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Published in:
Proceedings of the 3rd International Operational Modal Analysis conference - IOMAC

Publication date:
2009

Document version
Publisher's PDF, also known as Version of record

Citation for published version (APA):

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Download date: 05. feb. 2019
Frequency Domain Decomposition Revisited

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ABSTRACT: In this paper the mathematical background of the Frequency Domain Decomposition (FDD) for Operational Modal Analysis (OMA) is reviewed and complemented. First the theory of decomposition of the spectral density matrix is considered under white noise assumptions and under the assumption of uncorrelated modal coordinates. The numerical decomposition of the spectral is then discussed, and it is shown that instead of using the traditional technique of Singular Value Decomposition (SVD), also the eigenvalue decomposition (EVD) can be used for decomposing the spectral matrix. This is true for square matrices, as well as for cases when the spectral estimation problem is simplified by omitting some of the columns in the spectral matrix. In the latter case, in order to perform the needed decomposition, two eigenvalue problems are solved as an alternative to SVD. Finally the modal identification procedure is discussed. It is shown how closely spaced modes can be identified using an orthogonal projection technique, and it is explained how the modal parameters can be identified using a parametric technique in the frequency domain.

NOMENCLATURE

<table>
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<tr>
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<td>Imaginary unit</td>
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<tr>
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<td>$\Phi$</td>
<td>Mode shape matrix, $N \times M$</td>
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<td>$\tilde{\Phi}$</td>
<td>Extended mode shape matrix, $N \times 2M$</td>
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<tr>
<td>$q$</td>
<td>Modal coordinate</td>
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</table>
\( q \) Modal coordinate vector, \( M \times 1 \)
\( \bar{q} \) Extended modal coordinate vector, \( 2M \times 1 \)
\( \omega \) Natural frequency (rad/s)
\( \lambda \) Continuous time pole
\( C \) Response covariance matrix, \( N \times N \)
\( G \) Response power spectral density (PSD) matrix, \( N \times N \)
\( H \) Frequency response function (FRF) matrix, \( N \times N \)

1 INTRODUCTION

The FDD technique was first introduced by Brincker et al \[1\], in year 2000, as a technique for spectral decomposition based on SVD. However using the SVD for decomposition of the frequency response function matrix and plotting the singular values has been known as “the modal indicator function”, Shih et al \[2\], and decomposing the spectral matrix using EVD has been done for instance by Prevosto \[3\] or Otte et al \[4\]. However, to the knowledge of the authors, it was the first time that the singular values of a spectral matrix was interpreted as being an estimate of the modal coordinates of the system.

The first formulation of the technique, Brincker et al \[1\], did explain how to estimate natural frequencies and mode shapes, however it is was not explained how to estimate the corresponding damping value, this was explained the following year, Brincker et al \[5\], where a non-parametric technique for the damping estimation was introduced.

In the original introduction to the FDD, the modal coordinate was identified by sorting the set of singular values according to the singular vectors. Once a modal peak has been identified, the first singular value at this peak was chosen as the mode shape estimate. Then for adjacent frequency lines, the matching singular value was chosen as singular value where the MAC value between the mode shape estimate and the corresponding singular vector was maximum. However, this approach might not always work well, especially in cases with noise and closely spaced modes.

In Zhang et al \[6\] (2005), another way of estimating the modal coordinate was introduced. Instead of sorting existing singular values, the single-degree-of-system (SDOF) system response was isolated by using a projection technique, called spatial decomposition. This provided better estimates of the SDOF system and thus made the subsequent modal identification easier.

In this paper, we will review the basic theory of FDD in the light of some more general/improved ways of looking at the technique. We will consider the theory of decomposing the spectral density matrix, the theory of numerical decomposition of spectral density matrix, especially we will consider how the spectral matrix can also be decomposed using the EVD, and finally we will review the modal identification problem with focus on an improved projection technique for identification of the SDOF system and a parametric technique for modal parameter estimation.

2 THEORY OF SPECTRAL MATRIX DECOMPOSITION

In the original first presentation of the FDD technique, the spectral decomposition was presented as an approximate decomposition under the assumption of a broad-banded but not-white excitation spectrum. It is however worth considering the more simple case of a perfect flat excitation spectrum, as this case leads to a better insight into the properties of the response spectral density matrix.

2.1 Flat excitation spectrum

The theory of the FDD technique is based on the formula of input and output PSD relationship for a stochastic process, Bendat & Piersol \[7\].
\[ G_{y_j}(\omega) = H(\omega)G_{x_i}(\omega)H(\omega)^H \]  

(1)

where \( G_{x_i}(\omega) \), \( G_{y_j}(\omega) \) are input and output Power Spectral Density (PSD) matrices, respectively, \( H(\omega) \) is the Frequency Response Function (FRF) matrix, which can be expressed as a partial fraction form via poles \( \lambda_m \) and residues \( R_m \),

\[ H(\omega) = \sum_{m=1}^{M} \left( \frac{R_m}{i\omega - \lambda_m} + \frac{R^*_m}{i\omega - \lambda_m^*} \right) \]  

(2)

where \( \lambda_m = -\sigma_m + i\omega_m \) is the 1\textsuperscript{st} pole, and \( R_m \) is the corresponding residue matrix,

\[ R_m = \phi_m \gamma_m \]  

(3)

and where \( \phi_m = [\phi_{1m}, \phi_{2m}, \cdots, \phi_{Nm}]^T \); \( \gamma_m = [\gamma_{1m}, \gamma_{2m}, \cdots, \gamma_{Nm}]^T \) are the 1\textsuperscript{st} mode shape and modal participation factor vector, respectively, \( R \) is the number of inputs (or references). When all output measurements points are taken as references, then \( H(\omega) \) is a square matrix and \( \phi_m = \gamma_m \). Assuming that the excitation spectrum is flat (the input is assumed to be white noise), then its PSD is a constant matrix, i.e. \( G_{x_i}(\omega) = G_{xx} \), and Eq. (1) becomes

\[ G_{y_j}(\omega) = \sum_{m=1}^{M} \sum_{i=1}^{M} \left( \frac{R_m}{i\omega - \lambda_m} + \frac{R^*_m}{i\omega - \lambda_m^*} \right) G_{xx} \left( \frac{R_i}{i\omega - \lambda_i} + \frac{R^*_i}{i\omega - \lambda_i^*} \right)^H \]  

(4)

where superscript \( H \) denotes complex conjugate and transpose. Multiplying the two partial fraction factors and making use of the Heaviside partial fraction theorem, after some mathematical manipulations, the output PSD can be reduced to a pole/residue form as follows

\[ G_{y_j}(\omega) = \sum_{m=1}^{M} \left( \frac{A_m}{i\omega - \lambda_m} + \frac{A^*_m}{i\omega - \lambda_m^*} \right) \left( \frac{R_i}{i\omega - \lambda_i} + \frac{R^*_i}{i\omega - \lambda_i^*} \right)^H \]  

(5)

where \( A_m \) is the residue matrix of the output PSD corresponding to the 1\textsuperscript{st} pole \( \lambda_m \). As the output PSD itself, the residue matrix is an \( N \times N \) hermitian matrix. When taking all the measurement points as references, the residue matrix can be derived as

\[ A_m = R_m G_{xx} \sum_{i=1}^{M} \left( \frac{R^*_i}{i\omega - \lambda_i} + \frac{R^*_i}{i\omega - \lambda_i^*} \right) \]  

(6)

Taking Eq. (5) into account the output PSD can be modally decomposed as

\[ G_{y_j}(\omega) = \sum_{m=1}^{M} \left( \frac{\Phi_m \lambda_m^T}{i\omega - \lambda_m} + \frac{\Phi^*_m \lambda_m^H}{i\omega - \lambda_m^*} + \frac{\chi_m \Phi_m^T}{i\omega - \lambda_m} + \frac{\chi^*_m \Phi^*_m}{i\omega - \lambda_m^*} \right) \]  

(7)

where \( \Phi_m \) and \( \chi_m \) are the 1\textsuperscript{st} mode shape and operational reference vector, respectively. The reference vector is a function of the modal parameters and the PSD matrix of the unknown random input forces. It is noticed that the modal participation factors and by consequence the modal scale factors cannot be determined from OMA. The PSD derived above have a 4-quadrant symmetry i.e. the OMA modal model contains \( \lambda_m, -\lambda_m, \lambda_m^* \) and \( -\lambda_m^* \) as poles, positive \( \lambda_m, \lambda_m^* \) and negative \( -\lambda_m, -\lambda_m^* \), where \( m = 1, 2, \cdots, M \). The time domain counterpart of the PSD is the correlation function matrix

\[ C_{y_j}(\tau) = \sum_{m=1}^{M} \left( A_m e^{\lambda_m \tau} + A^*_m e^{\lambda_m^* \tau} - A_m e^{\lambda_m \tau} - A^*_m e^{\lambda_m^* \tau} \right) \]  

(8)
where the positive poles correspond to positive time lags (first two terms), and the negative poles correspond to negative time lags (last two terms).

Since the PSD is normally estimated via the Welch approach, a tradeoff must be made between the stochastic uncertainties and the bias errors introduced by leakage. Since the PSD is estimated from a limited amount of data, in OMA the spectral density matrix is typically characterized by high noise levels compared to FRF’s. Therefore, an additional noise reduction would be preferable. The noise problem can be overcome by using time-domain identification algorithms starting from the first \( N_o \) positive lags of the correlation function given by

\[
C_{yy}(\tau) = \sum_{m=1}^{M} (A_m e^{i\lambda_m \tau} + A_m^* e^{-i\lambda_m \tau})
\]  

(9)

Its counterpart in the frequency domain can be written as

\[
G_{yy}(\omega) = \sum_{m=1}^{M} \left( \frac{\Phi_m^T}{i\omega - \lambda_m} + \frac{\Phi_m^H}{i\omega - \lambda_m^*} \right)
\]

(10)

In matrix form the above equation can be re-written as

\[
G_{yy}(\omega) = \Phi \left[ i\omega I - \Lambda \right]^{-1} \Phi^T + \Phi \left[ i\omega I - \Lambda^* \right]^{+} \Phi^H = \Phi \left[ i\omega I - \Lambda_{2M} \right]^{+} \Phi^T
\]

(11)

The output PSD with only positive poles is called "positive" PSD, or "half" PSD, which, in form, is similar to the modally decomposed FRF matrix.

2.2 Uncorrelated input

The decomposition of the spectral matrix can also be seen as a result of uncorrelated modal coordinates. This view of the FDD technique was first introduced by Brincker et al [8], and is based on realizing that any dynamic response can be written in modal co-ordinates

\[
y(t) = \varphi_1 q_1(t) + \varphi_2 q_2(t) + \ldots
\]

(12)

Thus collecting the mode shapes as column vectors in the mode shape matrix \( \Phi \) and the modal coordinates in the column vector \( \mathbf{q}(t) \) we obtain

\[
y(t) = \Phi \mathbf{q}(t)
\]

(13)

In case of complex modes, poles and mode shapes appear in complex conjugate pairs, and thus, since there will be a contribution from each one, however assuming only one modal coordinate for each mode, Sestieri and Ibrahim [9]

\[
y(t) = \varphi_1 q_1(t) + \varphi_2 q_2(t) + \cdots + \varphi_1^* q_1(t)^* + \varphi_2^* q_2(t)^* + \ldots
\]

(14)

securing a non-complex response. Collecting the mode shape pairs in the extended mode shape matrix \( \tilde{\Phi} \) and the modal coordinates in the extended modal coordinate vector \( \tilde{\mathbf{q}}(t) \)

\[
\tilde{\Phi} = \begin{bmatrix} \Phi & \Phi^* \end{bmatrix}, \quad \tilde{\mathbf{q}}(t) = \begin{bmatrix} \mathbf{q}(t) \\ \mathbf{q}(t)^* \end{bmatrix}
\]

(15)

we obtain

\[
y(t) = \tilde{\Phi} \tilde{\mathbf{q}}(t)
\]

(16)

Now, obtaining the covariance matrix of the responses, Papoulis [10]

\[
C_{yy}(\tau) = E\left\{ y(t + \tau) y(t)^* \right\}
\]

(17)

where \( E\{ \} \) is the expectation operator and combining with Eq. (16) leads to
\[
\tilde{C}_{qq}(\tau) = E\{\tilde{q}(t + \tau)\tilde{q}^H(t)\tilde{\Phi}^H\} = \tilde{\Phi}E\{\tilde{q}(t + \tau)\tilde{q}^H(t)\}\tilde{\Phi}^H = \tilde{\Phi}\tilde{C}_{qq}(\tau)\tilde{\Phi}^H
\]

where \(\tilde{C}_{qq}(\tau)\) is the covariance matrix of the extended modal coordinates. Now, as it is normal in stochastic dynamics, if we assume that the modal coordinates are uncorrelated, then in \(\tilde{C}_{qq}(\tau)\) all off-diagonal elements are zero, thus, \(\tilde{C}_{qq}(\tau)\) is a diagonal matrix \(\tilde{C}_{qq} = [c_m]\) containing two identical contributions from each mode, \(c_m = c_{M+m} = C_m(\tau), m = 1 \cdots M\), where \(C_m(\tau)\) is the auto covariance function of the modal coordinate of mode \(m\). Taking the Fourier transform we obtain the corresponding response PSD matrix

\[
G_{\tilde{q}q}(\omega) = \tilde{\Phi}\tilde{G}_{qq}(\omega)\tilde{\Phi}^H
\]

where \(\tilde{G}_{qq}(\omega)\) is the power spectral density matrix of the extended modal coordinates. This matrix is diagonal with positive real components, \(\tilde{G}_{qq} = [g_m]\), containing two identical contributions from each mode, \(g_m = g_{M+m} = G_m(\omega), m = 1 \cdots M\), where \(G_m(\omega)\) is the auto spectral density of the modal coordinate of mode \(m\). Therefore, the response power spectral matrix decomposes into modal contributions of the form

\[
G_{\tilde{q}q}(\omega) = G_1(\omega)\phi_1\phi_1^H + G_2(\omega)\phi_2\phi_2^H + \cdots + G_i(\omega)\phi_i\phi_i^H + G_j(\omega)\phi_j\phi_j^H + \cdots
\]

\[
= G_1(\omega)\left(\begin{array}{c} \phi_1 \phi_1^T \end{array}\right) + G_2(\omega)\left(\begin{array}{c} \phi_2 \phi_2^T \end{array}\right) + \cdots
\]

showing that the influence of mode shape complexity vanishes on the response PSD matrix. For the case of non-complex mode shapes we obtain the simplified expression

\[
G_{\tilde{q}q}(\omega) = \Phi\tilde{G}_{qq}(\omega)\Phi^T
\]

\[
= G_1(\omega)\phi_1\phi_1^T + G_2(\omega)\phi_2\phi_2^T + \cdots
\]

3 NUMERICAL SPECTRAL MATRIX DECOMPOSITION

Originally, in Brincker et al [1], the spectral decomposition of the FDD technique was introduced as

\[
G_{\tilde{q}q} = USU^H
\]

where the factorization \(USU^H\) was described as a singular value decomposition where \(U\) is a unitary matrix holding the singular vectors and \(S\) is a diagonal matrix holding the singular values. In general, the singular value decomposition of an arbitrary matrix \(A\) is given by

\[
A = USV^H
\]

where \(U\) and \(V\) are unitary matrices holding the left and right singular vectors, and \(S = [s_k]\) is a rectangular matrix holding the non-zero singular values \(s_k\) in the diagonal starting from its upper left corner. Normally the singular values are sorted in descending order. The rank of \(A\) is the number of non-zero elements of \(S\), or in practice; the number of elements large enough to be considered non-zero.

For a Hermitian matrix \(A = A^H\) then \(USV^H = VSV^H\), but this does not mean that \(U = V\), since, if the matrix \(A\) has negative eigenvalues, then the only way they can be introduced is by having different signs on the corresponding singular vectors of \(U\) and \(V\) (see Eq. (26) below). However, if all eigenvalues are positive, then \(A\) is positive definite, and \(U = V\).
Since all spectral density matrices are describing energy – as it is expressed by the Parseval theorem – all spectral matrices are not only Hermitian, but also positive definite; thus, for a SVD decomposition of a square PSD matrix at a certain frequency line \( U = V \), and therefore the SVD of a spectral matrix can always be expressed as given by Eq. (22).

This fact, that a spectral matrix can always be decomposed by a factorization as given by Eq. (22) has been known long before the FDD and even before the SVD. The factorization is known as the spectral theorem, Halmos, [11]. Thus, the existence of the factorization does not rely on the SVD, but it can obtained using the SVD algorithm. It is worth considering if the factorization can be performed also by other algorithms, and actually, a much simpler and more commonly available algorithm such as the eigenvalue decomposition (EVD) can also be used for the factorization.

Noting the fact that \( U \) is unitary, thus \( U^{-1} = U^H \), we obtain from Eq. (22) by multiplying by \( U \) from the right

\[
G_{jj}, U = US
\]

which we recognize as an eigenvalue equation. The matrix \( U \) is holding the eigenvectors and the diagonal matrix \( S \) is holding the (real and positive) eigenvalues. Thus, the singular values can also be interpreted as the eigenvalues of \( G_{jj} \) and the singular vectors can be interpreted as the eigenvectors of \( G_{jj} \). This mean that we can use the EVD as a basis for FDD, the result is “the same” as indicated by Eq. (24). The procedure is well known as principal component analysis, see for instance Otte et al [4]. In this case it is normal to use the terms “principal values” and “principal vectors” instead of “singular values” and “singular vectors”.

In practice it is not exactly the same, as normally SVD sorts the singular values in descending order and the EVD usually does not. Another difference is some arbitrary phases on the singular vectors and the eigenvectors. For the case of a Hermitian and positive definite matrix let the singular vectors \( u_1, u_2, \cdots \) of \( A \) be defined as the column vectors of \( U \)

\[
U = [u_1, u_2, \cdots]
\]

then decomposing Eq. (22) into outer products of the singular vectors

\[
G_{jj} = s_1 u_1 u_1^H + s_2 u_2 u_2^H + \cdots
\]

This clearly shows that the factorization in Eq. (22) tolerates an arbitrary phase on the singular vectors. Thus an arbitrary angular multiplication on a singular vector \( u \) of the form \( u \exp(i\theta) \) will not influence the spectral matrix. Normally the arbitrary phase problem is solved in the SVD algorithm by forcing the phase to vanish for one of the components of the singular vector. A similar problem exists for the EVD. In order to have the two different estimation procedures giving the same result we need to have an objective procedure for adjusting the arbitrary phase.

A simple and good answer to that problem could be to choose a minimum phase solution. Such solution can be obtained by solving the least square problem that minimizes the imaginary part of the vectors (eigenvectors and singular vectors). If we do that – or something similar – then the SVD and EVD give the same results.

This brings us to the conclusion, that in order to decompose the spectral density matrix, we need a factorization of the form given by Eq. (22). This kind of factorization can be realized by using SVD, or it can be realized by using the EVD. The two techniques are numerically different, thus will have minor differences, but if we sort the eigenvalues of the spectral density matrix (that are always positive since a spectral matrix is positive definite) in descending order, and if we remove the arbitrary phase on the singular vector and the eigenvectors, then in practice, the SVD and the EVD will give the same results.

Now, since we can obtain the factorization either by SVD or by EVD, it does not seem like at good idea to refer to “singular vectors” and “singular values” nor to “eigen vectors” and “eigen values”; but it seems like more appropriate to use terms known from principal component analysis and just use the terms “principal vectors” and “principal values”.

In some cases it is of value to consider second order forms of the spectral matrix like
\[ B_1 = G_{yy}^H G_{yy}^H, \quad B_2 = G_{yy}^H G_{yy}^H \]  
\[ \text{(27)} \]

One reason is that in some cases we are actually dealing with spectral matrices of this kind. A good example is spectral matrices based on Random decrement functions, Asmussen [12].

Usually we would divide the time series into data segments \( y_s(t); \ s = 1, \cdots, S, \) where \( S \) is the number of data segments estimating the spectral density matrix by Welch averaging

\[ \hat{G}_{yy} = \frac{1}{S} \sum_{i=1}^{S} Y_s(\omega) Y_s^H(\omega) \]  
\[ \text{(28)} \]

where \( Y_s(\omega) \) is the Fourier transform of data segment number \( s \). Now, let us say that the data segments are not representing stochastic responses but rather estimates of correlation functions like random decrement functions, then, it might still be practical to use an estimation procedure like Eq. (28), and in this case we get the factorization

\[ B_1 = USU^H USU^H = US^2U^H \; ; \quad B_2 = U^H S U U^H S U = US^2U^H \]  
\[ \text{(29)} \]

thus, for the common second order spectral matrix \( B = B_1 = B_2 \) the singular values are squared and the singular vectors remain unchanged.

Turning to considering if the EVD can also be used for factorization of rectangular spectral matrices, the second forms becomes even more useful. A square spectral density matrix has a size that is the number of measured channel times the number of channels measured, \( N \times N \). However, it is often not necessary to consider the whole matrix, since at a given frequency, the physical rank of the spectral matrix is limited to the number of active modes at that frequency, and there is no reason to have an appearing rank (min of columns and rows) that is much larger than the actual physical rank of the matrix. Thus, if we have measured a large number of channels, then we do not necessarily want to estimate the full matrix, since throwing away a number of columns reducing the appearing rank to be something close to the actual rank of the matrix, will not significantly reduce the amount of information in the matrix. Therefore, in many cases we like to consider a spectral density matrix with a reduced number of (for instance) columns like

\[ \hat{G}_{1yy} = \frac{1}{S} \sum_{i=1}^{S} Y_{1s}(\omega) Y_{1s}^H(\omega) \]  
\[ \text{(30)} \]

where only the first \( R \) components in \( Y_{1s}(\omega) \) has been kept, thus \( \hat{G}_{1yy} \) is \( N \times R, \) \( R \) should be larger than the rank of \( \hat{G}_{yy} \). Now let us consider the factorization

\[ G_{1yy} = U_1 S_1 V_1^H \]  
\[ \text{(31)} \]

Again we can perform the factorization using EVD, in this case we need to solve two eigenvalue problems. Similar to Eq. (29)

\[ B_1 = G_{1yy}^H G_{1yy} = V_1 S_1^H S_1 V_1^H \; ; \quad B_2 = G_{1yy}^H G_{1yy} = U_1 S_1 S_1^H U_1^H \]  
\[ \text{(32)} \]

thus the matrices \( B_1 \) and \( B_2 \) are Hermitian and positive definite, and, therefore, the unitary square matrices \( U_2 \) and \( V_1 \) (\( U_2 \) of the size \( R \times R \), \( V_1 \) of the size \( N \times N \)), and the corresponding diagonal matrices \( S_1^H S_1 \) and \( S_1 S_1^H \) can be found by EVD as explained above. In order to arrive to the final factorization, we just need to take the square root of the diagonal elements of the “small version” of the diagonal matrix (in this case \( S_1^H S_1 \), since we are assuming that we have more rows than columns) and expand the matrix downwards by padding zeros to reach the needed size \( N \times R \).

Some examples of FDD are shown in Fig. 1 on the Heritage Court Building data, [13], data set one. This data set has 6 simultaneous channels, thus a full spectral matrix will be \( 6 \times 6 \), the corresponding considered rectangular matrix was formed by omitting the last three columns of the square matrix, resulting in a \( 6 \times 3 \) reduced spectral matrix. For both cases the principal values was obtained by SVD and EVD, and as it appears, the results are the same.

Before leaving the issue of numerical decomposition of the spectral matrix, a few words on the rank of the estimated spectral matrix is given below.
It is elementary, that the rank of the spectral matrix is limited by the number of measurement points, since the square spectral matrix is $N \times N$, and the rank can never be larger than the smallest dimension of the matrix. However, if two measurement point are close to each other, the two measurement points are representing the same information, and thus, the added channel does not contribute to the matrix rank. Therefore, it is essential, first of all, that the number of measurement channels are significantly larger than the physical rank of the spectral matrix, and secondly, that the measurement points are spread over the structure in such a way, that each individual measurement point does not repeat any information in other channels.

If only one data segment is present in the average given by Eqs. (28), (30) then obviously, the rank is limited to unity no matter what signal we are dealing with, since there is only one contribution to the matrix in the sum given by Eq. (26). Similarly, if there are two contributions to the average, then the maximum rank is two and so on. Therefore, the rank of the spectral matrix is limited by the number of averages $S$. Thus, the number of averages must by significantly larger than the physical rank of the spectral matrix, otherwise the physical rank of the matrix will not be present in the spectral estimate.

Finally, it is important to remember – as it is well known from traditional modal analysis – that the rank of the response spectral matrix is limited by the number of independent force inputs. It is not enough to have several inputs, the inputs must also be stochastically independent. If several inputs are strongly correlated, they will influence the spectral density matrix like a smaller number of independent inputs, and thus, even though many correlated inputs are present they might in fact limit the rank of the spectral density matrix.

4 MODAL IDENTIFICATION

4.1 Modal coordinate estimation

As explained in section 3, the key of the FDD technique is to conduct a factorization like Eq. (22) or Eq. (31) of the output PSD estimated at a discrete frequencies $\omega = \omega_k$, thus principal values and principal vectors are a function of frequency, $S(\omega_k) = \text{diag}(s_{k1}, s_{k2}, \ldots s_{kr})$ and $\mathbf{U}_k = \mathbf{U}(\omega_k) = [\mathbf{u}_{k1}, \mathbf{u}_{k2}, \ldots \mathbf{u}_{kr}]$, $\mathbf{V}_k = \mathbf{V}(\omega_k) = [\mathbf{v}_{k1}, \mathbf{v}_{k2}, \ldots \mathbf{v}_{kr}]$. When the frequency approaches a dominating modal frequency $\omega_{md}$, the rank of the PSD matrix approximates rank one since only the first principal vector will contribute to the spectral matrix, thus

$$\Delta \hat{G}_{yy}(\omega_k) \approx s_k(\omega_k) \mathbf{u}_k(\omega_k) \mathbf{v}_k(\omega_k)^H$$

and therefore, this contribution is an estimate of the contributions from the dominant mode, the first principal value that reaches maximum at the natural frequency $\omega_n = \omega_{dn}$ describes the modal coordinate and the corresponding (largest) principal vector $\mathbf{u}_1(\omega_{md})$ is an estimate of the $m^{th}$ mode shape, thus $\hat{\phi}_m \approx \mathbf{u}_1(\omega_{md})$ with unitary normalization. In the repeated mode case, the rank of PSD matrix will be equal to the number of multiplicity of the modes. In this case, the principal value function can suitably be adopted as a modal indicator function (MIF). Modal frequencies can be located by the peaks of the principal value plots, and from the corresponding principal vectors, mode shapes can be estimated. Since principal value analysis (either by using SVD or EVD) has the ability of separating signal space from noise space, the modes can be identified from the principal value plots also in case of noisy measurements. Further, since principal value analysis can handle full rank matrices, closely spaced modes or even modes with repeated modal frequencies can be detected.
The first generation of FDD, Brincker et al [1], can only estimate modal frequencies and mode shapes. The second generation of FDD, which is called as Enhanced FDD or EFDD, was formulated for estimation of not only modal frequencies and mode shapes, but also damping ratios, Brincker et al [5]. To estimate damping ratios, principal values data near the peak (all principal values considered) with corresponding singular vectors having the highest MAC value with the mode shape estimated at the peak \( \Phi_{\omega} = u_i(\omega_{\text{est}}) \) are transferred back to time domain via inverse FFT. This time domain function is an estimate of the correlation function of the single degree-of-freedom (S-DOF) system (modal coordinate). From this free decay function of the modal coordinate S-DOF system, the modal frequency and the damping ratio can be calculated by the logarithmic decrement technique.

Since truncated data are used (only the data near the peak is used) when taking the inverse FFT to calculate the approximate correlation function of the corresponding S-DOF system, the EFDD might introduce bias errors in the damping estimation. Moreover, when dealing with closely spaced modes, beat phenomena will often occur, which can leads inaccurate estimation of damping ratio by the logarithmic decrement technique. The third generation of FDD the Frequency-Spatial Domain Decomposition (FSDD) introduced by Zhang et al [6], was developed to eliminate these shortcomings. FSDD makes use of the property of the unitary singular matrix to derive an enhanced output PSD \( \Delta \tilde{G}_{\text{ykw}}(\omega_k) \) for the modal coordinate via modal filtering, i.e. pre- and post-multiplying a principal vector corresponding to the \( m^{\text{th}} \) damped natural frequency,

\[
\Delta \tilde{G}_{\text{ykw}}(\omega_k) = u_i^H(\omega_k) \tilde{G}_{\text{ykw}}(\omega_k) v_i(\omega_k) \approx \Re \left( \frac{2c_{\text{sp}}}{i\omega - \lambda_{\omega}} \right)
\]

It is seen that the output PSD is enhanced in the vicinity of the \( m^{\text{th}} \) modal frequency and behaves like an S-DOF system under the assumption that the principal vectors constitute an orthogonal set not only for spectral matrix at the natural frequency, but also for spectral matrices around the considered natural frequency. In other words, approximately, the first principal vector corresponding to a modal frequency acts as a modal filter. In most cases the enhanced PSD in the specific frequency range (narrow band) can be approximated as an S-DOF system, and therefore an S-DOF curve fitter making use of the spectral lines in the vicinity of a mode can be adopted to estimate the modal frequency and damping ratio.

It should be noted that since the modal identification for FDD is only conducted in the narrow frequency band, the requirement of white noise assumption, or flat excitation, can be re-
duced to broadband excitation, i.e. approximately flat in the narrow frequency band around the considered mode.

However, even though the FSDD estimates a smoother S-DOF curve than the simple sorting algorithm of the original EFDD, and also avoids some other problems introduced in the original FDD technique like beating phenomena, this approach also introduces some bias error. The reason is that the set of active modes in a certain frequency band does not always constitute an orthogonal set, thus when multiplying the PSD matrix by one of the first principal vectors at a considered natural frequency, the modal filter might not be perfect, since this vector is not necessarily orthogonal to the mode shape vectors of adjacent modes. The problem can be overcome by making the projection on an orthogonal set, this can be done in the following way. Based on the factorization given by Eq. (22) we form the modal coordinate response vector

\[ s = \text{diag}(S) \]  \hspace{1cm} (35)

This response vector is useful to look at (like we can look at the principal values), but for modal identification it is not useful since the information of the different modes is mixed together. In order to obtain the response of one single mode, we can do the following. A set of physical responses can be defined as

\[ r_i = Us \]  \hspace{1cm} (36)

This set of physical responses contain all modes in the signal and also all noise components. An orthogonal projection that is selecting the contributions from the individual modes can be performed in a limited frequency band (considering a limited number of modes). Let us say, that in the considered (reduced) frequency band \( B \), we have identified \( P \) modes \( \varphi_1, \varphi_2 \cdots \varphi_P \) gathered in the reduced mode shape matrix

\[ \Phi_B = [\varphi_1, \varphi_2 \cdots \varphi_P] \]  \hspace{1cm} (37)

To this representation of the mode shapes is the corresponding representation of the modal coordinates \( s_B \) so that we can define the corresponding set of physical responses

\[ r_z = \Phi Bs_B \]  \hspace{1cm} (38)

The modal coordinate responses \( s_B \) are the responses that we are looking for; a one degree of freedom response for each mode shape. By making the requirement that the two defined responses are the same, we isolate the modal coordinates and obtain

\[ s_B = \Phi_B^{-1}Us \]  \hspace{1cm} (39)

The number of modes must be smaller than or equal to the number of measurement points. If the number of modes is smaller than the number of measurement points, the inverse of \( \Phi_B \) is found by regression or by SVD. Multiplying the inverse of \( \Phi_B \) onto the set of physical responses \( Us \) makes sure that there is an optimal decoupling in such a way that only the contribution from the considered mode (and not from the other modes or from the noise) defines the estimated modal response for the considered mode. Some results are shown in Fig. 2.

![Figure 2](image_url)

Figure 2. Identification of the three closely spaced modes in the frequency band \( B = [0, 3] \) Hz for the Heritage Court Building data, data set one, by orthogonal projection.
4.2 Parametric identification in the frequency domain

As seen from the Eq. (5) the response PSD can be decomposed into four terms expressed by the residue matrices \( \mathbf{A}_m \) and the poles \( \lambda_m = -\sigma_m + i\omega_m \). In the narrow band with spectrum lines in the vicinity of a modal frequency, only the first two terms of the Eq. (5) are dominating. Taking this into account Eq. (5) can be simplified as

\[
\mathbf{G}_{yy}(\omega) \approx \sum_{m=1}^{M} \left( \frac{A_m}{i\omega - \lambda_m} + \frac{A_m^H}{-i\omega - \lambda_m^*} \right)
\]  

(40)

When the structure is lightly damped, i.e. \( \sigma_m < \omega_{dn} \), in the vicinity of the \( m^{th} \) modal frequency the residue can be derived approximately from Eq. (6) as

\[
\mathbf{A}_m \approx \frac{R_{xx} \mathbf{G}_{xx} \mathbf{R}_{xx}^H}{2\sigma_m} \quad c_m \mathbf{\Phi}_m \mathbf{\Phi}_m^H
\]  

(41)

where \( c_m = \gamma_m^H \mathbf{G}_{xx} \gamma_m \) is a real scaling factor. Substituting Eq. (41) into Eq. (40) yields

\[
\mathbf{G}_{yy}(\omega) \approx \sum_{m=1}^{M} \left( \frac{c_m \mathbf{\Phi}_m \mathbf{\Phi}_m^H}{i\omega - \lambda_m} + \frac{c_m \mathbf{\Phi}_m \mathbf{\Phi}_m^H}{-i\omega - \lambda_m^*} \right) = \mathbf{\Phi} \left[ \text{diag} \left( 2\text{Re} \left( \frac{c_m}{i\omega - \lambda_m} \right) \right) \right] \mathbf{\Phi}^H
\]  

(42)

Compared to the modal decomposition of the FRF matrix, we can see the similarity. The major difference is that the middle term of PSD decomposition is real compared to the FRF decomposition that is complex. The contribution to the spectral density matrix from a single mode can be expressed as

\[
\Delta \mathbf{G}_{yy}(\omega) \approx \mathbf{\Phi}_m \left[ \text{diag} \left( 2\text{Re} \left( \frac{c_m}{i\omega - \lambda_m} \right) \right) \right] \mathbf{\Phi}_m^H = \alpha_m \mathbf{\Phi}_m \mathbf{\Phi}_m^H
\]  

(43)

The above equations for the response PSD and its residue are based on the case of taking all the measurement points as references (inputs), and therefore, the output PSD and its residue are square matrices. In the case of considering a limited number of columns \( R < N \), the output PSD decomposition in the narrow band considering only the dominant mode can be written

\[
\Delta \mathbf{G}_{yy}(\omega) \approx \mathbf{\Phi}_m \left[ \text{diag} \left( 2\text{Re} \left( \frac{c_m}{i\omega - \lambda_m} \right) \right) \right] \mathbf{\Phi}_m^H = \alpha_m \mathbf{\Phi}_m \mathbf{\Phi}_m^H
\]  

(44)

Substituting the \( m^{th} \) pole into Eq. (34) yields

\[
\Delta \mathbf{\hat{G}}_{yy}(\omega_k) = 2\text{Re} \left( \frac{2c_m}{\sigma_m + i(\omega - \omega_{dn})} \right) = \frac{2c_m \sigma_m}{\sigma_m^2 + (\omega_k - \omega_{dn})^2}
\]  

(45)

Making use of the estimated output PSD data of the S-DOF system using only the spectrum lines in the vicinity of a modal frequency \( \omega_{ad} \), the real and imaginary parts of the pole \( \lambda_m = -\sigma_m + i\omega_m \) can be obtained via least squares estimation (LSE), and \( m^{th} \) the modal frequency and damping ratio can be calculated.

When positive or half PSD is obtained via unbiased estimation of the correlation function with positive time lags

\[
\mathbf{G}_{yy}(\omega) = \sum_{m=1}^{M} \left( \frac{\mathbf{\Phi}_m \mathbf{\Phi}_m^T}{i\omega - \lambda_m} + \frac{\mathbf{\Phi}_m^H \mathbf{\Phi}_m^T}{-i\omega - \lambda_m^*} \right) = \mathbf{\Phi} \left[ \omega \mathbf{I} - \mathbf{\Lambda}_{2M} \right]^{-1} \mathbf{X}^T
\]  

(46)

The enhanced PSD for the modal coordinate becomes
\[
\Delta \hat{G}_{jw}(\omega_k) = u_i(\omega_k) \hat{G}_{jw}(\omega_k) v_i(\omega_k) / (i\omega - \lambda_m) \quad (47)
\]
which is similar to the FRF of an S-DOF system, and thus, making use of principal values as MIFs and modal filtering, classical identification techniques for EMA, Shih et al [2], can be adopted for OMA.

REFERENCES


