IM}} - P_{j\text{IM}} \right)$$

where $K_j$ is a mass transfer coefficient [mol/(Pa·m$^2$·s)]

$P_{j\text{IM}}$ is the partial pressure of component j on the membrane surface in module chamber I [Pa]

$P_{j\text{IM}}$ is the partial pressure of component j on the membrane surface in module chamber II [Pa]

Assuming steady state and no reactions during the process design equations (2.37) for the membrane unit are applicable:

<table>
<thead>
<tr>
<th>System:</th>
</tr>
</thead>
</table>
| I: Plug flow chamber: | \[
\frac{dF_{jI}}{dx} = -\Omega_i(x) \cdot J_{jI}
\]  
| II: Perfect mixed chamber: | \[
0 = \frac{F_{jI\text{in}} - F_{jI\text{out}}}{V_{II}} - \frac{\int J_{jII} \cdot \Omega_{II}(x) \, dx}{V_{II}}
\]  
| M: Membrane: | \[
0 = \frac{\partial J_{jI}}{\partial y} = J_{jI} = -J_{jII} \cdot \frac{\Omega_{II}(x)}{\Omega_i(x)}
\]  

For the feed chamber the mass balance becomes

$$\frac{dF_{jI}}{dx} = -\Omega_i(x) \cdot J_{jI} = -\Omega_i(x) \cdot K_j \cdot \left( P_{jI} - P_{jII} \right) = -\Omega_i(x) \cdot K_j \cdot \left( x_{jI} \cdot P_I - x_{jII} \cdot P_{II} \right)$$

$$ = -\Omega_i(x) \cdot K_j \cdot \left( \frac{F_{jI}}{n} - \sum_{i=1}^{n} F_{jII} \cdot x_{jII} \cdot P_{II} \right)$$
Dividing through with the design equation for the key component yields

\[
\frac{dF_{jl}}{dF_{kl}} = \frac{\sum_{i=1}^{n} F_{il} \cdot P_{i} - x_{jl} \cdot P_{ll}}{\sum_{i=1}^{n} F_{il}} = \frac{\sum_{i=1}^{n} F_{il} \cdot x_{kl} \cdot P_{ll}}{\sum_{i=1}^{n} F_{il}} = \alpha_{jk} \cdot \frac{F_{jl} - P_{r} \cdot x_{jl} \cdot \sum_{i=1}^{n} F_{il}}{F_{kl} - P_{r} \cdot \sum_{i=1}^{n} F_{il}}
\]

where \( \alpha_{jk} = \frac{K_{j}}{K_{k}} \)

\[
p_{r} = \frac{P_{ll}}{P_{l}}
\]

These \( n-1 \) ordinary couple differential equations then have to be solved iteratively by a numerical method such as a 4th order Runge-Kutta routine, since the molecular fraction in the permeate stream is equal to

\[
x_{jl} = \frac{F_{jl} - F_{jr}}{\sum_{i=1}^{n} (F_{il} - F_{ir})}
\]

The necessary membrane area can then be found as

\[
\frac{dF_{jl}}{dx} = -\Omega(x) \cdot K_{j} \cdot \left( \frac{\sum_{i=1}^{n} F_{il} \cdot P_{i} - x_{jl} \cdot P_{ll}}{\sum_{i=1}^{n} F_{il}} \right)
\]

\[
A_{l} = \frac{1}{\Omega_{l}(x)} \int_{0}^{1} \frac{1}{F_{ji} \cdot x_{kl} \cdot P_{ll}} \cdot dF_{kl}
\]

This integral will in most cases has to be solved numerically by for instance Simpsons method.
Example 2.6: Plug flow-mixed flow membrane module for binary feed mixture

For binary mixtures the equations of example E2.5 have an analytical solution. For the problem shown in figure E2.7 the design equations are reduced to

\[
\frac{dF_{II}}{dF_{II}} = \alpha_{II} \cdot \frac{F_{II} - p_r \cdot x_{lp} \cdot (F_{II} + F_{II})}{F_{II} - p_r \cdot x_{lp} \cdot (F_{II} + F_{II})} \\
= \alpha_{II} \cdot \frac{F_{II} \cdot (1 - p_r \cdot x_{lp}) - p_r \cdot x_{lp} \cdot F_{II}}{F_{II} \cdot (1 - p_r \cdot x_{lp}) - p_r \cdot x_{lp} \cdot F_{II}} \\
= \alpha_{II} \cdot \frac{B \cdot F_{II} - C \cdot F_{II}}{D \cdot F_{II} - A \cdot F_{II}}
\]

where the constants A, B, C and D are defined as

\[
A = p_r \cdot x_{lp} \\
B = 1 - p_r \cdot x_{lp} \\
C = p_r \cdot x_{lp} \\
D = 1 - p_r \cdot x_{lp}
\]

This differential equation can be solved by using the following rather elegant substitution method whereby a separable differential equation is obtained.
Setting

\[ \zeta = \frac{F_{1I}}{F_{2I}} \]

leads to

\[ \frac{d\zeta}{dF_{2I}} = \frac{d}{dF_{2I}} \left( \frac{F_{1I}}{F_{2I}} \right) = \frac{1}{F_{2I}} \left( \frac{dF_{1I}}{dF_{2I}} - \frac{F_{1I}}{F_{2I}} \right) = \frac{1}{F_{2I}} \left( \frac{dF_{1I}}{dF_{2I}} - \frac{F_{1I}}{F_{2I}} \right) \]

Substituting the equation for \( \frac{dF_{1I}}{dF_{2I}} \) into this differential equation yields

\[ \frac{d\zeta}{dF_{2I}} = \frac{1}{F_{2I}} \left( \alpha_{12} \frac{B F_{1I} - C F_{2I} + F_{1I}}{D F_{1I} - A F_{1I}} - \frac{F_{1I}}{F_{2I}} \right) = \frac{1}{F_{2I}} \left( \alpha_{12} \frac{B F_{1I} - C}{D - A \zeta} - \zeta \right) \]

Separating the variables and integrating gives

\[ \alpha_{12} \int_{F_{2r}}^{F_{2f}} \frac{1}{F_{2I}} dF_{2I} = \int_{\zeta}^{\zeta_f} \frac{1}{B \zeta - C - \zeta} d\zeta \]

\[ \alpha_{12} \ln \left( \frac{F_{2f}}{F_{2r}} \right) = \int_{\zeta}^{\zeta_f} \frac{1}{B \zeta - C - \zeta} d\zeta \]

where \( \zeta = \frac{F_{1r}}{F_{2r}} \)

\( \zeta_f = \frac{F_{1f}}{F_{2f}} \)
Integration of the right side of the equation takes a little extra effort:

\[
\int_{\zeta}^{\xi} \frac{1}{B \cdot (\zeta - C)} \, d\zeta = \int_{\zeta}^{\xi} \frac{D - A \cdot \zeta}{B \cdot (\zeta - C) - D \cdot \zeta + A \cdot \zeta^2} \, d\zeta = \int_{\zeta}^{\xi} \frac{D - A \cdot \zeta}{A \cdot \zeta^2 + (B - D) \cdot \zeta - C} \, d\zeta \\
= D \int_{\zeta}^{\xi} \frac{1}{A \cdot \zeta^2 + (B - D) \cdot \zeta - C} \, d\zeta - E \int_{\zeta}^{\xi} \frac{\zeta}{A \cdot \zeta^2 + (B - D) \cdot \zeta - C} \, d\zeta
\]

The solution to these two integrals can be found in standard tables of integrals [3]:

\[
\int_{\zeta}^{\xi} \frac{1}{A \cdot \zeta^2 + (B - D) \cdot \zeta - C} \, d\zeta = \frac{1}{2} \ln \left( A \cdot \zeta^2 + (B - D) \cdot \zeta - C \right)_{\zeta}^{\xi}
\]

\[
\int_{\zeta}^{\xi} \frac{\zeta}{A \cdot \zeta^2 + (B - D) \cdot \zeta - C} \, d\zeta = -\frac{B - D}{2 \cdot A \sqrt{(B - D)^2 + 4 \cdot A \cdot C}} \left[ \ln \left( \frac{2 \cdot A \cdot \zeta + (B - D) - \sqrt{(B - D)^2 + 4 \cdot A \cdot C}}{2 \cdot A \cdot \zeta + (B - D) + \sqrt{(B - D)^2 + 4 \cdot A \cdot C}} \right)_{\zeta}^{\xi} \right]
\]

The right side integral therefore becomes

\[
\int_{\zeta}^{\xi} \frac{1}{B \cdot (\zeta - C)} \, d\zeta = \frac{D}{2 \cdot A \sqrt{(B - D)^2 + 4 \cdot A \cdot C}} \left[ \ln \left( \frac{2 \cdot A \cdot \zeta + (B - D) - \sqrt{(B - D)^2 + 4 \cdot A \cdot C}}{2 \cdot A \cdot \zeta + (B - D) + \sqrt{(B - D)^2 + 4 \cdot A \cdot C}} \right)_{\zeta}^{\xi} \right]
- \frac{A}{2 \cdot A} \left[ \ln \left( A \cdot \zeta^2 + (B - D) \cdot \zeta - C \right)_{\zeta}^{\xi} \right]
+ \frac{(B - D) \cdot A}{2 \cdot A \sqrt{(B - D)^2 + 4 \cdot A \cdot C}} \left[ \ln \left( \frac{2 \cdot A \cdot \zeta + (B - D) - \sqrt{(B - D)^2 + 4 \cdot A \cdot C}}{2 \cdot A \cdot \zeta + (B - D) + \sqrt{(B - D)^2 + 4 \cdot A \cdot C}} \right)_{\zeta}^{\xi} \right]
- \frac{(B + D) - \sqrt{(B - D)^2 + 4 \cdot A \cdot C}}{2 \cdot A \sqrt{(B - D)^2 + 4 \cdot A \cdot C}} \left[ \ln \left( \frac{2 \cdot A \cdot \zeta + (B - D) - \sqrt{(B - D)^2 + 4 \cdot A \cdot C}}{2 \cdot A \cdot \zeta + (B - D) + \sqrt{(B - D)^2 + 4 \cdot A \cdot C}} \right)_{\zeta}^{\xi} \right]
- \frac{1}{2} \left[ \ln \left( A \cdot \zeta^2 + (B - D) \cdot \zeta - C \right)_{\zeta}^{\xi} \right]
\]
Using that

\[ x_{2p} = 1 - x_{ip} \]

leads to

\[
\begin{align*}
A &= p_r^* (1 - x_{ip}) \\
B &= 1 - p_r^* x_{ip} \\
C &= p_r^* x_{ip} \\
D &= 1 - p_r^* + p_r^* x_{ip} \\
B - D &= 1 - p_r^* x_{ip} - 1 + p_r^* - p_r^* x_{ip} = p_r^* (1 - 2 \cdot x_{ip}) \\
B + D &= 1 - p_r^* x_{ip} + 1 - p_r^* + p_r^* x_{ip} = 2 - p_r^* \\
(B - D)^2 + 4 \cdot A \cdot C &= p_r^* (1 + 4 \cdot x_{ip}^2 - 4 \cdot x_{ip}) + 4 \cdot p_r^2 \cdot x_{ip}^2 (1 - x_{ip}) = p_r^* \end{align*}
\]

from which the integrals simplify to

\[
\frac{1}{B \cdot C - C} \int_0^1 \frac{d\zeta}{D - A \cdot \zeta - \zeta} = \frac{2 - p_r^*}{2 \cdot p_r^*} \left[ \ln \left( \frac{2 \cdot p_r^* (1 - x_{ip}) \cdot \zeta + p_r^* (1 - 2 \cdot x_{ip}) - p_r^*}{2 \cdot p_r^* (1 - x_{ip}) \cdot \zeta + p_r^* (1 - 2 \cdot x_{ip}) + p_r^*} \right) \right]_0^1
\]

Inserting this result in the design equation lead to

\[
a_{12} \cdot \ln \left( \frac{F_{2f}}{F_{2r}} \right) = \frac{2 - p_r^*}{2 \cdot p_r^*} \left[ \ln \left( \frac{2 \cdot p_r^* (1 - x_{ip}) \cdot \zeta + p_r^* (1 - 2 \cdot x_{ip}) - p_r^*}{2 \cdot p_r^* (1 - x_{ip}) \cdot \zeta + p_r^* (1 - 2 \cdot x_{ip}) + p_r^*} \right) \right] - \ln \left( \frac{2 \cdot p_r^* (1 - x_{ip}) \cdot \zeta + p_r^* (1 - 2 \cdot x_{ip}) + p_r^*}{2 \cdot p_r^* (1 - x_{ip}) \cdot \zeta + p_r^* (1 - 2 \cdot x_{ip}) + p_r^*} \right)
\]

Since

\[
x_{ip} = \frac{F_{if} - F_{ir}}{F_{if} + F_{2f} - F_{ir} - F_{2r}} = \frac{F_{if} - \zeta_r}{F_{2f} + F_{2r} - \zeta_r - 1}
\]
\( \zeta \) is the only unknown which can be found by iteration. The necessary membrane area \( A_I \) can afterwards be determined from

\[
A_I = \int^L_0 \Omega_{i}(x) \, dx = \int^F_{F_2} \frac{1}{K_2 (x_{22} - x_{21})}\left( \frac{F_2}{F_1} + \frac{F_2}{F_2} \right) \, dF_2 
\]

which has to be solved numerically using for example Simpsons method.

### 2.4 Mole Balance and Design Equation for Cocurrent Plug Flow Membrane Module (coPF-MM)

#### 2.4.1 Cocurrent Plug Flow Membrane Module with Reaction

In Cocurrent Plug Flow Membrane Modules (coPF-MM) the fluid in both module chambers move in plug flow and in the same direction as shown schematically in figure 2.6. This type of flow often occur in hollow fiber, capillary and tubular membrane modules used for gas separation and ultrafiltration, and can be accomplice in flat sheet membrane modules.

![Figure 2.6 Cocurrent plug flow membrane module](image-url)
From the general mole balance (1.8)

\[
\text{System: Accumulated} = \text{In} - \text{Out} + \text{Generated}
\]

\[
\begin{align*}
\text{I: } & \quad \frac{dN_{ji}}{dt} = F_{ji}\big|_{\text{In}} - F_{ji}\big|_{\text{Out}} - \int J_{ji} dA_i + \int r_{ji} dV_i \\
\text{II: } & \quad \frac{dN_{ji}}{dt} = F_{ji}\big|_{\text{In}} - F_{ji}\big|_{\text{Out}} - \int J_{ji} dA_{ii} + \int r_{ji} dV_{ii} \\
\text{M: } & \quad \frac{dN_{ji}}{dt} = \int J_{ji} dA_i + \int J_{ji} dA_{ii} - 0 + \int r_{ji} dV_{M}
\end{align*}
\] (1.8)

the mole balance for system I still simplifies to equation (2.27)

\[
\frac{\partial C_{ji}}{\partial t} = - \frac{1}{A_{Tf}(x)} \frac{\partial F_{ji}}{\partial x} - \frac{\Omega_i(x)}{A_{Tf}(x)} \cdot J_{ji} + r_{ji}
\] (2.27)

as shown in paragraph 2.3. In similar fashion the mole balance for system II simplifies to

\[
\frac{\partial C_{ji}}{\partial t} = - \frac{1}{A_{Tf}(x)} \frac{\partial F_{ji}}{\partial x} - \frac{\Omega_{ii}(x)}{A_{Tf}(x)} \cdot J_{ji} + r_{ji}
\] (2.38)

while the mole balance for the membrane remains as (2.34)

\[
\frac{\partial C_{ji}}{\partial t} = - \frac{\partial J_{ji}}{\partial y} + r_{M}
\] (2.34)

The general design equations for cocurrent plug flow membrane module therefore becomes

\[
\begin{align*}
\text{System:} \\
\text{I: Plug flow chamber: } & \quad \frac{\partial C_{ji}}{\partial t} = - \frac{1}{A_{Tf}(x)} \frac{\partial F_{ji}}{\partial x} - \frac{\Omega_i(x)}{A_{Tf}(x)} \cdot J_{ji} + r_{ji} \\
\text{II: Plug flow chamber: } & \quad \frac{\partial C_{ji}}{\partial t} = - \frac{1}{A_{Tf}(x)} \frac{\partial F_{ji}}{\partial x} - \frac{\Omega_{ii}(x)}{A_{Tf}(x)} \cdot J_{ji} + r_{ji} \\
\text{M: Membrane: } & \quad \frac{\partial C_{ji}}{\partial t} = - \frac{\partial J_{ji}}{\partial y} + r_{M}
\end{align*}
\] (2.39)
Under steady-state conditions these design equations simplify to

\[
\begin{align*}
\text{System:} & \\
I: \text{Plug flow chamber:} & \quad \frac{dF_{ji}}{dx} = -\Omega_i(x) \cdot J_{ji} + r_{ji} \cdot A_{Ti}(x) \\
II: \text{Plug flow chamber:} & \quad \frac{dF_{ji}}{dx} = -\Omega_{ii}(x) \cdot J_{ji} + r_{ji} \cdot A_{Tii}(x) \\
M: \text{Membrane:} & \quad \frac{\partial J_{jm}}{\partial y} = r_{jm}
\end{align*}
\]

(2.40)

2.4.2 Cocurrent Plug Flow Membrane Module without Reaction

If no reaction occurs in the membrane module the steady state design equations reduces to

\[
\begin{align*}
\text{System:} & \\
I: \text{Plug flow chamber:} & \quad \frac{dF_{ji}}{dx} = -\Omega_i(x) \cdot J_{ji} \\
II: \text{Plug flow chamber:} & \quad \frac{dF_{ji}}{dx} = -\Omega_{ii}(x) \cdot J_{ji} = -\frac{dF_{ji}}{dx} \\
M: \text{Membrane:} & \quad 0 = \frac{\partial J_{jm}}{\partial y} = J_{ji} = -J_{ji} \cdot \frac{\Omega_{ii}(x)}{\Omega_i(x)}
\end{align*}
\]

(2.41)
Example 2.7: cocurrent Plug Flow Membrane Module for Gas Separation

For a gas separation process where the two membrane module chambers behave as cocurrent plug flow chambers the following method can be used to solve the design equations. The gas separation module is shown in figure E2.8.

The following quantities are assumed to be specified in advance:

- $F_{\text{tot}I}$: the total molecular feed flow to chamber I \( \text{mol/s} \)
- $F_{\text{tot}II}$: the total molecular feed flow to chamber II \( \text{mol/s} \)
- $F_{kI}$: the molecular product flow for a key component $k$ from chamber I \( \text{mol/s} \)
- $P_I$: the total pressure in chamber I \([\text{Pa}]\)
- $P_{II}$: the total pressure in chamber II \([\text{Pa}]\)
- $T_I$: the temperature in chamber I \([\text{K}]\)
- $T_{II}$: the temperature in chamber II \([\text{K}]\)
- $x_{jI}$: the molecular fraction of component $j$ in the feed to chamber I
- $x_{jII}$: the molecular fraction of component $j$ in the feed to chamber II

while the following quantities are to be determined:

- $A_I$: the membrane area facing chamber I \([\text{m}^2]\)
- $A_{II}$: the membrane area facing chamber II \([\text{m}^2]\)
- $F_{\text{tot}IP}$: the total molecular product flow from chamber I \( \text{mol/s} \)
- $F_{\text{tot}IIp}$: the total molecular product flow from chamber II \( \text{mol/s} \)
- $x_{jIP}$: the molecular fraction of component $j$ product from chamber I
- $x_{jIIp}$: the molecular fraction of component $j$ product from chamber II
Furthermore the flux through the membrane of component \( j \) is assumed to be described by an equation of the form

\[ J_j = K_j \left( P_{jIM} - P_{jIIM} \right) \]

where \( K_j \) is a mass transfer coefficient \( \left[ \frac{\text{mol}}{\text{Pa} \cdot \text{m}^2 \cdot \text{s}} \right] \).

\( P_{jIM} \) is the partial pressure of component \( j \) on the membrane surface in module chamber I [Pa]

\( P_{jIIM} \) is the partial pressure of component \( j \) on the membrane surface in module chamber II [Pa]

Assuming steady state and no reactions during the process design equations (2.41) for the membrane unit are applicable:

\[
\begin{align*}
\text{System:} \\
\text{I: Plug flow chamber:} & \quad \frac{dF_{JI}}{dx} = -\Omega_I(x) \cdot J_{JI} \\
\text{II: Plug flow chamber:} & \quad \frac{dF_{JI}}{dx} = -\Omega_{II}(x) \cdot J_{JI} = -\frac{dF_{JI}}{dx} \\
\text{M: Membrane:} & \quad 0 = \frac{\partial J_{JM}}{\partial y} = J_{JM} = -J_{JM} \cdot \frac{\Omega_{II}(x)}{\Omega_I(x)} 
\end{align*}
\]

For chamber I and II the mass balance becomes

\[
\frac{dF_{JI}}{dx} = -\Omega_I(x) \cdot J_{JI} = -\Omega_I(x) \cdot K_j \left( P_{JI} - P_{JII} \right) = -\Omega_I(x) \cdot K_j \left( x_{JI} \cdot P_I - x_{JII} \cdot P_{II} \right)
\]

\[
\frac{dF_{JI}}{dx} = -\Omega_I(x) \cdot K_j \left\{ \frac{\sum_{i=1}^{n} F_{JI}}{\sum_{i=1}^{n} F_{II}} - \frac{F_{JI}}{F_{II}} \cdot P_{II} \right\}
\]

\[
\frac{dF_{JI}}{dx} = \Omega_{II}(x) \cdot J_{JI} = \Omega_{II}(x) \cdot K_j \left\{ \frac{\sum_{i=1}^{n} F_{II}}{\sum_{i=1}^{n} F_{II}} - \frac{F_{JI}}{F_{II}} \cdot P_{II} \right\}
\]

This apparently gives \( 2n \) ordinary coupled differential equations to solve. A daunting task even with todays computers if the number of components \( n \) are more than 4 to 5. The number of differential equations that need to be solved though can be halved by setting up a molar balance over the dotted box shown in figure E2.8:

\[
\begin{align*}
\text{In} & = \text{Out} \\
F_{JI} + F_{JII} & = F_{JII} + F_{JI}
\end{align*}
\]

This eliminates the need for the differential equations describing the flow in chamber II.

To eliminate the as yet unknown length of the membrane module the design equations for chamber I is divided through
with the chamber I design equation for the key component:

\[ \frac{dF_{ji}}{dF_{kl}} = \frac{F_{ji} \cdot P_1 - F_{jil} \cdot P_{il}}{K_j \sum_{i=1}^{n} F_{il} - K_k \sum_{i=1}^{n} F_{il}} = \frac{F_{ji} \cdot P_1 - F_{jil} \cdot P_{il}}{K_j \sum_{i=1}^{n} F_{il} - K_k \sum_{i=1}^{n} F_{il}} \]

where \( \alpha_{jk} = \frac{K_j}{K_k} \)

\[ p_r = \frac{P_{il}}{P_1} \]

Using the molar balances for chamber II leads to

\[ \sum_{i=1}^{n} F_{jll} = \sum_{i=1}^{n} F_{jlf} + \sum_{i=1}^{n} F_{jllf} - \sum_{i=1}^{n} F_{jil} = F_{totlf} + F_{totlf} - \sum_{i=1}^{n} F_{jil} \]

which inserted in the design equations gives

\[ \frac{dF_{ji}}{dF_{kl}} = \frac{F_{jil} + F_{jllf} - F_{ji} \cdot P_{il} + F_{jil} - F_{jil} \cdot P_{il}}{F_{totlf} + F_{totlf} - \sum_{i=1}^{n} F_{il}} \]

These \( n-1 \) ordinary couple differential equations then have to be solved iteratively by a numerical method such as a 4th order Runge-Kutta routine.

The necessary membrane area can then be found as

\[ \frac{dF_{ji}}{dx} = -\Omega_1(x) \cdot K_j \left( \frac{F_{ji} \cdot P_1 - F_{jil} \cdot P_{il}}{\sum_{i=1}^{n} F_{il} - \sum_{i=1}^{n} F_{il}} \right) \]

\[ A_1 = \int_0^1 \Omega_1(x) dx = \int_{k_{hi}}^{k_{ho}} 1 \cdot \left( \frac{F_{jil} \cdot P_1 - F_{jil} \cdot P_{il}}{\sum_{i=1}^{n} F_{il} - \sum_{i=1}^{n} F_{il}} \right) dF_{kl} \]

This integral will in most cases have to be solved numerically by for instance Simpsons method.
2.5 Mole Balance and Design Equation for Countercurrent Plug Flow Membrane Module (counterPF-MM)

2.5.1 Countercurrent Plug Flow Membrane Module with Reaction

In Countercurrent Plug Flow Membrane Modules (counterPF-MM) the fluid in both module chambers move in plug flow but in opposite directions as shown schematically in figure 2.7. This type of flow is often seen in hollow fiber, capillary and turbular membrane modules for membrane distillation, gas stripping and gas separation and can be achieved in flat sheet membrane modules.

![Figure 2.7 Countercurrent plug flow membrane module](image)

From the general mole balance (1.8)

\[
\begin{align*}
\text{System: } & \quad \text{Accumulated} = \frac{dN_j}{dt} = F_{j\text{ln}} - F_{j\text{out}} - \int J_{j\text{ln}} \, dA_j + \int r_{j\text{ln}} \, dV_j \\
\text{I: } & \quad \frac{dN_{j\text{I}}}{dt} = F_{j\text{ln}} - F_{j\text{out}} - \int J_{j\text{ln}} \, dA_j + \int r_{j\text{ln}} \, dV_j \\
\text{II: } & \quad \frac{dN_{j\text{II}}}{dt} = F_{j\text{ln}} - F_{j\text{out}} - \int J_{j\text{ln}} \, dA_j + \int r_{j\text{ln}} \, dV_j \\
\text{M: } & \quad \frac{dN_{j\text{M}}}{dt} = \int J_{j\text{ln}} \, dA_j + \int J_{j\text{II}} \, dA_j - 0 + \int r_{j\text{M}} \, dV_M
\end{align*}
\]

the mole balance for system I still simplifies to equation (2.27)

\[
\frac{\partial C_j}{\partial t} = -\frac{1}{A_{Tj}(x)} \frac{\partial F_j}{\partial x} - \frac{\Omega_j(x)}{A_{Tj}(x)} \cdot J_j + r_j
\]  

as shown in paragraph 2.3. The mole balance for system II though is a little different since the direction of flow now is opposite the direction of integration, x.
For system II the mass balance can be rewritten as

\[
\frac{dN_{j\text{II}}}{dt} = \frac{d}{dt} \left( \int C_{j\text{II}} \, dV_{\text{II}} \right) = F_{j\text{II}}|_{\text{in}} - F_{j\text{II}}|_{\text{out}} - \int J_{j\text{II}} \, dA_{\text{II}} + \int r_{j\text{II}} \, dV_{\text{II}}
\]

(2.42)

Differentiation with the volume \(V_{\text{II}}\) leads to

\[
\frac{\partial}{\partial V_{\text{II}}} \left( \frac{d}{dt} \left( \int C_{j\text{II}} \, dV_{\text{II}} \right) \right) = \frac{\partial}{\partial V_{\text{II}}} \left( F_{j\text{II}}|_{\text{in}} - F_{j\text{II}}|_{\text{out}} - \int J_{j\text{II}} \, dA_{\text{II}} + \int r_{j\text{II}} \, dV_{\text{II}} \right)
\]

(2.43)

For further simplification the differential volume and differential wetted membrane area in chamber II are expressed as

\[
dV_{\text{II}} = A_{\text{II}}(x) \cdot dx \\
dA_{\text{II}} = \Omega_{\text{II}}(x) \cdot dx
\]

(2.44)

where \(A_{\text{II}}(x)\) is the cross sectional area of the membrane chamber perpendicular to the direction of flow [m²]

dx is the differential length in the direction of flow [m].

\(\Omega_{\text{II}}(x)\) is wetted periferi of the membrane in chamber II [m].

Inserted in the mass balance for chamber II this gives

\[
\frac{\partial C_{j\text{II}}}{\partial t} = \frac{1}{A_{\text{II}}(x)} \frac{\partial F_{j\text{II}}}{\partial x} - \frac{1}{A_{\text{II}}(x)} \frac{\partial}{\partial x} \left( \int J_{j\text{II}} \cdot \Omega_{\text{II}}(x) \, dx \right) + r_{j\text{II}}
\]

(2.45)
The mole balance for the membrane remains as (2.34)

\[ \frac{\partial C_{JM}}{\partial t} = - \frac{\partial J_{JM}}{\partial y} + r_{JM} \] (2.34)

The general design equations for a countercurrent plug flow membrane module therefore are

System:

I: Plug flow chamber:

\[ \frac{\partial C_{JI}}{\partial t} = - \frac{1}{A_{JI}(x)} \frac{\partial F_{JI}}{\partial x} - \frac{\Omega_{I}(x)}{A_{JI}(x)} J_{JI} + r_{JI} \]

II: Plug flow chamber:

\[ \frac{\partial C_{JI}}{\partial t} = - \frac{1}{A_{JI}(x)} \frac{\partial F_{JI}}{\partial x} - \frac{\Omega_{II}(x)}{A_{JI}(x)} J_{JI} + r_{JI} \] (2.46)

M: Membrane:

\[ \frac{\partial C_{JM}}{\partial t} = - \frac{\partial J_{JM}}{\partial y} + r_{JM} \]

Under steady-state conditions these design equations simplify to

System:

I: Plug flow chamber:

\[ \frac{dF_{JI}}{dx} = - \Omega_{I}(x) J_{JI} + r_{JI} A_{JI}(x) \]

II: Plug flow chamber:

\[ \frac{dF_{JI}}{dx} = \Omega_{II}(x) J_{IJ} - r_{JI} A_{JI}(x) \] (2.47)

M: Membrane:

\[ \frac{\partial J_{JM}}{\partial y} = r_{JM} \]

2.5.2 Countercurrent Plug Flow Membrane Module without Reaction

If no reactions occur in the membrane module the steady state design equations reduce to

System:

I: Plug flow chamber:

\[ \frac{dF_{JI}}{dx} = - \Omega_{I}(x) J_{JI} \]

II: Plug flow chamber:

\[ \frac{dF_{JI}}{dx} = \Omega_{II}(x) J_{IJ} = \frac{dF_{JI}}{dx} \] (2.48)

M: Membrane:

\[ 0 = \frac{\partial J_{JM}}{\partial y} = J_{JI} - J_{IJ} \frac{\Omega_{II}(x)}{\Omega_{I}(x)} \]
Example 2.8: countercurrent Plug Flow Membrane Module for Gas Separation

Figure E2.9 countercurrent Plug Flow Membrane Module

For a gas separation process where the two membrane module chambers behave as countercurrent plug flow chambers, the following method can be used to solve the design equations. The gas separation module is shown in figure E2.9.

The following quantities are assumed to be specified in advance:

- $F_{\text{tot}I}$: the total molecular feed flow to chamber I \( \text{mol s}^{-1} \)
- $F_{\text{tot}II}$: the total molecular feed flow to chamber II \( \text{mol s}^{-1} \)
- $F_{\text{tot}pI}$: the molecular product flow for a key component $k$ from chamber I \( \text{mol s}^{-1} \)
- $P_I$: the total pressure in chamber I [Pa]
- $P_{II}$: the total pressure in chamber II [Pa]
- $T_I$: the temperature in chamber I [K]
- $T_{II}$: the temperature in chamber II [K]
- $x_{jI}$: the molecular fraction of component $j$ in the feed to chamber I
- $x_{jII}$: the molecular fraction of component $j$ in the feed to chamber II

while the following quantities are to be determined:

- $A_I$: the membrane area facing chamber I [m$^2$]
- $A_{II}$: the membrane area facing chamber II [m$^2$]
- $F_{\text{tot}pI}$: the total molecular product flow from chamber I \( \text{mol s}^{-1} \)
- $F_{\text{tot}pII}$: the total molecular product flow from chamber II \( \text{mol s}^{-1} \)
- $x_{jIpI}$: the molecular fraction of component $j$ product from chamber I
- $x_{jIpII}$: the molecular fraction of component $j$ product from chamber II
Furthermore the flux through the membrane of component $j$ is assumed to be described by an equation of the form

$$J_j = K_j \cdot (P_{jIM} - P_{jIM})$$

where $K_j$ is a mass transfer coefficient

$$\text{mol m}^{-2} \text{s}^{-1} \text{Pa}^{-1}$$

$P_{jIM}$ is the partial pressure of component $j$ on the membrane surface in module chamber I [Pa]

$P_{jIM}$ is the partial pressure of component $j$ on the membrane surface in module chamber II [Pa]

Assuming steady state and no reactions during the process design equations (2.48) for the membrane unit are applicable:

For chamber I and II the mass balance becomes

$$\begin{align*}
\frac{dF_{jI}}{dx} &= -\Omega_I(x) \cdot J_{jI} \\
\frac{dF_{jII}}{dx} &= -\Omega_I(x) \cdot J_{jII} = -\frac{dF_{jI}}{dx}
\end{align*}$$

This apparently gives $2n$ ordinary coupled differential equations to solve. The number of differential equations that needs to be solved can be halved by setting up a molar balance over the dotted box in figure E2.9:

$$\text{In} = \text{Out}$$

$$F_{jIt} + F_{jII} = F_{jI} + F_{jIIP}$$

$$F_{jII} = F_{jIIP} - F_{jIr} + F_{jI}$$

This eliminates the need for the differential equations describing the flow in chamber II.
To eliminate the as yet unknown length of the membrane module the design equations for chamber I is divided through with the chamber I design equation for the key component:

\[
\frac{dF_{jI}}{dF_{kI}} = \sum_{i=1}^{n} F_{il} \cdot \frac{F_{jl} \cdot P_i - F_{jll} \cdot P_{II}}{\sum_{i=1}^{n} F_{II}} \cdot \frac{F_{jl} - p_r \cdot F_{jll} \cdot \sum_{i=1}^{n} F_{II}}{F_{kl} - p_r \cdot \sum_{i=1}^{n} F_{II}} = \alpha_{jk}
\]

where \( \alpha_{jk} = \frac{K_j}{K_k} \)

\[p_r = \frac{P_{II}}{P_I}\]

Using the molar balances for chamber II leads to

\[\sum_{i=1}^{n} F_{II} = \sum_{i=1}^{n} F_{II} \cdot n - \sum_{i=1}^{n} F_{II} + \sum_{i=1}^{n} F_{II} = F_{totIIp} - F_{totIIf} + \sum_{i=1}^{n} F_I\]

which inserted in the design equations gives

\[
\frac{dF_{II}}{dF_{kl}} = \sum_{i=1}^{n} F_{II} \cdot \frac{F_{jllp} - F_{jle} + F_{jll}}{F_{totIIp} - F_{totIIf} + \sum_{i=1}^{n} F_I} - \frac{n}{\sum_{i=1}^{n} F_I} = \alpha_{jk} \cdot \frac{F_{kllp} - F_{kle} + F_{kl}}{F_{totIIp} - F_{totIIf} + \sum_{i=1}^{n} F_I} - \frac{n}{\sum_{i=1}^{n} F_I}
\]

These n-1 ordinary couple differential equations then have to be solved iteratively by a numerical method such as a 4. order Runge-Kutta routine. The only problem is, that \( F_{jIIp} \) and \( F_{totIIp} \) are unknown. The procedure is now to guess a value for \( F_{jIIp} \), solve the design equations and see if the found values for \( F_{jIIf} \) are identical to the given values. This has to be done iteratively. When the correct values for \( F_{jIIp} \) have been determined the necessary membrane area can be found as

\[
\frac{dF_{II}}{dx} = -\Omega(x) \cdot K_j \cdot \left( \frac{F_{jll} \cdot P_I - F_{jll} \cdot P_{II}}{\sum_{i=1}^{n} F_{II}} \right)
\]

\[
A_1 = \int_{0}^{l_1} \Omega(x) \cdot dx = \int_{F_{jlp}}^{F_{jIIp}} \left( \frac{F_{kllp} - F_{kle} + F_{kl}}{\sum_{i=1}^{n} F_I} \right) - \frac{n}{\sum_{i=1}^{n} F_I} \cdot dF_{II}
\]

This integral will in most cases have to be solved numerically by for instance Simpsons method.
2.6 Mole Balance and Design Equation for Plug Flow - Differential Cross Flow Membrane Module (PF&DCF-MM)

2.6.1 Plug Flow - Differential Cross Flow Membrane Module (PF&DCF-MM) with Reaction

In a Plug Flow - Differential Cross Flow Membrane Module (PF&DCF-MM) the fluid in chamber I, the feed chamber, flows in plug flow, while the fluid in chamber II, the permeate chamber, flows perpendicular away from the membrane as shown schematically in figure 2.8. This flow configuration is seldom seen in practice though it can occur in microfiltration, ultrafiltration, nanofiltration, pervaporation and vacuum membrane processes when flat sheet membranes are used.

\[
\frac{dN_{jI}}{dt} = F_{jI}\big|_{\text{In}} - F_{jI}\big|_{\text{Out}} - \int J_{jI} dA_I + \int r_{jI} dV_I
\]

\[
\frac{dN_{jII}}{dt} = F_{jII}\big|_{\text{In}} - F_{jII}\big|_{\text{Out}} - \int J_{jII} dA_{II} + \int r_{jII} dV_{II}
\]

\[
\frac{dN_{jM}}{dt} = \int J_{jI} dA_I + \int J_{jII} dA_{II} - 0 + \int r_{jM} dV_M
\]

the mole balance for system I still simplifies to equation (2.27)

\[
\frac{\partial C_{jI}}{\partial t} = -\frac{1}{A_{jI}(x)} \frac{\partial F_{jI}}{\partial x} - \frac{\Omega_j(x)}{A_{jI}(x)} J_{jI} + r_{jI}
\]

as shown in paragraph 2.3.
The mole balance for system II though is different since the fluid flows away from the membrane. For system II the mass balance can be rewritten as equation (2.42)

\[
\frac{dN_{jII}}{dt} = \frac{d}{dt} \left( \int C_{jII} \, dV_{II} \right) = F_{jII}^\text{in} - F_{jII}^\text{out} - \int J_{jII} \, dA_{II} + \int r_{jII} \, dV_{II}
\]  

(2.42)

Differentiation with the volume \(V_{II}\) leads to (2.49)

\[
\frac{\partial}{\partial V_{II}} \left( \frac{d}{dt} \left( \int C_{jII} \, dV_{II} \right) \right) = \frac{\partial}{\partial V_{II}} \left( F_{jII}^\text{in} - F_{jII}^\text{out} \right) - \frac{\partial}{\partial V_{II}} \left( \int J_{jII} \, dA_{II} \right) + \frac{\partial}{\partial V_{II}} \left( \int r_{jII} \, dV_{II} \right)
\]

(2.49)

For further simplification the differential volume and differential wetted membrane area in chamber II are expressed as

\[
\frac{dV_{II}}{dt} = dx \cdot dy \cdot dz \\
\frac{dA_{II}}{dt} = dx \cdot dz 
\]

(2.50)

while the molecular flow away from the membrane is expressed as the area integral of the product between velocity and concentration of component \(j\):

\[
F_{jII} = \int \int u_{II}(y) \cdot C_{jII} \, dx \, dz
\]

(2.51)

where \(u_{II}(y)\) is the velocity of the fluid [m·s\(^{-1}\) ].
Inserted in the mass balance for chamber II this gives

\[
\frac{\partial C_{ji}}{\partial t} = -\frac{\partial}{\partial x} \left( \int \int u_i(y) \cdot C_{ji} \, dx \, dz \right) - \frac{\partial}{\partial y} \left( \int \int J_{ji} \, dx \, dz \right) + r_{ji}
\]

(2.52)

Since the flux from chamber II to the membrane, \( J_{ji} \), does not depend on \( y \), equation (2.52) simplifies further to

\[
\frac{\partial C_{ji}}{\partial t} = -\frac{\partial (u_i(y) \cdot C_{ji})}{\partial y} + r_{ji}
\]

(2.53)

The mole balance for the membrane remains as (2.34)

\[
\frac{\partial C_{JM}}{\partial t} = -\frac{\partial J_{JM}}{\partial y} + r_{JM}
\]

(2.34)

The general design equations for a plug flow- differential cross flow membrane module therefore are

System:

I: Plug flow chamber:

\[
\frac{\partial C_{ji}}{\partial t} = -\frac{1}{A_{T_1}(x)} \frac{\partial F_{ji}}{\partial x} - \frac{\Omega_i(x)}{A_{T_1}(x)} \cdot J_{ji} + r_{ji}
\]

II: Cross flow chamber:

\[
\frac{\partial C_{ji}}{\partial t} = -\frac{\partial (u_i(y) \cdot C_{ji})}{\partial y} + r_{ji}
\]

M: Membrane:

\[
\frac{\partial C_{JM}}{\partial t} = -\frac{\partial J_{JM}}{\partial y} + r_{JM}
\]

(2.54)

Under steady-state conditions these design equations simplify to

System:

I: Plug flow chamber:

\[
\frac{dF_{ji}}{dx} = -\Omega_i(x) \cdot J_{ji} + r_{ji} \cdot A_{T_1}(x)
\]

II: Cross flow chamber:

\[
\frac{\partial (u_i(y) \cdot C_{ji})}{\partial y} = r_{ji}
\]

M: Membrane:

\[
\frac{\partial J_{JM}}{\partial y} = r_{JM}
\]

(2.54)
2.6.2 Plug Flow - Differential Cross Flow Membrane Module (PF&DCF-MM) without Reaction

If no reaction occurs in the membrane module the steady state design equations reduce to

\[
\begin{align*}
\text{System:} & \\
\text{I: Plug flow chamber:} & \quad \frac{dF_{jI}}{dx} = -\Omega_I(x) \cdot J_{jI} \\
\text{II: Cross flow chamber:} & \quad \frac{\partial (u_{II}(y) \cdot C_{jII})}{\partial y} = 0 \Rightarrow F_{jII}\big|_{\text{out}} = -\int J_{jII} \, dA_{II} \tag{2.54} \\
\text{M: Membrane:} & \quad 0 = \frac{\partial J_{jM}}{\partial y} \Rightarrow J_{jM} = -J_{jII} \cdot \frac{\Omega_{II}(x)}{\Omega_I(x)}
\end{align*}
\]

Example 2.9: Plug Flow-Differential Cross Flow Membrane Module for gas separation

For a gas separation process where the feed membrane module chamber behaves as a plug flow chamber and the permeate flow in the membrane module chamber is differential cross flow the following method can be used to solve the design equations. The gas separation module is shown in figure E2.10.

The following quantities are assumed to be specified in advance:

- \( F_{\text{totf}} \): the total molecular feed flow [mol/s]
- \( F_k \): the molecular retentate flow for a key component \( k \) [mol/s]
- \( P_I \): the total pressure in chamber I [Pa]
- \( P_{II} \): the total pressure in chamber II [Pa]
- \( T_I \): the temperature in chamber I [K]
- \( T_{II} \): the temperature in chamber II [K]
- \( x_j \): the molecular fraction of component \( j \) in the feed
System:

I: Plug flow chamber:
\[
\frac{\mathrm{d}F_{jI}}{\mathrm{d}x} = -\Omega_I(x) \cdot J_{jI}
\]

II: Cross flow chamber:
\[
\frac{\partial}{\partial y} (u_{II}(y) \cdot C_{jII}) = 0 = F_{jII,\text{Out}} = - \int J_{jII} \mathrm{d}A_{II}
\]

M: Membrane:
\[
0 = \frac{\partial J_{jM}}{\partial y} \Rightarrow J_{jM} = -J_{jM} \cdot \frac{\Omega_{M}(x)}{\Omega_I(x)}
\]

For the feed chamber, chamber I, the molar balance becomes
\[
\frac{\mathrm{d}F_{jI}}{\mathrm{d}x} = -\Omega_I(x) \cdot J_{jI} = -\Omega_I(x) \cdot K_{jI} \cdot (P_{jI} - P_{jII}) = -\Omega_I(x) \cdot K_{jI} \cdot (x_{jI} \cdot P_{jI} - x_{jII} \cdot P_{jII}) \\
= -\Omega_I(x) \cdot K_{jI} \left( \sum_{i=1}^{n} \frac{F_{jI}}{F_{jI}} \cdot P_{jI} - x_{jII} \cdot P_{jII} \right)
\]
The mole fraction in the permeate chamber, chamber II, is as yet unknown, but can at a specific point just outside the membrane surface be calculated as

\[ x_{ji} = \frac{dF_{ji}}{\sum_{i=1}^{n} dF_{ii}} \]

since the flow in chamber II is away from the membrane.

Inserted in the molar balance for chamber I this leads to

\[ \frac{dF_{ji}}{dx} = -\Omega(x)K_j \left( \frac{F_{ji} - P_i}{\sum_{i=1}^{n} F_{ii}} - \frac{dF_{ji}}{\sum_{i=1}^{n} dF_{ii}} \cdot P_i \right) \]

Dividing through with the design equation for the key component yields

\[ \frac{dF_{ji}}{dF_{ki}} = \frac{K_i}{K_k} \cdot \frac{\sum_{i=1}^{n} F_{ii} - \sum_{i=1}^{n} dF_{ii}}{\sum_{i=1}^{n} F_{ii} - \sum_{i=1}^{n} dF_{ii}} \cdot \frac{\sum_{i=1}^{n} dF_{ii}}{\sum_{i=1}^{n} dF_{ii}} \cdot \frac{F_{ji} - P_i}{\sum_{i=1}^{n} F_{ii}} - \frac{dF_{ji}}{dF_{ki}} \cdot \frac{P_i}{P_i} \]

where \( \alpha_{jk} = \frac{K_j}{K_k} \)

\( p_r = \frac{P_{ii}}{P_i} \)

d\( F_{ji} \) is not known nor can it be calculated directly, but using that

\[ \frac{dF_{ji}}{\sum_{i=1}^{n} dF_{ii}} = \frac{\sum_{i=1}^{n} dF_{ii}}{\sum_{i=1}^{n} dF_{ii}} \]

changes the molar balance into

\[ \frac{dF_{ji}}{dF_{ki}} = \alpha_{jk} \cdot \frac{\sum_{i=1}^{n} F_{ii} - \sum_{i=1}^{n} dF_{ii}}{\sum_{i=1}^{n} F_{ii} - \sum_{i=1}^{n} dF_{ii}} \cdot \frac{F_{ji} - P_i}{\sum_{i=1}^{n} F_{ii}} - \frac{dF_{ji}}{dF_{ki}} \cdot \frac{1}{P_r} \]
These \( n-1 \) ordinary couple differential equations then has to be solved iteratively by a numerical method such as a simple Eulers method, treating \( \frac{dF_{ji}}{dF_{ki}} \) as a single variable during each iterative integration step.

The necessary membrane area can then be found as

\[
\frac{dF_{ji}}{dx} = -\Omega_i(x) \cdot K_j \cdot \left( \frac{F_{ji}}{\sum_{i=1}^{n} F_{il}} \cdot P_{ii} - \frac{x_{ji}}{\sum_{i=1}^{n} F_{il}} \cdot P_{ii} \right)
\]

\[
A_i = \int_{x_0}^{x_1} \Omega_i(x) \, dx = \int_{F_{ki}}^{F_{ki}} K_j \cdot \left( \frac{F_{ki}}{\sum_{i=1}^{n} F_{il}} \cdot P_{ii} - \frac{1}{\sum_{i=1}^{n} dF_{ki}} \cdot P_{ii} \right) \, dF_{ki}
\]

This integral in most cases has to be solved numerically by for instance Simpsons method.
Example 2.10: Plug Flow-Differential Cross Flow Membrane Module for binary feed mixture

For binary mixtures the equations of example E2.9 have an analytical solution. For the problem shown in figure E2.11 the design equations are reduced to

\[
\frac{dF_{11}}{dF_{21}} = \frac{F_{11}}{F_{21}} - \frac{dF_{11}}{dF_{21}} + \frac{1}{F_{21}} \cdot \rho_r = \frac{F_{11}}{F_{21}} + 1 - \frac{dF_{11}}{dF_{21}} + \frac{1}{F_{21}} \cdot \rho_r
\]

This differential equation can be solved by using the following substitution method whereby a separable differential equation is obtained.

Setting

\[
\zeta = \frac{F_{11}}{F_{21}}
\]

leads to

\[
\frac{d\zeta}{dF_{21}} = \frac{d\left( \frac{F_{11}}{F_{21}} \right)}{dF_{21}} = \frac{1}{F_{21}} \cdot \frac{dF_{11}}{dF_{21}} - \frac{F_{11}}{F_{21}} \left( \frac{dF_{11}}{dF_{21}} - \frac{F_{11}}{F_{21}} \right)
\]

Substituting \( \zeta \) into the design equation gives
This can be rearranged into a second order equation in \( \frac{dF_{ll}}{dF_{2l}} \) and solved

\[
0 = \left( \frac{dF_{ll}}{dF_{2l}} \right)^2 + \left( 1 - p_{r'}(\zeta + 1) - a_{12}'(\zeta - p_{r'}(\zeta + 1)) \right) \frac{dF_{ll}}{dF_{2l}} - \zeta a_{12}
\]

To ease the mathematical manipulations to follow this solution is rearranged to

\[
\frac{dF_{ll}}{dF_{2l}} = \frac{1}{2} \left[ -1 + p_{r'}(a_{12} - 1) + \zeta (a_{12} - 1) p_{r} - a_{12} \right]
\]

\[
+ \sqrt{\left[ 1 + p_{r'}(a_{12} - 1) \right]^2 + \left( a_{12} - 1 \right)^2 \cdot \zeta^2 + \left( 1 + p_{r'}(a_{12} - 1) \right) \left( a_{12} - 1 \right) \cdot p_{r} + \left( a_{12} - 1 \right) \cdot a_{12} + 4 \cdot \zeta \cdot a_{12} + 4 \cdot a_{12}^2}
\]

where the constants A, B and C are defined as

\[
A = \frac{1}{2} \left[ a_{12} + (1 - a_{12}) p_{r} \right]
\]

\[
C = \frac{1}{2} \left[ 1 + (a_{12} - 1) p_{r} \right]
\]

\[
2B = -2 \cdot A \cdot C + a_{12}
\]

Substituting the equation for \( \frac{dF_{ll}}{dF_{2l}} \) into this differential equation gives

\[
\frac{d\zeta}{dF_{2l}} = \frac{1}{F_{2l}} \left( A \cdot \zeta - C + \sqrt{A^2 \cdot \zeta^2 + 2 \cdot B \cdot \zeta + C^2} - \zeta \right)
\]
Separating the variables and integrating yields

\[
\int_{\zeta_r}^{\zeta_f} \frac{1}{F_{2f}} dF_{2l} = \int_{\zeta_r}^{\zeta_f} \frac{1}{(A - 1) \cdot \zeta - C + \sqrt{A^2 \cdot \zeta^2 + 2 \cdot B \cdot \zeta + C^2}} d\zeta
\]

\[
\ln \left( \frac{F_{2r}}{F_{2f}} \right) = \int_{\zeta_r}^{\zeta_f} \frac{1}{(A - 1) \cdot \zeta - C + \sqrt{A^2 \cdot \zeta^2 + 2 \cdot B \cdot \zeta + C^2}} d\zeta
\]

where \( \zeta_r = \frac{F_{1r}}{F_{2r}} \)

\( \zeta_f = \frac{F_{1f}}{F_{2f}} \)

Integration of the right side of the equation takes a lot of ingenuity, but the result, according to Weller and Steiner [2] and corrected in Weller and Steiner [4], is:

\[
\zeta_f \left( A - 1 \right) \cdot \zeta - C + \sqrt{A^2 \cdot \zeta^2 + 2 \cdot B \cdot \zeta + C^2}
\]

\[
- \frac{1}{2 \cdot A - 1} \cdot \ln \left( \frac{A \left( \sqrt{A^2 \cdot \zeta_r^2 + 2 \cdot B \cdot \zeta_r + C^2} - A \cdot \zeta_r \right) - B}{A \left( \sqrt{A^2 \cdot \zeta_f^2 + 2 \cdot B \cdot \zeta_f + C^2} - A \cdot \zeta_f \right) - B} \right)
\]

\[
- \frac{2 \cdot a_{12} \cdot (A - 1) + 2 \cdot C}{(2 \cdot A - 1) \cdot (a_{12} - 2 \cdot C)} \cdot \ln \left( \frac{\sqrt{A^2 \cdot \zeta_r^2 + 2 \cdot B \cdot \zeta_r + C^2} - A \cdot \zeta_r}{\sqrt{A^2 \cdot \zeta_f^2 + 2 \cdot B \cdot \zeta_f + C^2} - A \cdot \zeta_f} \right) - \alpha_{12} + C
\]

\[
\frac{C}{C \cdot (1 - A) - B} \cdot \ln \left( \frac{\sqrt{A^2 \cdot \zeta_r^2 + 2 \cdot B \cdot \zeta_r + C^2} - A \cdot \zeta_r}{\sqrt{A^2 \cdot \zeta_f^2 + 2 \cdot B \cdot \zeta_f + C^2} - A \cdot \zeta_f} \right) - \alpha_{12} + C
\]

Inserting this result in the design equation lead to

\[
\ln \left( \frac{F_{2r}}{F_{2f}} \right) =
\]

\[
\frac{1}{2 \cdot A - 1} \cdot \ln \left( \frac{A \left( \sqrt{A^2 \cdot \zeta_r^2 + 2 \cdot B \cdot \zeta_r + C^2} - A \cdot \zeta_r \right) - B}{A \left( \sqrt{A^2 \cdot \zeta_f^2 + 2 \cdot B \cdot \zeta_f + C^2} - A \cdot \zeta_f \right) - B} \right)
\]

\[
+ \frac{2 \cdot a_{12} \cdot (A - 1) + 2 \cdot C}{(2 \cdot A - 1) \cdot (a_{12} - 2 \cdot C)} \cdot \ln \left( \frac{\sqrt{A^2 \cdot \zeta_r^2 + 2 \cdot B \cdot \zeta_r + C^2} - A \cdot \zeta_r}{\sqrt{A^2 \cdot \zeta_f^2 + 2 \cdot B \cdot \zeta_f + C^2} - A \cdot \zeta_f} \right) - \alpha_{12} + C
\]

\[
+ \frac{C}{C \cdot (1 - A) - B} \cdot \ln \left( \frac{\sqrt{A^2 \cdot \zeta_r^2 + 2 \cdot B \cdot \zeta_r + C^2} - A \cdot \zeta_r}{\sqrt{A^2 \cdot \zeta_f^2 + 2 \cdot B \cdot \zeta_f + C^2} - A \cdot \zeta_f} \right) - \alpha_{12} + C
\]
Since \( \zeta \) is the only unknown it can now be found by iteration.

The necessary membrane area \( A_1 \) can afterwards be determined from

\[
\frac{dF_{II}}{dx} = -\Omega(x) \cdot K_2 \left( \sum_{i=1}^{n} \frac{F_{ji} \cdot P_i - x_{ji} \cdot P_{II}}{F_{II}} \right)
\]

\[
A_1 = \int_0^1 \Omega_i(x) \, dx = \int_{\zeta_f}^{\zeta_r} F_{II} \left( \frac{F_{2I}}{F_{II} + F_{2I}} \cdot P_i - \frac{1}{\frac{dF_{II}}{dF_{2I}}} - P_{II} \right) \, dF_{2I}
\]

\[
= \int_{\zeta_f}^{\zeta_r} K_2 \left( \frac{F_{2I}}{\frac{1}{\zeta} + 1 - P_1 - \frac{1}{\frac{A \cdot \zeta}{\sqrt{A^2 \cdot \zeta^2 + 2 \cdot B \cdot \zeta + C^2 + 1}} - P_{II}} \right) \, d\zeta
\]

which has to be solved numerically using for example Simpson's method.
2.7 Mole Balance and Design Equation for Cross Flow Membrane Module (CF-MM)

2.7.1 Cross Flow Membrane Module (CF-MM) with Reaction

In a Cross Flow Membrane Module (CF-MM) the fluid in chamber I and II, flows in plug flow but perpendicular to each other as shown schematically in figure 2.9. This flow is the one encountered in spiralwound membrane modules as used for ultrafiltration, nanofiltration and reverse osmosis and can be accomplished in flat sheet membrane modules.

The general mole balance (1.8)

\[
\begin{align*}
\text{System: Accumulated} & \quad \frac{dN_{jI}}{dt} = \left[ F_{jI}|_{\text{In}} - F_{jI}|_{\text{Out}} + \int_{\Lambda_I} J_{jI} \, d\Lambda_I + \int_{V_I} r_{jI} \, dV_I \right] \\
\text{II:} & \quad \frac{dN_{jII}}{dt} = \left[ F_{jII}|_{\text{In}} - F_{jII}|_{\text{Out}} + \int_{\Lambda_{II}} J_{jII} \, d\Lambda_{II} + \int_{V_{II}} r_{jII} \, dV_{II} \right] \\
\text{M:} & \quad \frac{dN_{jM}}{dt} = \int_{\Lambda_I} J_{jI} \, d\Lambda_I + \int_{\Lambda_{II}} J_{jII} \, d\Lambda_{II} - 0 + \int_{V_M} r_{jM} \, dV_M
\end{align*}
\]
can for system I be simplified by using equation (2.21)

\[
\frac{dN_{ji}}{dt} = \frac{d}{dV} \left( \int C_{ji} dV \right) = F_{ji,\text{in}} - F_{ji,\text{out}} - \int J_{ji} dA_{i} + \int r_{ji} dV_{i} \tag{2.21}
\]

and differentiating with the volume as in equation (2.23)

\[
\frac{\partial}{\partial V_i} \left( \frac{d}{dt} \left( \int C_{ji} dV_i \right) \right) = \frac{\partial}{\partial V_i} \left( F_{ji,\text{in}} - F_{ji,\text{out}} \right) - \frac{\partial}{\partial V_i} \left( \int J_{ji} dA_i \right) + \frac{\partial}{\partial V_i} \left( \int r_{ji} dV_i \right) - \frac{\partial C_{ji}}{\partial V_i} = \frac{\partial}{\partial V_i} \left( \int J_{ji} dA_i \right) + r_{ji} \tag{2.23}
\]

A further simplification comes from considering the geometry of the membrane chambers as seen from figure 2.9

\[
dV_{I} = L_{I}(x,z) \cdot dx \cdot dz \\
dV_{II} = L_{II}(x,z) \cdot dx \cdot dz \\
dA_{II} = L_{I}(x,z) \cdot dx \\
dA_{III} = L_{II}(x,z) \cdot dx \\
dA_{I} = dx \cdot dz \\
dA_{II} = dx \cdot dz \tag{2.55}
\]

Where \( L_{I}(x,z) \) is the channel width in chamber I as defined in figure 2.9 [m]

\( L_{II}(x,z) \) is the channel width in chamber II as defined in figure 2.9 [m]

and using that

\[
F_{ji} = \int_{x_{ji}}^{x_{ji}} u_i(x,y) \cdot C_{ji} \cdot L_{I}(x,z) dz \tag{2.56}
\]

Where \( u_i(x,y) \) is the fluid velocity in the x-direction in chamber I [m/s]
From this equation (2.23) reduces to

$$\frac{\partial C_{ji}}{\partial t} = - \frac{1}{L_i(x,z)} \frac{\partial}{\partial x} \left( C_{ji} \cdot u_i(x,z) \cdot L_i(x,z) \right) - \frac{1}{L_i(x,z)} \frac{\partial}{\partial z} \int J_{ji} \, dx \, dz + r_{ji} \quad (2.57)$$

Using a similar approach the mole balance for system II becomes

$$\frac{\partial C_{jii}}{\partial t} = - \frac{1}{L_{ii}(x,z)} \frac{\partial}{\partial z} \left( C_{jii} \cdot u_{ii}(x,z) \cdot L_{ii}(x,z) \right) - \frac{J_{ji}}{L_{ii}(x,z)} + r_{jii} \quad (2.57)$$

Where \( u_{ii}(x,y) \) is the fluid velocity in the z-direction in chamber II [m/s]

The mole balance for the membrane remains as (2.34)

$$\frac{\partial C_{jm}}{\partial t} = - \frac{\partial J_{jm}}{\partial y} + r_{jm} \quad (2.34)$$

The general design equations for a cross flow membrane module therefore are

**System:**

I: Plug flow chamber:

$$\frac{\partial C_{ji}}{\partial t} = - \frac{1}{L_i(x,z)} \frac{\partial}{\partial x} \left( C_{ji} \cdot u_i(x,z) \cdot L_i(x,z) \right) - \frac{J_{ji}}{L_i(x,z)} + r_{ji}$$

II: Plugflow chamber:

$$\frac{\partial C_{jii}}{\partial t} = - \frac{1}{L_{ii}(x,z)} \frac{\partial}{\partial z} \left( C_{jii} \cdot u_{ii}(x,z) \cdot L_{ii}(x,z) \right) - \frac{J_{ji}}{L_{ii}(x,z)} + r_{jii} \quad (2.58)$$

M: Membrane:

$$\frac{\partial C_{jm}}{\partial t} = - \frac{\partial J_{jm}}{\partial y} + r_{jm}$$
Under steady-state conditions these design equations simplify to

\[
\text{System:}
\]

\[
\begin{align*}
\text{I: Plug flow chamber:} & \quad \frac{\partial}{\partial x} \left( C_{ji} u_i(x,z) \cdot L_i(x,z) \right) = - J_{ji} + r_{ji} \cdot L_i(x,z) \\
\text{II: Plugflow chamber:} & \quad \frac{\partial}{\partial z} \left( C_{ji} u_i(x,z) \cdot L_i(x,z) \right) = - J_{ji} + r_{ji} \cdot L_i(x,z) \\
\text{M: Membrane:} & \quad \frac{\partial J_{JM}}{\partial y} = f_{JM}
\end{align*}
\]

(2.59)

2.7.2 Cross Flow Membrane Module (CF-MM) without Reaction

If no reaction occurs in the membrane module the steady state design equations reduce to

\[
\text{System:}
\]

\[
\begin{align*}
\text{I: Plug flow chamber:} & \quad \frac{\partial}{\partial x} \left( C_{ji} u_i(x,z) \cdot L_i(x,z) \right) = - J_{ji} \\
\text{II: Plugflow chamber:} & \quad \frac{\partial}{\partial z} \left( C_{ji} u_i(x,z) \cdot L_i(x,z) \right) = - J_{ji} \\
\text{M: Membrane:} & \quad \frac{\partial J_{JM}}{\partial y} = 0 \Rightarrow J_{ji} = - J_{ji} \cdot \frac{\Omega_i(x)}{\Omega_{ji}(x)}
\end{align*}
\]

(2.60)
3. General Energy Balance for a Membrane System

The general energy balance for a generalized membrane module consisting of a membrane, M, system I and system II as shown in figure 3.1 is:

\[
\begin{align*}
\text{System: } & \quad \text{In } = \text{Out } + \text{Accumulated } \\
\text{I: } & \quad \dot{E}_I|_{\text{in}} \cdot \Delta t + \dot{Q}_I|_{\text{in}} \cdot \Delta t + \dot{Q}_M|_{\text{M}} \cdot \Delta t = \dot{E}_I|_{\text{out}} \cdot \Delta t + \dot{E}_I|_{\text{M}} \cdot \Delta t + \dot{W}_I^\text{tot} \cdot \Delta t + \dot{W}_I^\text{eff} \cdot \Delta t + \Delta E_{\text{sys I}} \\
\text{II: } & \quad \dot{E}_II|_{\text{in}} \cdot \Delta t + \dot{Q}_II|_{\text{in}} \cdot \Delta t + \dot{Q}_M|_{\text{M}} \cdot \Delta t = \dot{E}_II|_{\text{out}} \cdot \Delta t + \dot{E}_II|_{\text{M}} \cdot \Delta t + \dot{W}_II^\text{tot} \cdot \Delta t + \dot{W}_II^\text{eff} \cdot \Delta t + \Delta E_{\text{sys II}} \\
\text{M: } & \quad \dot{E}_M|_{\text{in}} \cdot \Delta t + \dot{Q}_M|_{\text{M}} \cdot \Delta t + \dot{W}_I^\text{tot} \cdot \Delta t + \dot{W}_II^\text{tot} \cdot \Delta t \cdot \Delta t = \quad \dot{Q}_M|_{\text{M}} \cdot \Delta t + \dot{Q}_M|_{\text{M}} \cdot \Delta t + \Delta E_M
\end{align*}
\]
Where

\( E_M \) is the energy stored in the membrane [J]

\( E_{\text{sysI}} \) is the energy stored in system I [J]

\( E_{\text{sysII}} \) is the energy stored in system II [J]

\( \dot{E}_{\text{I}|\text{in}} \) is the energy flow rate to system I with the molar feed flow [J s\(^{-1}\)]

\( \dot{E}_{\text{I}|\text{M}} \) is the energy flow rate from system I with the molar flow to the membrane [J s\(^{-1}\)]

\( \dot{E}_{\text{II}|\text{in}} \) is the energy flow rate from system II with the molar feed flow [J s\(^{-1}\)]

\( \dot{E}_{\text{II}|\text{M}} \) is the energy flow rate from system II with the molar flow to the membrane [J s\(^{-1}\)]

\( \dot{E}_{\text{II}|\text{rad}} \) is the energy flow rate from system II with the molar product flow [J s\(^{-1}\)]

\( \dot{Q}_I \) is the heat effect transferred to system I from the surroundings [J s\(^{-1}\)]

\( \dot{Q}_{I|M} \) is the heat effect transferred to system I from the membrane [J s\(^{-1}\)]

\( \dot{Q}_{II} \) is the heat effect transferred to system II from the surroundings [J s\(^{-1}\)]

\( \dot{Q}_{II|M} \) is the heat effect transferred to system II from the membrane [J s\(^{-1}\)]

\( W_{\text{totI}} \) is power removed from system I as work to surroundings [J s\(^{-1}\)]

\( W_{\text{totII}} \) is power removed from system II as work to the surroundings [J s\(^{-1}\)]

\( W_{\text{totI}|M} \) is power removed from system I as work to the membrane [J s\(^{-1}\)]

\( W_{\text{totII}|M} \) is power removed from system II as work to the membrane [J s\(^{-1}\)]

Concentrating first on system I the energy balance can be rewritten into an apparent ordinary first order differential equations:

\[
\frac{\Delta E_{\text{sysI}}}{\Delta t} = \dot{E}_{\text{I}|\text{in}} - \dot{E}_{\text{I}|\text{Out}} - \dot{E}_{\text{I}|\text{M}} + \dot{Q}_I + \dot{Q}_{I|M} - \dot{W}_{\text{totI}} - \dot{W}_{\text{totI}|M}
\]

\[
\frac{dE_{\text{sysI}}}{dt} = \dot{E}_{\text{I}|\text{in}} - \dot{E}_{\text{I}|\text{Out}} - \dot{E}_{\text{I}|\text{M}} + \dot{Q}_I + \dot{Q}_{I|M} - \dot{W}_{\text{totI}} - \dot{W}_{\text{totI}|M}
\]
The same procedure applies for system II:

\[
\frac{\Delta E_{\text{sysII}}}{\Delta t} = \dot{E}_{\text{II}|\text{in}} - \dot{E}_{\text{II}|\text{out}} - \dot{E}_{\text{II}|M} + \dot{Q}_{\text{II}|M} - \dot{W}_{\text{II}}^{\text{tot}} - \dot{W}_{\text{II}}^{\text{tot}}|_M
\]

\[
\frac{dE_{\text{sysII}}}{dt} = \dot{E}_{\text{II}|\text{in}} - \dot{E}_{\text{II}|\text{out}} - \dot{E}_{\text{II}|M} + \dot{Q}_{\text{II}|M} - \dot{W}_{\text{II}}^{\text{tot}} - \dot{W}_{\text{II}}^{\text{tot}}|_M
\]

while the energy balance for the membrane becomes

\[
\frac{\Delta E_{\text{M}}}{\Delta t} = \dot{E}_{\text{M}|\text{in}} + \dot{E}_{\text{M}|\text{out}} - \dot{E}_{\text{M}|M} + \dot{W}_{\text{M}}^{\text{tot}} + \dot{W}_{\text{M}}^{\text{tot}}|_M - \dot{Q}_{\text{M}|M} - \dot{Q}_{\text{M}|M}
\]

\[
\frac{dE_{\text{M}}}{dt} = \dot{E}_{\text{M}|\text{in}} + \dot{E}_{\text{M}|\text{out}} - \dot{E}_{\text{M}|M} + \dot{W}_{\text{M}}^{\text{tot}} + \dot{W}_{\text{M}}^{\text{tot}}|_M - \dot{Q}_{\text{M}|M} - \dot{Q}_{\text{M}|M}
\]

The energy term generally is divided into internal, kinetic and potential energy

\[
E = U + E_{\text{pot}} + E_{\text{kin}} = U + E_{\text{pot, elec}}
\]

Where \( E \) is the total energy [J]

- \( E_{\text{kin}} \) is the kinetic energy [J]
- \( E_{\text{pot}} \) is the potential energy [J]
- \( E_{\text{pot, elec}} \) is the potential electrical energy [J]
- \( U \) is the total internal energy [J]

Which, written as energy flow rate becomes [7]:

\[
\dot{E} = \dot{U} + \dot{E}_{\text{pot, elec}} = \sum_{j=1}^{n} \{ F_j \cdot U_j \} + \mathcal{F} \cdot \sum_{j=1}^{n} \{ z_j \cdot F_j \} \Phi
\]

Where \( \dot{E} \) is the total energy flow rate [J \cdot s^{-1}]

- \( \mathcal{F} \) is Faraday’s constant 9.64853 \cdot 10^4 [C \cdot mol^{-1}]
- \( U_j \) is the internal energy of component j [J \cdot mole^{-1}]
- \( \dot{U} \) is the total internal energy flow rate [J \cdot mole^{-1}]
- \( z_j \) is the electrical charge of component j
- \( \Phi \) is the electrical potential [V]
Similarly the work term will normally be divided into PV-work (pressure-volume term), mechanical shaft work and an electric work term.

\[ W_{\text{tot}} = (P \cdot V)_{\text{Out}} - (P \cdot V)_{\text{In}} + W_s + W_{\text{elec}} \] (3.7)

Where \( P \) is total pressure [Pa]
\( V \) is total volume [m³]

\( W_{\text{elec}} \) is the electrical work done by the system on its surroundings [J]
\( W_s \) is the shaft work done by the system on its surroundings [J]

Which written as power produced becomes

\[ \dot{W}_{\text{tot}} = \sum_{j=1}^{n} (P \cdot V_j \cdot F_j)_{\text{Out}} - \sum_{j=1}^{n} (P \cdot V_j \cdot F_j)_{\text{In}} + \dot{W}_s + \dot{W}_{\text{elec}} \] (3.8)

Where \( \dot{W}_{\text{tot}} \) is the total power removed from system as work [J \cdot s⁻¹]

\( \dot{W}_s \) is the total power removed from system to surroundings as shaft work [J]

\( \dot{W}_{\text{elec}} \) is the total electrical power removed from system [J]

Inserting equations (3.6) and (3.8) into the general energy balance (3.2) for system I leads to:

\[ \frac{dE_{\text{sys}}}{dt} = \left( \sum_{j=1}^{n} [F_j \cdot U_j] \right)_{\text{ln}} + \mathcal{F} \left( \sum_{j=1}^{n} [F_j \cdot U_j] \right)_{\text{ln}} - \left( \sum_{j=1}^{n} [F_j \cdot U_j] \right)_{\text{out}} - \mathcal{F} \left( \sum_{j=1}^{n} [F_j \cdot U_j] \right)_{\text{out}} + \bar{Q}_l + \dot{Q}_l - W_s - W_{\text{elec}} \]

\[ - \left( \sum_{j=1}^{n} (P \cdot V_j \cdot F_j) \right)_{\text{ln}} + \left( \sum_{j=1}^{n} (P \cdot V_j \cdot F_j) \right)_{\text{ln}} - \left( \sum_{j=1}^{n} (P \cdot V_j \cdot F_j) \right)_{\text{ml}} \] (3.9)
Rearranging equation (3.9) leads to

\[
\frac{dE_{\text{sysI}}}{dt} = \sum_{j=1}^{n} \left( [F_{ji}(U_j + P \cdot V_j)]_{\text{in}} - [F_{ji}(U_j + P \cdot V_j)]_{\text{out}} - [F_{ji}(U_j + P \cdot V_j)]_{\text{MII}} \right) \\
+ \mathcal{G} \cdot \sum_{j=1}^{n} \left( [z_j \cdot F_{ji} \cdot \Phi]_{\text{in}} - [z_j \cdot F_{ji} \cdot \Phi]_{\text{out}} - [z_j \cdot F_{ji} \cdot \Phi]_{\text{MII}} \right) \\
+ \dot{Q}_I + \dot{Q}_M - \dot{W}_{sl} - \dot{W}_{\text{elec}} \\
= \sum_{j=1}^{n} \left( [F_{ji} \cdot H_j]_{\text{in}} - [F_{ji} \cdot H_j]_{\text{out}} - [F_{ji} \cdot H_j]_{\text{MII}} \right) \\
+ \mathcal{G} \cdot \sum_{j=1}^{n} \left( [z_j \cdot F_{ji} \cdot \Phi]_{\text{in}} - [z_j \cdot F_{ji} \cdot \Phi]_{\text{out}} - [z_j \cdot F_{ji} \cdot \Phi]_{\text{MII}} \right) \\
+ \dot{Q}_I + \dot{Q}_M - \dot{W}_{sl} - \dot{W}_{\text{elec}} \tag{3.10}
\]

Where \( H_j \) is the molar enthalpy for component \( j \) [J·mole\(^{-1}\)]

The same procedure applies to system II:

\[
\frac{dE_{\text{sysII}}}{dt} = \sum_{j=1}^{n} \left( [F_{ji} \cdot H_j]_{\text{in}} - [F_{ji} \cdot H_j]_{\text{out}} - [F_{ji} \cdot H_j]_{\text{MII}} \right) \\
+ \mathcal{G} \cdot \sum_{j=1}^{n} \left( [z_j \cdot F_{ji} \cdot \Phi]_{\text{in}} - [z_j \cdot F_{ji} \cdot \Phi]_{\text{out}} - [z_j \cdot F_{ji} \cdot \Phi]_{\text{MII}} \right) \\
+ \dot{Q}_II + \dot{Q}_M - \dot{W}_{sl} - \dot{W}_{\text{elecII}} \tag{3.11}
\]

The energy equation for the membrane can be rewritten in the form

\[
\frac{dE_{M}}{dt} = \left( \sum_{j=1}^{n} [F_{ji} \cdot U_j] \right)_{\text{MII}} + \mathcal{G} \left( \sum_{j=1}^{n} [z_j \cdot F_{ji} \cdot \Phi] \right)_{\text{MII}} \\
+ \left( \sum_{j=1}^{n} [F_{ji} \cdot U_j] \right)_{\text{MII}} + \mathcal{G} \left( \sum_{j=1}^{n} [z_j \cdot F_{ji} \cdot \Phi] \right)_{\text{MII}} \\
+ \left( \sum_{j=1}^{n} [P \cdot V_j \cdot F_{ji}] \right)_{\text{MII}} + \left( \sum_{j=1}^{n} [P \cdot V_j \cdot F_{ji}] \right)_{\text{MII}} \\
- \dot{Q}_I - \dot{Q}_M \\
= \sum_{j=1}^{n} \left( [F_{ji} \cdot H_j]_{\text{MII}} + [F_{ji} \cdot H_j]_{\text{MII}} \right) + \mathcal{G} \cdot \sum_{j=1}^{n} \left( [z_j \cdot F_{ji} \cdot \Phi]_{\text{MII}} + [z_j \cdot F_{ji} \cdot \Phi]_{\text{MII}} \right) \\
- \dot{Q}_I - \dot{Q}_M \tag{3.12}
\]
Leading to the general energy balances for the membrane unit

\[
\begin{align*}
\frac{dE_{\text{sysI}}}{dt} &= \sum_{j=1}^{n} \left( (F_{jI} - F_{jI})_{\text{in}} - (F_{jI} - F_{jI})_{\text{out}} - (F_{jI} - F_{jI})_{\text{M}} \right) \\
&\quad + \mathcal{J} \cdot \sum_{j=1}^{n} \left( (z_{j} - z_{j})_{\text{in}} - (z_{j} - z_{j})_{\text{out}} - (z_{j} - z_{j})_{\text{M}} \right) \\
&\quad + \dot{Q}_{I} + \dot{Q}_{I|M} - \dot{W}_{sl} - \dot{W}_{elcI}
\end{align*}
\]

\[
\frac{dE_{\text{sysII}}}{dt} = \sum_{j=1}^{n} \left( (F_{jII} - F_{jII})_{\text{in}} - (F_{jII} - F_{jII})_{\text{out}} - (F_{jII} - F_{jII})_{\text{MII}} \right) \\
&\quad + \mathcal{J} \cdot \sum_{j=1}^{n} \left( (z_{j} - z_{j})_{\text{in}} - (z_{j} - z_{j})_{\text{out}} - (z_{j} - z_{j})_{\text{MII}} \right) \\
&\quad + \dot{Q}_{II} + \dot{Q}_{II|M} - \dot{W}_{slII} - \dot{W}_{elcII}
\]

\[
\frac{dE_{\text{M}}}{dt} = \sum_{j=1}^{n} \left( (F_{jI} - F_{jI})_{\text{M}} + (F_{jII} - F_{jII})_{\text{MII}} \right) + \mathcal{J} \cdot \sum_{j=1}^{n} \left( (z_{j} - z_{j})_{\text{M}} + (z_{j} - z_{j})_{\text{MII}} \right) \\
&\quad - \dot{Q}_{I|M} - \dot{Q}_{II|M}
\]

These energy equations can thereafter be simplified depending on the membrane configuration and membrane process.
4. Symbols

A Constant defined in the example where the constant appears

$A_{xy}$ Cross sectional area of membrane in the y-direction [m$^2$]

$A_{FC}$ Contact area between filter cake and filter [m$^2$]

$A_I$ Contact area between system I and membrane [m$^2$]

$A_{II}$ Contact area between system II and membrane [m$^2$]

B Constant defined in the example where the constant appears

C Constant defined in the example where the constant appears

$C_j$ Concentration of component j [mole $\cdot$ m$^{-3}$]

$C_{jF}$ Concentration of component j in filter [mole $\cdot$ m$^{-3}$]

$C_{jFC}$ Concentration of component j in filter cake [mole $\cdot$ m$^{-3}$]

$C_{jF}$ Concentration of component j in feed [mole $\cdot$ m$^{-3}$]

$C_{jp}$ Concentration of component j in product [mole $\cdot$ m$^{-3}$]

$C_{jI}|_I$ Concentration of component j in stream to chamber I [mole $\cdot$ m$^{-3}$]

$C_{jI}|_O$ Concentration of component j in stream from chamber I [mole $\cdot$ m$^{-3}$]

$C_{jII}|_I$ Concentration of component j in stream to chamber II [mole $\cdot$ m$^{-3}$]

$C_{jII}|_O$ Concentration of component j in stream from chamber II [mole $\cdot$ m$^{-3}$]

D Constant defined in the example where the constant appears

$d_i$ Inner diameter of hollow fiber, capillary or turbular membrane [m]

$d_o$ Outer diameter of hollow fiber, capillary or turbular membrane [m]

E Energy [J]

$E_{kin}$ Kinetic energy [J]

$E_M$ Energy stored in membrane [J]

$E_{pot}$ Potential energy [J]

$E_{pot,elec}$ Potential electrical energy [J]

$E_{sysI}$ Energy stored in system I [J]

$E_{sysII}$ Energy stored in system II [J]

$\dot{E}$ Energy flow rate [J $\cdot$ s$^{-1}$]

$\dot{E}_{pot,elec}$ Potential electrical energy flow [J $\cdot$ s$^{-1}$]

$\dot{E}_{I}|_I$ Energy flow rate to system I with molar feed flow [J $\cdot$ s$^{-1}$]

$\dot{E}_{I}|_M$ Energy flow rate from system I in molar flow to membrane [J $\cdot$ s$^{-1}$]

$\dot{E}_{I}|_O$ Energy flow rate from system I with molar product flow [J $\cdot$ s$^{-1}$]

$\dot{E}_{II}|_I$ Energy flow rate to system II with molar feed flow [J $\cdot$ s$^{-1}$]
\( \dot{E}_{\text{II}}^{\text{M}} \)  Energy flow rate from system II with molar flow to membrane \([\text{J} \cdot \text{s}^{-1}]\)

\( \dot{E}_{\text{II}}^{\text{Out}} \)  Energy flow rate from system II with molar product flow \([\text{J} \cdot \text{s}^{-1}]\)

\( \mathcal{F} \)  Faraday’s constant \(9.64853 \cdot 10^4 \ [\text{C} \cdot \text{mol}^{-1}]\)

\( F_j \)  Molecular flow rate of component j \([\text{mole} \cdot \text{s}^{-1}]\)

\( F_p \)  Molecular flow rate of component j in permeate \([\text{mole} \cdot \text{s}^{-1}]\)

\( F_j^{\text{r}} \)  Molecular flow rate of component j in retentate \([\text{mole} \cdot \text{s}^{-1}]\)

\( F_j^{\text{f}} \)  Molecular feed flow for component j to chamber I \([\text{mole} \cdot \text{s}^{-1}]\)

\( F_j^{\text{fII}} \)  Molecular feed flow for component j to chamber II \([\text{mole} \cdot \text{s}^{-1}]\)

\( F_j^{\text{p}} \)  Molecular product flow for component j from chamber I \([\text{mole} \cdot \text{s}^{-1}]\)

\( F_j^{\text{pII}} \)  Molecular product flow for component j from chamber II \([\text{mole} \cdot \text{s}^{-1}]\)

\( F_j^{\text{In}} \)  Molecular flow rate of component j to system I \([\text{mole} \cdot \text{s}^{-1}]\)

\( F_j^{\text{IIin}} \)  Molecular flow rate of component j to system II \([\text{mole} \cdot \text{s}^{-1}]\)

\( F_j^{\text{IIin}} \)  Molecular flow rate of component j from system I to membrane \([\text{mole} \cdot \text{s}^{-1}]\)

\( F_j^{\text{Iin}} \)  Molecular flow rate of component j from system II to membrane \([\text{mole} \cdot \text{s}^{-1}]\)

\( F_j^{\text{I}} \)  Molecular flow rate of component j from system I \([\text{mole} \cdot \text{s}^{-1}]\)

\( F_j^{\text{II}} \)  Molecular flow rate of component j from system II \([\text{mole} \cdot \text{s}^{-1}]\)

\( F_{\text{totf}} \)  Total molecular feed flow rate \([\text{mole} \cdot \text{s}^{-1}]\)

\( F_{\text{top}} \)  Total molecular permeate flow rate \([\text{mole} \cdot \text{s}^{-1}]\)

\( F_{\text{totr}} \)  Total molecular retentate flow rate \([\text{mole} \cdot \text{s}^{-1}]\)

\( G_j^{\text{M}} \)  Mole j generated in membrane \([\text{mole} \cdot \text{s}^{-1}]\)

\( G_j^{\text{I}} \)  Mole j generated in system I \([\text{mole} \cdot \text{s}^{-1}]\)

\( G_j^{\text{II}} \)  Mole j generated in system II \([\text{mole} \cdot \text{s}^{-1}]\)

\( H_j \)  Enthalpy for component j \([\text{J} \cdot \text{mole}^{-1}]\)

\( j \)  Component number

\( J_j \)  Flux of component j \([\text{mole} \cdot \text{m}^2 \cdot \text{s}^{-1}]\)

\( J_{\text{FC}} \)  Flux of component j from filter cake to filter \([\text{mole} \cdot \text{m}^2 \cdot \text{s}^{-1}]\)

\( J_g \)  Flux of component j from system I to membrane \([\text{mole} \cdot \text{m}^2 \cdot \text{s}^{-1}]\)

\( J_{\text{II}} \)  Flux of component j from system II to membrane \([\text{mole} \cdot \text{m}^2 \cdot \text{s}^{-1}]\)

\( K_j \)  Mass transfer coefficient for component j \([\text{mole} \cdot \text{Pa}^{-1} \cdot \text{m}^2 \cdot \text{s}^{-1}]\)

\( L \)  Length \([\text{m}]\)

\( L_{\text{st}} \)  Channel length in chamber I \([\text{m}]\)

\( L_{\text{st}} \)  Channel height in chamber I \([\text{m}]\)

\( L_{\text{ef}} \)  Channel width in chamber I \([\text{m}]\)

\( L_{\text{I(x,z)}} \)  Channel width in chamber I as defined in figure 2.9 \([\text{m}]\)

\( L_{\text{II(x,z)}} \)  Channel width in chamber II as defined in figure 2.9 \([\text{m}]\)
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l_{FC}$</td>
<td>Filter cake depth [m]</td>
</tr>
<tr>
<td>$M_{wj}$</td>
<td>Molecular weight of component $j$ [kg·mol$^{-1}$]</td>
</tr>
<tr>
<td>$N_j$</td>
<td>Number of moles of component $j$ [mole $j$]</td>
</tr>
<tr>
<td>$N_{jF}$</td>
<td>Number of moles of component $j$ in the filter [mole $j$]</td>
</tr>
<tr>
<td>$N_{jFC}$</td>
<td>Number of moles of component $j$ in filter cake [mole $j$]</td>
</tr>
<tr>
<td>$N_{jM}$</td>
<td>Mole $j$ in membran [mole $j$]</td>
</tr>
<tr>
<td>$N_{jI}$</td>
<td>Mole $j$ in system I [mole $j$]</td>
</tr>
<tr>
<td>$N_{jII}$</td>
<td>Mole $j$ in system II [mole $j$]</td>
</tr>
<tr>
<td>$n$</td>
<td>Total number of components in system</td>
</tr>
<tr>
<td>$P$</td>
<td>Pressure [Pa]</td>
</tr>
<tr>
<td>$P_{jm}$</td>
<td>Partial pressure for component $j$ at system I- membrane interface [Pa]</td>
</tr>
<tr>
<td>$P_{jIM}$</td>
<td>Partial pressure for component $j$ at system II- membrane interface [Pa]</td>
</tr>
<tr>
<td>$P_I$</td>
<td>Pressure in system I [Pa]</td>
</tr>
<tr>
<td>$P_{IM}$</td>
<td>Pressure at system I- membrane interface [Pa]</td>
</tr>
<tr>
<td>$P_{II}$</td>
<td>Pressure in system II [Pa]</td>
</tr>
<tr>
<td>$p_r$</td>
<td>Pressure ratio $p_r = \frac{P_{II}}{P_I}$</td>
</tr>
<tr>
<td>$\dot{Q}_I$</td>
<td>Heat effect transferred to system I from the surroundings [J·s$^{-1}$]</td>
</tr>
<tr>
<td>$\dot{Q}_I</td>
<td>_M$</td>
</tr>
<tr>
<td>$\dot{Q}_{II}$</td>
<td>Heat effect transferred to system II from the surroundings [J·s$^{-1}$]</td>
</tr>
<tr>
<td>$\dot{Q}_{II}</td>
<td>_M$</td>
</tr>
<tr>
<td>$R_F$</td>
<td>Resistance to flow in filter [m$^{-1}$]</td>
</tr>
<tr>
<td>$R_{FC}$</td>
<td>Resistance to flow in filter cake [m$^{-1}$]</td>
</tr>
<tr>
<td>$R_M$</td>
<td>Resistance to flow in membrane [m$^{-1}$]</td>
</tr>
<tr>
<td>$R_o$</td>
<td>Overall resistance to flow over membrane system [m$^{-1}$]</td>
</tr>
<tr>
<td>$r_j$</td>
<td>Rate of formation of component $j$ [mole $j$·m$^{-3}$·s$^{-1}$]</td>
</tr>
<tr>
<td>$r_{jF}$</td>
<td>Rate of formation of component $j$ in filter [mole $j$·m$^{-3}$·s$^{-1}$]</td>
</tr>
<tr>
<td>$r_{jFC}$</td>
<td>Rate of formation of component $j$ in filter cake [mole $j$·m$^{-3}$·s$^{-1}$]</td>
</tr>
<tr>
<td>$r_{jM}$</td>
<td>Rate of formation of component $j$ in membrane [mole $j$·m$^{-3}$·s$^{-1}$]</td>
</tr>
<tr>
<td>$r_{jI}$</td>
<td>Rate of formation of component $j$ in system I [mole $j$·m$^{-3}$·s$^{-1}$]</td>
</tr>
<tr>
<td>$r_{jII}$</td>
<td>Rate of formation of component $j$ in system II [mole $j$·m$^{-3}$·s$^{-1}$]</td>
</tr>
<tr>
<td>$S$</td>
<td>Recovery or stage cut, dimensionless</td>
</tr>
<tr>
<td>$T_I$</td>
<td>Temperature in chamber I [$^\circ$C]</td>
</tr>
<tr>
<td>$T_{II}$</td>
<td>Temperature in chamber II [$^\circ$C]</td>
</tr>
<tr>
<td>$t$</td>
<td>Time [s]</td>
</tr>
<tr>
<td>$U$</td>
<td>Internal energy [J]</td>
</tr>
</tbody>
</table>
Symbols

\( U_j \)  Internal energy of component j [J·mole\(^{-1}\)]
\( \dot{U} \)  Internal energy flow rate [J·s\(^{-1}\)]
\( u_j(x,z) \)  Fluid velocity in the x-direction in chamber I [m·s\(^{-1}\)]
\( u_j(x,z) \)  Fluid velocity in the z-direction in chamber II [m·s\(^{-1}\)]
\( V \)  Volume [m\(^3\)]
\( V_F \)  Volume of filter [m\(^3\)]
\( V_{FC} \)  Volume of filter cake [m\(^3\)]
\( V_j \)  Molar volume of component j [m\(^3\)·mole\(^{-1}\)]
\( V_M \)  Volume of membrane [m\(^3\)]
\( V_I \)  Volume of system I [m\(^3\)]
\( V_{II} \)  Volume of system II [m\(^3\)]
\( v \)  Volumetric flow rate [m\(^3\)·s\(^{-1}\)]
\( v_r \)  Volumetric feed flow rate [m\(^3\)·s\(^{-1}\)]
\( v_p \)  Volumetric product flow rate [m\(^3\)·s\(^{-1}\)]
\( v_{I\text{In}} \)  Volumetric flow rate to chamber I [m\(^3\)·s\(^{-1}\)]
\( v_{I\text{Out}} \)  Volumetric flow rate from chamber I [m\(^3\)·s\(^{-1}\)]
\( v_{II\text{In}} \)  Volumetric flow rate to chamber II [m\(^3\)·s\(^{-1}\)]
\( v_{II\text{Out}} \)  Volumetric flow rate from chamber II [m\(^3\)·s\(^{-1}\)]
\( W \)  Work done by system on its surroundings [J]
\( W_s \)  Shaft work done by system on its surroundings [J]
\( W_{elec} \)  Electrical work done by system on its surroundings [J]
\( \dot{W}_{\text{tot}} \)  Power removed from system as work [J·s\(^{-1}\)]
\( \dot{W}_s \)  Power removed from system to surroundings as shaft work [J]
\( W_{elec} \)  Electrical power removed from system [J]
\( \dot{W}_{\text{elec}I} \)  Electrical power removed from system I to surroundings as work [J·s\(^{-1}\)]
\( \dot{W}_{\text{elecII}} \)  Electrical power removed from system II to surroundings as work [J·s\(^{-1}\)]
\( \dot{W}_{sl} \)  Power removed from system I to surroundings as shaft work [J·s\(^{-1}\)]
\( \dot{W}_{sll} \)  Power removed from system II to surroundings as shaft work [J·s\(^{-1}\)]
\( \dot{W}_{\text{tot}} \)  Power removed from system I as work to surroundings [J·s\(^{-1}\)]
\( \dot{W}_{\text{tot}} \)  Power removed from system II as work to surroundings [J·s\(^{-1}\)]
\( \dot{W}_{tot, I} \)  Power removed from system I as work to membrane [J·s\(^{-1}\)]
$W^\text{tot}_{II|M}$  Power removed from system II as work to membrane [J·s$^{-1}$]

$x$  Cartesian coordinate vector [m]

$x_{jf}$  Mole fraction of component $j$ in feed

$x_p$  Mole fraction of component $j$ in permeate

$x_r$  Mole fraction of component $j$ in retentate

$y$  Cartesian coordinate vector [m]

$z$  Cartesian coordinate vector [m]

$z_j$  Electrical charge of component $j$

$\alpha_{spC}$  Specific cake resistance [m·kg$^{-1}$]

$\alpha_{jk}$  is the ideal gas separation factor

$\alpha^*_{jk,\text{Mulder}}$  Intrinsic membrane selectivity as defined in Marcel Mulder [5]

$\alpha^*_{jk,\text{S&H}}$  Intrinsic membrane selectivity as defined in Seader and Henley [6]

$\Delta$  Denotes a finite interval

$\Phi$  Electrical potential [V]

$\mu_{\text{liquid}}$  Liquid viscosity [Pa·s]

$\rho_{j\text{FC}}$  Bulk density of component $j$ in filter cake [kg·m$^{-3}$]

$\zeta$  Molar flow ratio defined in examples as $\zeta = \frac{F_{1I}}{F_{2I}}$

$\zeta_f$  Molar flow ratio defined in examples as $\zeta_f = \frac{F_{1f}}{F_{2f}}$

$\zeta_r$  Molar flow ratio defined in examples as $\zeta_r = \frac{F_{1r}}{F_{2r}}$
5. Litterature


Postscript

Life's a deal, you knew it,
when you signed the dotted line
They nail you to the paper, put a rope around your neck
And so we sing along, the boy stood on the burning deck!

John Lennon