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ON ENDOMORPHISMS OF THE CUNTZ ALGEBRA WHICH PRESERVE THE CANONICAL UHF-SUBALGEBRA, II

TOMOHIRO HAYASHI, JEONG HEE HONG, AND WOJCIECH SZYMAŃSKI

Abstract. It was shown recently by Conti, Rørdam and Szymański that there exist endomorphisms \( \lambda_u \) of the Cuntz algebra \( O_n \) such that \( \lambda_u(F_n) \subseteq F_n \) but \( u \notin F_n \), and a question was raised if for such a \( u \) there must always exist a unitary \( v \in F_n \) with \( \lambda_u|_{F_n} = \lambda_v|_{F_n} \). In the present paper, we answer this question to the negative. To this end, we analyze the structure of such endomorphisms \( \lambda_u \) for which the relative commutant \( \lambda_u(F_n)' \cap F_n \) is finite dimensional.

1. Introduction and preliminaries

This paper is devoted to continuation of the line of investigation of exotic endomorphisms of the Cuntz algebras initiated in [4]. Our main result is solution of a question raised therein, see below for details. Our startegy is based on a detailed analysis of such endomorphisms \( \lambda_u \) of \( O_n \) that globally preserve the core UHF subagebra \( F_n \) and have finite dimensional relative commutant \( \lambda_u(F_n)' \cap O_n \), and builds on the earlier results in this direction obtained in [10].

The Cuntz algebra \( O_n \), \( n \geq 2 \), is the \( C^* \)-algebra generated by isometries \( S_1, \ldots, S_n \) satisfying \( \sum_{i=1}^n S_i S_i^* = 1 \). It is a purely infinite, simple \( C^* \)-algebra, independent of the choice of generating isometries, [7]. We denote by \( W_n^k \) the set of \( k \)-tuples \( \mu = (\mu_1, \ldots, \mu_k) \) with \( \mu_m \in \{1, \ldots, n\} \), and by \( W_n \) the union \( \bigcup_{k=0}^\infty W_n^k \), where \( W_n^0 = \{0\} \). If \( \mu \in W_n^k \) then \( |\mu| = k \) is the length of \( \mu \). If \( \mu = (\mu_1, \ldots, \mu_k) \in W_n \) then \( S_\mu = S_{\mu_1} \cdots S_{\mu_k} \) (\( S_0 = 1 \) by convention) is an isometry in \( O_n \). Every word in \( \{S_i, S_i^* \mid i = 1, \ldots, n\} \) can be uniquely expressed as \( S_\mu S_\nu^* \), for \( \mu, \nu \in W_n \) [7, Lemma 1.3].

The gauge action \( \gamma \) of the circle group \( \mathbb{T} \) on \( O_n \) is defined by \( \gamma_z(S_i) = z S_i, \ z \in \mathbb{T} \). Let \( F_n \) be the fixed point algebra of \( \gamma \). Denote \( F_n^{(k)} := \text{span}\{S_\mu S_\nu^* \mid \mu, \nu \in W_n^k\} \). Then \( F_n \) is generated by \( F_n^{(k)} \), \( k = 1, 2, \ldots, \) and each \( F_n^{(k)} \) is isomorphic to the matrix algebra \( M_{n^k}(\mathbb{C}) \). Thus \( F_n \) is isomorphic to the UHF-algebra of type \( n^\infty \), and hence it has a

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unique tracial state $\tau$. There exists a faithful conditional expectation $E : \mathcal{O}_n \to \mathcal{F}_n$, defined by integration with respect to the Haar measure on $\mathbb{T}$ as

$$E(x) = \int_{\mathbb{T}} \gamma_z(x)dz.$$  

For each $k \in \mathbb{Z}$ we denote by $\mathcal{O}_n^{(k)}$ the corresponding spectral subspace for $\gamma$ in $\mathcal{O}_n$,

$$\mathcal{O}_n^{(k)} := \{x \in \mathcal{O}_n | \gamma_z(x) = z^k, \forall z \in \mathbb{T}\}.$$  

Thus, in particular, $\mathcal{O}_n^{(0)} = \mathcal{F}_n$.

The $C^*$-subalgebra of $\mathcal{O}_n$ generated by projections $P_\mu := S_\mu S_\mu^*$, $\mu \in W_n$, is a MASA (maximal abelian subalgebra) in $\mathcal{O}_n$. We call it the diagonal and denote $\mathcal{D}_n$, also writing $\mathcal{D}_k$ for $\mathcal{D}_n \cap \mathcal{F}_n^{(k)}$.

The canonical shift endomorphism $\varphi : \mathcal{O}_n \to \mathcal{O}_n$ is defined by

$$\varphi(x) = \sum_{i=1}^{n} S_i x S_i^*.$$  

It is easy to see that $S_i x = \varphi(x) S_i$ and $x S_i^* = S_i^* \varphi(x)$ for all $x \in \mathcal{O}_n$.

As shown by Cuntz in [8], there exists a bijective correspondence between unitaries in $\mathcal{O}_n$ (whose collection is denoted $U(\mathcal{O}_n)$) and unital $*$-endomorphisms of $\mathcal{O}_n$, determined by

$$\lambda_u(S_i) = u S_i, \quad i = 1, \ldots, n.$$  

We have $\text{Ad}(u) = \lambda_{u \varphi(u^*)}$ for all $u \in U(\mathcal{O}_n)$. If $u \in U(\mathcal{O}_n)$ then for each positive integer $k$ we denote

$$u_k = u \varphi(u) \cdots \varphi^{k-1}(u).$$  

Here $\varphi^0 = \text{id}$, and we agree that $u^*_k$ stands for $(u_k)^*$. If $\alpha$ and $\beta$ are multi-indices of length $k$ and $m$, respectively, then $\lambda_u(S_\alpha S_\beta^*) = u_k S_\alpha S_\beta^* u_m^*$.

The Cuntz correspondence between unitaries and endomorphisms of $\mathcal{O}_n$ provides a very efficient tool for investigations of the latter. In this note, we continue the study (by several authors) of those unital endomorphisms which globally preserve the UHF-subalgebra $\mathcal{F}_n$. For example, such endomorphisms were analyzed from the point of view of the Jones-Kosaki-Watatani index theory in [12] and [3], and in connection with Hopf algebra actions in [9] and [13]. More recently, interesting combinatorial approaches to the study of permutative endomorphisms of this type have been found (e.g. see [6], [2], and a survey article [1]).

It was observed by Cuntz in his groundbreaking paper [8] that an automorphism $\lambda_u$ globally preserves $\mathcal{F}_n$ if and only if $u \in \mathcal{F}_n$. The situation is more complex with proper endomorphisms. Clearly, $u \in \mathcal{F}_n$ implies $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$, [8], but the question if the converse is true remained open until very recently. Indeed, it was shown in [4] that
there exist unitaries $u$ in $O_n \setminus \mathcal{F}_n$ such that $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$. All such examples found therein were of the form $u = vw$ with $w \in \lambda_u(\mathcal{F}_n)' \cap O_n$ and $v \in \mathcal{F}_n$. In such a case, we also have $\lambda_u(x) = \lambda_u(x)$ for all $x \in \mathcal{F}_n$. Thus a natural question arises if such a factorization of $u$ is always possible whenever $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ (cf. [11 Problem 5.3]).

Some progress towards answering this question has been made recently in [10] and [11]. The main purpose of the present paper is to develop definite methods for analyzing endomorphisms $\lambda_u$ of $O_n$ satifying $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ and an additional condition that the relative commutant $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$ be finite dimensional. In particular, we give a verifiable criterion for determining if the aforementioned decomposition is possible, Corollary 3.4. Based on this criterion, in Section 3 we give an explicit example of a unitary $u \in O_2$ such that $\lambda_u(\mathcal{F}_2) \subseteq \mathcal{F}_2$ and $\dim \lambda_u(\mathcal{F}_2)' \cap \mathcal{F}_2 < \infty$ but there is no unitary $v \in \mathcal{F}_2$ such that $\lambda_u(\mathcal{F}_2) = \lambda_v(\mathcal{F}_2)$, see Example 3.6. In this way, we answer to the negative the question raised in [4] and [11].

2. The relative commutants

We begin by recording for future references a few simple facts, essentially contained in [4] and [10].

**Proposition 2.1.** Let $u \in \mathcal{U}(O_n)$. Then the following conditions are equivalent.

1. $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$,
2. $\lambda_{\gamma_z(u)}|\mathcal{F}_n = \lambda_u|\mathcal{F}_n$ for all $z \in \mathbb{T}$,
3. $w\gamma_z(u^*) \in \lambda_u(\mathcal{F}_n)' \cap O_n$ for all $z \in \mathbb{T}$.

**Proof.** Clearly, $\gamma_z\lambda_u\gamma_z^{-1} = \lambda_{\gamma_z(u)}$ for all $z \in \mathbb{T}$. Thus condition (2) above is equivalent to $\gamma_z\lambda_u|\mathcal{F}_n = \lambda_u|\mathcal{F}_n$ for all $z \in \mathbb{T}$. Obviously, this holds if and only if $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$. That is, (1) is equivalent to (2).

It is an immediate consequence of Proposition 2.1 and Proposition 4.7 from [11] that $\lambda_u|\mathcal{F}_n = \lambda_u|\mathcal{F}_n$ if and only if $vu^* \in \lambda_u(\mathcal{F}_n)' \cap O_n$. This gives (2) is equivalent to (3). □

**Proposition 2.2.** If $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ and $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n = \mathbb{C}1$, then $u \in \mathcal{F}_n$.

**Proof.** If $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ and $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n = \mathbb{C}1$, then $\lambda_u(\mathcal{F}_n)' \cap O_n = \mathbb{C}1$ as well, [10, Theorem 1.1]. As shown in [4], this implies that $u \in \mathcal{F}_n$. □

**Proposition 2.3.** Let $u$ be a unitary in $O_n$. Then $u = vw$ for some $w \in \lambda_u(\mathcal{F}_n)' \cap O_n$ and a unitary $v \in \mathcal{F}_n$ if and only if there exists a unitary $y \in \lambda_u(\mathcal{F}_n)' \cap O_n$ such that $w\gamma_z(u^*) = y\gamma_z(y^*)$ for all $z \in \mathbb{T}$.

**Proof.** If $u = vw$ for some $w \in \lambda_u(\mathcal{F}_n)' \cap O_n$ and $v \in \mathcal{U}(\mathcal{F}_n)$, then $w\gamma_z(u^*) = w\gamma_z(w^*)$, and it suffices to put $y = w$. □
Conversely, if there exists a unital \( y \in \lambda_u(F_n)' \cap O_n \) such that \( u\gamma_z(u^*) = y\gamma_z(y^*) \) for all \( z \in \mathbb{T} \) then \( y^*u \) is fixed by all \( \gamma_z \). Thus \( y^*u \in F_n \) and it suffices to put \( w = y \) and \( v = y^*u \).

From now on, we make a standing assumption that \( u \in \mathcal{U}(O_n) \) is such that
\[
\lambda_u(F_n) \subseteq F_n \quad \text{and} \quad \dim \lambda_u(F_n)' \cap F_n < \infty.
\]

As shown in [10], assumption (2) above entails a number of important consequences, which we summarize as follows.

- We also have \( \dim \lambda_u(F_n)' \cap O_n < \infty \).
- There exists a unitary group \( \{u_z\}_{z \in \mathbb{T}} \) in the center of \( \lambda_u(F_n)' \cap F_n \) such that \( \text{Ad}_{u_z}(x) = \gamma_z(x) \) for all \( x \in \lambda_u(F_n)' \cap O_n \).
- Minimal projections in \( \lambda_u(F_n)' \cap F_n \) are minimal in \( \lambda_u(F_n)' \cap O_n \) as well. Thus \( \lambda_u(F_n)' \cap O_n \) contains a MASA consisting of projections in \( \lambda_u(F_n)' \cap F_n \).

The proof of the following theorem is modeled after that of [10, Lemma 1.11].

**Theorem 2.4.** Let \( u \in \mathcal{U}(O_n) \) be such that \( \lambda_u(F_n) \subseteq F_n \) and \( \dim \lambda_u(F_n)' \cap F_n < \infty \). Then there exist unitaries \( w \in \lambda_u(F_n)' \cap O_n \) and \( v \in O_n \), and a unitary group \( \{v_z\}_{z \in \mathbb{T}} \subseteq \lambda_u(F_n)' \cap F_n \) satisfying \( u = vw \) and \( \gamma_z(v) = v_zv \) for all \( z \in \mathbb{T} \).

**Proof.** At first we note that \( u\gamma_z(u^*)u_z \) is a unitary group in \( \lambda_u(F_n)' \cap O_n \). Indeed,
\[
(u\gamma_z(u^*)u_z)(u\gamma_z(u^*)u_z) = u\gamma_z(u^*)(\text{Ad}_{u_z}(u)u_z\gamma_z(u^*)u_z)
\]
\[
= u\gamma_z(u^*)\gamma_z(u)\gamma_z(\gamma_z(u^*))u_zu_z = u\gamma_z(u^*)u_zu_z.
\]

Since \( \dim \lambda_u(F_n)' \cap O_n < \infty \), this unitary group may be diagonalized. On the other hand, \( \lambda_u(F_n)' \cap O_n \) contains a MASA composed of projections in \( \lambda_u(F_n)' \cap F_n \). Thus, there exists a unitary \( w \in \lambda_u(F_n)' \cap O_n \) such that \( y_z := w^*(u\gamma_z(u^*)u_z)w \) is a unitary group in \( \lambda_u(F_n)' \cap F_n \). Since each \( u_z \) is in the center of \( \lambda_u(F_n)' \cap F_n \), the unitary groups \( \{y_z\}_{z \in \mathbb{T}} \) and \( \{u_z\}_{z \in \mathbb{T}} \) commute.

Set \( v_z := u_zy_z^*z \in \mathbb{T} \), and \( v := w^*u \). Then \( v_z \) is a unitary group in \( \lambda_u(F_n)' \cap F_n \) and
\[
\gamma_z(v) = \gamma_z(w^*u) = u_z(u_z^*\gamma_z(w^*u)w^*)w^*u = u_zw^*u = v_zv.
\]

for all \( z \in \mathbb{T} \). This completes the proof. \( \square \)

We keep the notation from Theorem 2.1 assuming that unitaries \( w, v \) and \( v_z \) have the properties described therein. Thus, in particular, \( \lambda_u|_{F_n} = \lambda_u|_{F_n} \) by [4, Proposition 2.1]. Consequently, \( \text{Ad} \circ \varphi \) is an automorphism of \( \lambda_u(F_n)' \cap O_n \), by [4, Proposition 2.3 and Lemma 2.4].

**Lemma 2.5.** With unitaries \( u, v, v_z \) and \( u_z \) as above, put
\[
X_z := (\text{Ad} \circ \varphi)(u_z^*)u_z.
\]
Then \( \{X_z\}_{z \in T} \) is a unitary group in the center of \( \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n \), and we have

\[
\gamma_z(v) = X_z u_z (\text{Ad} v \circ \varphi)(u_z^*) v.
\]

Proof. For each \( x \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n \), we see that

\[
u_z(\text{Ad} v \circ \varphi)(x) u_z^* = \gamma_z(v \varphi(x) v^*) = \gamma_z(v) \gamma_z(\varphi(x)) \gamma_z(v)^* = \gamma_z(v) \varphi(\gamma_z(x)) \gamma_z(v)^* = v_z v \varphi(u_z) v^* v \varphi(x) v^* v \varphi(u_z^*) v^* v_z^* = v_z \text{Ad}(v \varphi(u_z) v^*)(\text{Ad} v \circ \varphi)(x)) v_z^*.
\]

Hence, we have

\[
\text{Ad}(v_z^* u_z)((\text{Ad} v \circ \varphi(x)) = \text{Ad}((\text{Ad} v \circ \varphi)(u_z))(\text{Ad} v \circ \varphi)(x)).
\]

Since \( \text{Ad} v \circ \varphi \) is an automorphism of \( \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n \), this shows that

\[
(3) \quad \text{Ad}(v_z^* u_z) = \text{Ad}((\text{Ad} v \circ \varphi)(u_z)) \text{ on } \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n.
\]

Consequently, \( X_z \) belongs to the center of \( \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n \).

Now, \( \{u_z\}_{z \in T} \) and \( \{v_z\}_{z \in T} \) are commuting unitary groups, and both commute with \( X_z \), by the above argument. Therefore the unitary group \( \text{Ad} v \circ \varphi(u_z) = X_z u_z v_z^* \) commutes with both of them. Consequently, \( X_z \) being a product of three mutually commuting unitary groups itself is a unitary group.

The final claim of the lemma now follows from the fact that \( \gamma_z(v) = v_z v \). \( \square \)

Before proceeding further, we introduce the following notation. For \( x \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n \) and \( k \in \mathbb{N} \), we set

\[
(4) \quad x^{(k)} := x(\text{Ad} v \circ \varphi)(x)(\text{Ad} v \circ \varphi)^2(x) \ldots (\text{Ad} v \circ \varphi)^{k-1}(x).
\]

**Lemma 2.6.** With unitaries \( u, v, v_z \) and \( u_z \) as above, and \( x \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n \), for all \( g \in U(\mathcal{O}_n) \), \( z \in T \) and \( k \in \mathbb{N} \) we have the following identities.

(i) \( \gamma_z(v_k) = v_z^{(k)} v_k \),

(ii) \( (\text{Ad} g \circ \varphi)^k(x) = g_k \varphi^k(x) g_k^* \),

(iii) \( v_z^{(k)} = X_z^{(k)} u_z(\text{Ad} v \circ \varphi)^k(u_z^*) \),

(iv) \( (g v)_k = g^{(k)} v_k \).

Proof. In all three cases, we proceed by induction on \( k \).

Ad (i). Case \( k = 1 \) is the identity \( \gamma_z(v_1) = \gamma_z(v) = v_z v = v_z^{(1)} v_1 \) from Theorem 2.4. For the inductive step, we calculate

\[
\gamma_z(v_{k+1}) = \gamma_z(v_k \varphi^k(v)) = \gamma_z(v_k) \varphi^k(\gamma_z(v)) = v_z^{(k)} v_k \varphi^k(v_z^* v_k \varphi^k(v)) = v_z^{(k+1)} v_{k+1}.
\]

In this calculation we used identity (ii) of the present lemma, whose proof does not depend on (i).
Ad (ii). Case $k = 1$ is clear. For the inductive step, we have

$$(\text{Ad } g \circ \varphi)^{k+1} = (\text{Ad } g \circ \varphi)(g_k \varphi^k(x)g_k^*) = g \varphi(g_k) \varphi^{k+1}(x) \varphi(g_k^*)g^* = g_{k+1} \varphi^{k+1}(x)g_{k+1}^*.$$ 

Ad (iii). Case $k = 1$ is clear. For the inductive step, we see that

$$v_z^{(k+1)} = v_z^{(k)}(\text{Ad } v \circ \varphi)^{k}(v_z) = X_z^{(k)} u_z (\text{Ad } v \circ \varphi)^{k}(u_z^*) (\text{Ad } v \circ \varphi)^{k}(v_z)$$

$$= X_z^{(k)} u_z (\text{Ad } v \circ \varphi)^{k}(u_z^*) = X_z^{(k)} u_z (\text{Ad } v \circ \varphi)^{k}(X_z (\text{Ad } v \circ \varphi)(u_z^*))$$

$$= X_z^{(k)} u_z (\text{Ad } v \circ \varphi)^{k}(X_z)(\text{Ad } v \circ \varphi)^{k+1}(u_z^*) = X_z^{(k+1)} u_z (\text{Ad } v \circ \varphi)^{k+1}(u_z^*).$$

Ad (iv). Case $k = 1$ is clear. For the inductive step, we calculate using part (ii) above,

$$(gv)^{k+1} = (gv)^{k} \varphi^{k}(gv) = g^{(k)} v_k \varphi^{k}(gv) = g^{(k)}(v_k \varphi^{k}(g)v_k^*)v_k \varphi^{k}(v) = g^{(k+1)} v_{k+1},$$

and this completes the proof. \hfill \qed

The following lemma provides a key step in the proof of our second main result, Theorem 2.4. Here we remark that since $v = w^* u$, $w \in \lambda_u(F_n)^/ \cap O_n$, and Ad $u \circ \varphi$ is an automorphism of $\lambda_u(F_n)^/ \cap O_n$, we see that Ad $v \circ \varphi = Ad w^* \circ (Ad u \circ \varphi)$ is an automorphism of $\lambda_u(F_n)^/ \cap O_n$ as well. We also note that for each positive integer $k$, \{X_z^{(k)}\}_{z \in \mathbb{T}} is a unitary group in the center of $\lambda_u(F_n)^/ \cap O_n$.

**Lemma 2.7.** With unitaries $u, v, v_z$ and $u_z$ as above, there exist a positive integer $k$ and a unitary $U \in \lambda_u(F_n)^/ \cap O_n$ such that

$$(\text{Ad } v \circ \varphi)^{k}(x) = \text{Ad } U(x) \text{ for all } x \in \lambda_u(F_n)^/ \cap O_n.$$ 

Then $X_z^{(k)} = 1$. Furthermore, for such $U$ and $k$, we have $U^* v_k \in F_n$.

**Proof.** Since Ad $v \circ \varphi$ is an automorphism of a finite dimensional C*-algebra $\lambda_u(F_n)^/ \cap O_n$, its restricts to the center has finite order. Thus there exists a positive integer $k$ and a unitary $U \in \lambda_u(F_n)^/ \cap O_n$ such that $(\text{Ad } v \circ \varphi)^{k} = \text{Ad } U$ on $\lambda_u(F_n)^/ \cap O_n$. We claim that $U^* v_k \in F_n$.

Indeed, by Lemma 2.6, for all $z \in \mathbb{T}$ we have

$$\gamma_z(v_k) = v_z^{(k)} v_k = X_z^{(k)} u_z (\text{Ad } v \circ \varphi)^{k}(u_z^*) v_k = X_z^{(k)} u_z U u_z^* U^* v_k = X_z^{(k)} \gamma_z(U) U^* v_k,$$

and this yields

$$(5) \quad \gamma_z(U^* v_k) = X_z^{(k)} U^* v_k.$$ 

Since \{X_z^{(k)}\}_{z \in \mathbb{T}} is a unitary group in the center of $\lambda_u(F_n)^/ \cap O_n$, there exists a partition of unity $1 = \sum p_i$ in $Z(\lambda_u(F_n)^/ \cap O_n)$ and integers $k_i$ such that

$$X_z^{(k)} = \sum_i z^{k_i} p_i.$$
We have \((\text{Ad} \circ \varphi)^k(p_i) = U p_i U^* = p_i\) for all \(i\). Combining this with part (ii) of Lemma 2.6, we get
\[
(6) \quad v_k^* p_i v_k = \varphi^k(p_i).
\]
We want to show that \(k_i = 0\) for all \(i\). Suppose for a moment this is not the case and let \(k_i > 0\) for some \(i\). We set \(K := p_i U^* v_k(S^*_1)^{k_i}\). Since \(p_i\) being in \(Z(\lambda_u(F_n) \cap O_n)\) belongs to \(F_n\) as well, it follows from identity (5) above that
\[
\gamma_z(K) = \gamma_z(p_i U^* v_k(S^*_1)^{k_i}) = p_i X_z^{(k)} U^* v_k \gamma_z((S^*_1)^{k_i}) = z^{k_i} p_i U^* v_k(z^{-k_i} (S^*_1)^{k_i}) = K.
\]
Hence \(K\) belongs to \(F_n\). We have \(KK^* = p_i\). On the other hand, using identity (6) we get
\[
K^* K = S^{k_i} v_k^* p_i v_k (S^*_1)^{k_i} = S^{k_i} \varphi^k(p_i) (S^*_1)^{k_i} = \varphi^{k+k_i} (p_i) S^{k_i} (S^*_1)^{k_i}.
\]
It easily follows that \(\tau(KK^*) > \tau(K^* K)\), which is a contradiction. A similar argument applies in the case \(k_i < 0\). Hence \(k_i = 0\) for all \(i\) and thus \(X_z^{(k)} = 1\). Now, identity (5) implies that \(U^* v_k\) is fixed by the gauge action and hence belongs to \(F_n\).

Now, we are ready to prove the second main result of this paper.

**Theorem 2.8.** Let \(u \in U(O_n)\) be such that \(\lambda_u(F_n) \subseteq F_n\) and \(\dim \lambda_u(F_n) \cap F_n < \infty\). Then there exist a positive integer \(k\) and unitaries \(W \in \lambda_u(F_n) \cap O_n\) and \(V \in F_n\) such that \(u_k = W V\).

**Proof.** By Theorem 2.4 and Lemma 2.7, there exist unitaries \(w, U \in \lambda_u(F_n) \cap O_n\), a unitary group \(\{v_z\}_{z \in \mathbb{T}}\) in \(\lambda_u(F_n) \cap F_n\) and a positive integer \(k\) satisfying \(u = wv\), \(\gamma_z(v) = v_z v\), \(U^* v_k \in F_n\). By part (iv) of Lemma 2.6 we have \(w^{(k)} v_k = u_k\). Thus to complete the proof, it suffices to put \(W := w^{(k)} U\) and \(V := U^* v_k\).

It was observed in [4] (just above Remark 4.4) that if \(\lambda_u(F_n) \subseteq F_n\) and \(\lambda_u(F_n) \cap O_n = \mathbb{C} 1\) then \(u \in F_n\). The following corollary gives a sharp strengthening of that result.

**Corollary 2.9.** Let \(u\) be a unitary in \(O_n\). If \(\lambda_u(F_n) \subseteq F_n\), \(\dim \lambda_u(F_n) \cap F_n < \infty\) and the automorphism \(\text{Ad} u \circ \varphi\) of \(\lambda_u(F_n) \cap O_n\) is inner, then there exist a unitary \(w \in \lambda_u(F_n) \cap O_n\) and a unitary \(v \in F_n\) such that \(u = wv\), and hence also \(\lambda_u|F_n = \lambda_v|F_n\). In particular, this is the case whenever \(\lambda_u(F_n) \cap O_n\) is a factor.

**Remark 2.10.** The assumption in Corollary 2.9 above that the automorphism \(\text{Ad} u \circ \varphi\) of \(\lambda_u(F_n) \cap O_n\) be inner, is equivalent to demanding existence of a unitary \(g\) in the relative commutant \(\lambda_u(F_n) \cap O_n\) such that
\[
\lambda_{gu}(F_n) \cap O_n = \lambda_{gu}(O_n) \cap O_n.
\]
Indeed, if \(\text{Ad} u \circ \varphi\) is inner then \(\text{Ad} gu \circ \varphi = \text{id}\) for a suitable unitary \(g\) in \(\lambda_u(F_n) \cap O_n\). Hence \(\lambda_{u}(F_n) \cap O_n = \lambda_{gu}(F_n) \cap O_n = \lambda_{gu}(O_n) \cap O_n\). Conversely, if \(\lambda_{gu}(F_n) \cap O_n = \lambda_{gu}(O_n) \cap O_n\)
\( \lambda_{g\mu}(\mathcal{O}_n)' \cap \mathcal{O}_n \) then \( \text{Ad} \ g \mu \circ \varphi = \text{id} \) on \( \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n = \lambda_{g\mu}(\mathcal{F}_n)' \cap \mathcal{O}_n \), and hence \( \text{Ad} \ u \circ \varphi \) is inner.

\[ \Box \]

**Remark 2.11.** We remark that the implication in Corollary 2.9 above cannot be reversed. In fact, there exist unitaries \( u \in \mathcal{F}_n \) such that \( \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n \) is finite dimensional and the automorphism \( \text{Ad} \ u \circ \varphi \) is outer on \( \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n \). For example, take

\[ u = S_{22}S_{11}^* + S_{12}^*S_{22} + S_{11}^*S_{12} + P_{21}. \]

a permutative unitary in \( \mathcal{F}_2 \). Then \( \text{Ad} \ u \circ \varphi \) is outer on \( \lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2 \). For otherwise let \( h \) be a unitary in \( \lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2 \) such that \( \text{Ad} \ u \circ \varphi = \text{Ad} \ h \) on \( \lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2 \). Then \( \text{Ad} \ u \circ \varphi(h) = h \) and thus \( h \in \lambda_u(\mathcal{O}_2)' \cap \mathcal{O}_2 \). But it can be shown that \( \lambda_u \) is irreducible on \( \mathcal{O}_2 \) (e.g., see [5], where this endomorphism is denoted \( P_{12} \)), and hence \( \lambda_u(\mathcal{O}_2)' \cap \mathcal{O}_2 = \mathbb{C}1 \). Thus \( h \) is a scalar and consequently \( \text{Ad} \ u \circ \varphi \) is identity on \( \lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2 \). This however is not the case, since one can calculate directly that \( \text{Ad} \ u \circ \varphi \) permutes \( P_1 \) and \( P_2 \), and both these projections are in \( \lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2 \).

We want to elaborate a little bit the statement of Theorem 2.8 above. We continue keeping our standing assumption (2).

**Lemma 2.12.** Let \( \alpha \) be an automorphism of \( \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n \) and let \( k \in \mathbb{N} \) be such that \( \alpha^k \) acts trivially on \( \mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n) \). Then there exists a MASA \( D \) of \( \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n \) and a unitary \( g \) in \( \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n \) such that

(i) \( (\text{Ad} \ g \circ \alpha)^k = \text{id} \), and

(ii) \( (\text{Ad} \ g \circ \alpha)(D) = D \).

**Proof.** Automorphism \( \alpha \) permutes the finitely many minimal central projections of \( \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n \). Write this permutation as a product of disjoint cycles. Clearly, it suffices to prove the lemma for each cycle separately. Thus we may simply assume that \( \alpha \) acts transitively on minimal projections \( p_1, p_2, \ldots, p_l \) in \( \mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n) \), so that \( \alpha(p_i) = p_{i+1} \), with \( p_{l+1} = p_1 \). Let \( \{e^{(i)}_{r,s}\} \) be matrix units of the full matrix algebra \( p_i(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n) \), such that all \( e^{(i)}_{r,s} \) are in \( \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n \). Then \( D := \text{span}\{e^{(i)}_{r,s}\} \) is a MASA in \( \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n \). Since \( p_i(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n) \cong p_{i+1}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n) \), we can find a unitary \( g_i \in p_{i+1}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n) \) such that \( (\text{Ad} \ g_i \circ \alpha)(e^{(i)}_{r,s}) = e^{(i+1)}_{r,s} \). Setting \( g := \sum_{i=1}^l g_i \) we obtain the desired result.

\[ \Box \]

**Lemma 2.13.** Let \( u \in \mathcal{U}(\mathcal{O}_n) \) be such that \( \lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n \) and \( \dim \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n < \infty \). Then there exist a positive integer \( k \), a unitary \( g \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n \), and a unitary group \( \{d_z\}_{z \in T} \subseteq \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n \) such that \( (gv)_k \in \mathcal{F}_n \) and \( \gamma_s(gv) = d_zgv \).

**Proof.** Put \( \alpha := \text{Ad} \ v \circ \varphi \), and let \( g \) and \( k \) be as in Lemma 2.12. Then we have

\[ (\text{Ad} \ gv \circ \varphi)^k = \text{id}, \]
and thus

\[ \operatorname{Ad} v_k \circ \varphi^k = (\operatorname{Ad} v \circ \varphi)^k = \operatorname{Ad}(g^{(k)})^* \]

by parts (ii) and (iv) of Lemma 2.6. Then arguing as in the proof of Lemma 2.7 (with \( g^{(k)} \) playing the role of \( U \)), we get

\[ (gv)_k = g^{(k)}v_k \in \mathcal{F}_n. \]

Now, let \( D \) be a MASA as in Lemma 2.12. For all \( x \in D \) and \( z \in \mathbb{T} \), we see that

\[ gv\varphi(x)v^*g^* = \gamma_z(gv\varphi(x)v^*g^*) = \gamma_z(g)v_z\varphi(x)v^*v_z^*\gamma_z(g^*) \]

\[ = (\gamma_z(g)v_zg^*)(gv\varphi(x)v^*g^*)(\gamma_z(g)v_zg^*), \]

which implies that \( \gamma_z(g)v_zg^* \) is in the commutant of MASA \( D \), and hence in \( D \) itself. Set \( d_z = \gamma_z(g)v_zg^* \), a unitary in \( D \). Now, \( d_z = u_zg^*v_zg^* \) implies \( u_z^*d_z = g(u_z^*v_z)g^* \). Since \( \{u_z\}_{z \in \mathbb{T}} \) and \( \{v_z\}_{z \in \mathbb{T}} \) are commuting unitary groups, so is \( \{u_z^*d_z\}_{z \in \mathbb{T}} \), and consequently also is \( \{d_z\}_{z \in \mathbb{T}} \). Finally, we see that \( \gamma_z(gv) = \gamma_z(g)v_zv = d_zgv. \]

Now, we are ready to prove the following result.

**Theorem 2.14.** Let \( u \in \mathcal{U}(\mathcal{O}_n) \). If \( \lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n \) and \( \dim \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n < \infty \), then there exists a unitary \( W \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n \) satisfying the following.

(i) There exists a unitary group \( \{d_z\}_{z \in \mathbb{T}} \subseteq \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n \) such that \( \gamma_z(Wu) = d_zWu \) for all \( z \in \mathbb{T} \).

(ii) There exists a positive integer \( k \) such that \( (Wu)_k \in \mathcal{F}_n \).

**Proof.** Let \( u = wv \) be a factorization as in Theorem 2.3 and let \( k \in \mathbb{N} \) and \( g \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n \) be as in Lemma 2.13 above. Then setting \( W := gw^* \) gives the claim. \( \square \)

**3. The criterion and examples**

In this section, we give a dynamic characterization of those unitaries \( u \in \mathcal{O}_n \) satisfying our standing assumptions which either belong to \( \mathcal{F}_n \) (Theorem 3.3) or admit a unitary \( v \in \mathcal{F}_n \) such that \( \lambda_u_{|\mathcal{F}_n} = \lambda_v_{|\mathcal{F}_n} \) (Corollary 3.4). Before proving these results, we still need one technical lemma about the structure of the relative commutants. We keep our standing assumptions (2).

**Lemma 3.1.** There exist a unitary group \( \{q_z\}_{z \in \mathbb{T}} \) in \( \mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n) \) such that

\[ X_z = q_z(\operatorname{Ad} v \circ \varphi)(q_z^*). \]

**Proof.** Since \( \operatorname{Ad} v \circ \varphi \) restricts to an automorphism of \( \mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n) \), there exist minimal projections \( p_{ij}^{(j)} \), \( j = 1, \ldots, N, \ i = 1, \ldots, n_j \), in \( \mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n) \) such that

\[ \mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n) = \bigoplus_{j=1}^N \bigoplus_{i=1}^{n_j} p_{ij}^{(j)}. \]
and 
\[(\text{Ad } v \circ \varphi)(p_i^{(j)}) = p_{i+1}^{(j)} \text{ for } i < n_j, \quad \text{and } (\text{Ad } v \circ \varphi)(p_n^{(j)}) = p_1^{(j)}\].

Then \(X_z\) from Lemma 2.3 can be written as 
\[X_z = \sum_{j=1}^{N} \sum_{i=1}^{n_j} z^{m_i^{(j)} p_i^{(j)}},\]
for some \(m_i^{(j)} \in \mathbb{N}\). Now, let \(k \in \mathbb{N}\) be such that \(\text{Ad } v \circ \varphi\) is an inner automorphism of \(\lambda_u(F_n)' \cap O_n\). Then 
\[X_z^{(k)} = X_z(\text{Ad } v \circ \varphi)(X_z)(\text{Ad } v \circ \varphi)^2(X_z) \ldots (\text{Ad } v \circ \varphi)^{k-1}(X_z) = 1\]
by Lemma 2.7. Since each \(n_j\) divides \(k\), this implies that 
\[\sum_{i=1}^{n_j} m_i^{(j)} = 0\]
for each \(j = 1, \ldots, N\). Now, we want to define \(q_z\) as follows,
\[q_z = \sum_{j=1}^{N} \sum_{i=1}^{n_j} z^{r_i^{(j)} p_i^{(j)}},\]
for suitable chosen integers \(r_i^{(j)}\), so that \(X_z = q_z(\text{Ad } v \circ \varphi)(q_z^*)\). To this end, it suffices to put 
\[r_1^{(j)} = 0, \quad j = 1, \ldots, N,\]
\[r_k^{(j)} = \sum_{r=2}^{k} m_r^{(j)}, \quad j = 1, \ldots, N, \quad k = 2, \ldots, n_j.\]
\[\square\]

**Theorem 3.2.** Let \(u \in U(O_n)\) be such that \(\lambda_u(F_n) \subseteq F_n\) and \(\dim \lambda_u(F_n)' \cap F_n < \infty\). Put \(\alpha := \text{Ad } u \circ \varphi\). If \(\alpha\) satisfies the following two conditions:

(i) \(\alpha(\lambda_u(F_n)' \cap F_n) = \lambda_u(F_n)' \cap F_n\), and

(ii) \(\alpha|_{\lambda_u(F_n) \cap F_n}\) preserves the \(\tau\)-trace,

then \(u \in F_n\).

**Proof.** At first, we observe that there exists a unitary group \(\{u'_z\}_{z \in \mathbb{T}}\) in \(Z(\lambda_u(F_n)' \cap F_n)\) such that \(\text{Ad } u'_z(x) = \gamma_z(x)\) for all \(x \in \lambda_u(F_n)' \cap O_n\) and \(\gamma_z(u) = u'_z \alpha(u'_z^*) u\). Indeed, it suffices to put \(u'_z := q_z u_z\), with \(q_z\) as in Lemma 3.1 above. Then \(\alpha(u'_z) \in Z(\lambda_u(F_n)' \cap F_n)\) by condition (i) of the theorem, and hence \(\{u'_z \alpha(u'_z^*)\}_{z \in \mathbb{T}}\) is a unitary group. Thus,
\[u'_z \alpha(u'_z^*) = \sum z^{k_j} p_{j}\]
for some integers \(k_j\) and a partition of unity by projections \(p_j\) from \(Z(\lambda_u(F_n)' \cap F_n)\).

Now, we claim that \(p_j = 0\) whenever \(k_j \neq 0\). To this end, suppose first that \(k_j > 0\) for some index \(j\), and put \(R := p_{k_j} u(S_1^{*})^{k_j}\). We have \(\gamma_z(R) = R\) for all
z ∈ ℤ, and thus R ∈ F_n. However, an easy calculation shows that RR* = pk_j and 
R*R = \varphi^{k+1}(\alpha^{-1}(p_k))S_1^{(S_1^*)k}. In view of condition (ii) of the theorem, this would 
implicitly show that \tau(RR*) ≠ \tau(R*R) if p_j ≠ 0, a contradiction. Therefore p_j = 0 for all k_j > 0. A 
similar argument shows that p_j = 0 if k_j < 0.

Consequently, u^*_z\alpha(u^*_z) = 1. But this gives γ_z(u) = u for all z ∈ ℤ. Hence u ∈ F_n 
and the theorem is proved. □

We note that Theorem 3.2 gives a necessary and sufficient condition for u ∈ F_n, since 
the reverse implication is trivial. Likewise, Corollary 3.3 below, gives a necessary and 
sufficient condition for uk ∈ F_n.

**Corollary 3.3.** Let u ∈ U(O_n) be such that λ_u(F_n) ⊆ F_n and \dim\lambda_u(F_n)' ∩ F_n < ∞. 
Put α := (Ad u ∘ \varphi)^k, for some positive integer k. If α satisfies the following two 
conditions:

(i) \alpha(λ_u(F_n)' ∩ F_n) = λ_u(F_n)' ∩ F_n, and 
(ii) α| λ_u(F_n)' ∩ F_n preserves the \tau\text{-trace},

then \ uk ∈ F_n.

Now, we are ready to give the following decomposability criterion.

**Corollary 3.4.** Let u ∈ U(O_n) be such that λ_u(F_n) ⊆ F_n and \dim\lambda_u(F_n)' ∩ F_n < ∞. 
Put α := Ad u ∘ \varphi. Then the following two conditions are equivalent:

(1) There exist unitaries w ∈ λ_u(F_n)' ∩ O_n and v ∈ F_n such that u = wv.

(2) For each minimal projection p ∈ Z(λ_u(F_n)' ∩ O_n) there exists a \tau\text{-preserving} 

isomorphism 

p(λ_u(F_n)' ∩ F_n) ≅ α(p)(λ_u(F_n)' ∩ F_n).

Now, we show how to construct examples of endomorphisms λ_u of O_n globally pre-
serving the core UHF-subalgebra F_n but such that no unitary v ∈ F_n exists for which 
λ_u|_{F_n} = λ_v|_{F_n}.

To begin with, take two non-zero, orthogonal projections q_1, q_2 in F_n such that 
\tau(q_2)/\tau(q_1) = n^r for some non-zero integer r. Let A_1 be a partial isometry in O_n^{(r)} 
with domain projection \varphi(q_1) and range projection q_2. Likewise, let A_2 be a partial 
isometry in O_n^{(r)} with domain projection \varphi(q_2) and range projection q_1. Finally, let A_3 
be a partial isometry in F_n with domain projection 1 - \varphi(q_1) - \varphi(q_2) and range projection 
1 - q_1 - q_2. Put u := A_1 + A_2 + A_3. Then u is a unitary in O_n such that 

(7) Ad u ∘ \varphi(q_1) = q_2 and Ad u ∘ \varphi(q_2) = q_1.

Now, u^*_z(u^*_z) = z^*q_1 + z^{-r}q_2 + 1 - q_1 - q_2 belongs to span\{1, q_1, q_2\}, and span\{1, q_1, q_2\} ⊆ 
λ_u(F_n)' ∩ \tau\text{-isomorphism} F_n by [4 Proposition 2.3] and (7) above. Thus λ_u(F_n) ⊆ F_n by Proposition 
2.1 above.
More generally, Let $1 = \sum q_j$ be a partition of unity by projections in $\mathcal{O}_n$. Let $u$ be any unitary in $\mathcal{O}_n$ such that $\text{Ad} u \circ \varphi$ permutes projections $\{q_j\}$ and for each $j$ there is a $k_j \in \mathbb{Z}$ such that $q_j u \in \mathcal{O}_n^{(k_j)}$. Then $u \gamma_z(u^*) \in \text{span}\{q_j\} \subseteq \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ for all $z \in \mathbb{T}$. This simple construction gives a large class of examples of unitaries $u \in \mathcal{O}_n \setminus \mathcal{F}_n$ such that $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$. However, to verify the conditions of Corollary 3.4 one needs more detailed information on the relative commutants $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n \subseteq \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$. Exact determination of these relative commutants is rather difficult and does not seem possible in general, despite the identity from [4, Proposition 2.3]. However, it is quite doable in concrete cases.

Now, we illustrate the above discussion with two concrete examples in $\mathcal{O}_2$. In these examples, along with the main algebra $C^*(S_1, S_2) \cong \mathcal{O}_2$, we consider its other subalgebras, also isomorphic to $\mathcal{O}_2$. For example, if $T_1, T_2$ are isometries in $C^*(S_1, S_2)$ generating a copy of $\mathcal{O}_2$, then we use subscript $T$ along with the standard notation to indicate that the object comes from $C^*(T_1, T_2)$ and its generators. Thus $\varphi_T$ denotes the usual shift on $C^*(T_1, T_2)$, that is a map $\varphi : C^*(T_1, T_2) \to C^*(T_1, T_2)$ such that $\varphi(x) = T_1 x T_1^* + T_2 x T_2^*$. Similarly, $D_T$ denotes the diagonal MASA of $C^*(T_1, T_2)$, and so on. The proof of one technical lemma needed in Example 3.6 is given afterwards.

**Example 3.5.** Take $q_1 = P_{11}, q_2 = P_{222}$, and set

$A_1 = S_{2221}S_{111}^* + S_{2222}S_{211}^*$

$A_2 = S_{111}S_{1222}^* + S_{112}S_{2222}^*$

$A_3 = S_{1222}S_{2221}^* + S_{211}S_{112}^* + P_{121} + P_{1221} + P_{211} + P_{212} + P_{221}$

We note that unitary $u := A_1 + A_2 + A_3$ falls within the class of polynomial unitaries considered in [4, Section 5]. In particular, its graph $E_u$, as defined therein, admits the $\{-1, 0, +1\}$ labelling:

This labelled graph satisfies the path condition defined in [4, p. 616], and this is an alternative way of showing that $\lambda_u(\mathcal{F}_2) \subseteq \mathcal{F}_2$. 
Now, we have $P_{11}\mathcal{O}_2P_{11} \cong \mathcal{O}_2 = C^*(T_1, T_2)$, under the isomorphism sending $T_1$ to $S_{111}S_{11}^*$ and $T_2$ to $S_{112}S_{11}^*$. Similarly, $P_{222}\mathcal{O}_2P_{222} \cong \mathcal{O}_2 = C^*(R_1, R_2)$, under the isomorphism sending $R_1$ to $S_{222}S_{222}^*$ and $R_2$ to $S_{2222}S_{222}^*$. Then an easy calculation shows that

$$\text{Ad} u \circ \varphi(T_j) = \varphi_R(R_j),$$

$$\text{Ad} u \circ \varphi(R_j) = \varphi_T(T_j),$$

for $j = 1, 2$. Consequently, the restriction of $(\text{Ad} u \circ \varphi)^2$ to $P_{11}\mathcal{O}_2P_{11}$ is conjugate to $\varphi_R^2$. Likewise, the restriction of $(\text{Ad} u \circ \varphi)^2$ to $P_{222}\mathcal{O}_2P_{222}$ is conjugate to $\varphi_T^2$. This immediately implies

$$\lambda_u(\mathcal{F}_2)' \cap P_{11}\mathcal{O}_2P_{11} \subseteq \bigcap_{k=1}^{\infty} (\text{Ad} u \circ \varphi)^{2k}(P_{11}\mathcal{O}_2P_{11}) = \mathbb{C}P_{11},$$

$$\lambda_u(\mathcal{F}_2)' \cap P_{222}\mathcal{O}_2P_{222} \subseteq \bigcap_{k=1}^{\infty} (\text{Ad} u \circ \varphi)^{2k}(P_{222}\mathcal{O}_2P_{222}) = \mathbb{C}P_{222}.$$

That is, both $P_{11}$ and $P_{222}$ are minimal projections in $\lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2$. One easily checks that $\text{Ad} u \circ \varphi(S_{11}S_{222}^*) = S_{222}S_{11}^*$. Thus $S_{11}S_{222}^*$ is in $\lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2$, and we see that $(P_{11} + P_{222})\lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2(P_{11} + P_{222}) \cong M_2(\mathbb{C})$. We remark that the restriction of $\text{Ad} u \circ \varphi$ to $(P_{11} + P_{222})\mathcal{O}_2(P_{11} + P_{222})$ is conjugate to endomorphism $\rho_{1342}$ from [5]. Let

$$w := S_{11}S_{222}^* + S_{222}S_{11}^* + 1 - P_{11} - P_{222}.$$

Then $w$ is a unitary in $\lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2$ such that $w^*u \in \mathcal{F}_2$. $\square$

**Example 3.6.** Take $q_1 = P_1$, $q_2 = P_{21}$, and set

$$A_1 = S_{211}S_{21}^* + S_{212}S_{112}^* + S_{2122}S_{111}^*,$$

$$A_2 = S_{12}S_{121}^* + S_{11}S_{221}^*,$$

$$A_3 = S_{221}S_{122}^* + P_{222}.$$

We put $u := A_1 + A_2 + A_3$. By construction, $\text{Ad} u \circ \varphi(P_1) = P_{21}$ and also $\text{Ad} u \circ \varphi(P_{21}) = P_1$. Hence $\text{Ad} u \circ \varphi(P_{22}) = P_{22}$ as well.

We have $P_{22}C^*(S_1, S_2)P_{22} \cong \mathcal{O}_2 = C^*(R_1, R_2)$, under the identification of $S_{221}S_{221}^*$ with $R_1$ and $S_{222}S_{222}^*$ with $R_2$. This isomorphism yields a conjugation between the restriction of $\text{Ad} u \circ \varphi$ to $P_{22}C^*(S_1, S_2)P_{22}$ and the shift $\varphi_R$. Consequently,

$$\lambda_u(\mathcal{F}_2)' \cap P_{22}C^*(S_1, S_2)P_{22} = \bigcap_{k=1}^{\infty} (\text{Ad} u \circ \varphi)^{k}(P_{22}C^*(S_1, S_2)P_{22}) = \mathbb{C}P_{22}.$$

We have $P_1C^*(S_1, S_2)P_1 \cong \mathcal{O}_2 = C^*(T_1, T_2)$, under the identification of $S_{11}S_{11}^*$ with $T_1$ and $S_{12}S_{12}^*$ with $T_2$. This isomorphism carries the restriction of $(\text{Ad} u \circ \varphi)^2$ to $P_1C^*(S_1, S_2)P_1$ to the endomorphism of $C^*(T_1, T_2)$ given as composition $\varphi_T \circ \psi_T$, where $\psi_T$ is an endomorphism of $C^*(T_1, T_2)$ such that

$$\psi_T(x) = T_1xT_1^* + T_2(\text{Ad} F_T(x))T_2^*,$$
where $F_T := T_2 T_1^* + T_1 T_2^*$. By Lemma 3.7 we have

$$\lambda_u(F_2)' \cap P_1 C^*(S_1, S_2) P_1 \subseteq \Lambda \lambda^k (\text{Ad } u \circ \varphi) (P_1 C^*(S_1, S_2) P_1) = \mathbb{C} P_1.$$  

We have $P_21 C^*(S_1, S_2) P_2 \cong \mathbb{O}_2 = C^*(V_1, V_2)$, under the identification of $S_{211} S_{21}$ with $V_1$ and $S_{212} S_{21}$ with $V_2$. This isomorphism carries the restriction of $(\text{Ad } u \circ \varphi)^2$ to $P_21 C^*(S_1, S_2) P_2$ to $\psi_V \circ \varphi_V$. An argument similar to that from Lemma 3.7 shows that $\lambda_u(F_2)' \cap P_21 C^*(S_1, S_2) P_21 = \mathbb{C} P_21$. Alternatively, this also easily follows from the preceding argument and the fact that $\text{Ad } u \circ \varphi(P_21) = P_1$.

In view of the above, either $\lambda_u(F_2)' \cap \mathbb{O}_2 = \text{span}\{P_1, P_21, P_22\} \cong \mathbb{C}^3$, or $P_1$ and $P_21$ are equivalent in $\lambda_u(F_2)' \cap \mathbb{O}_2$. In the latter case, $\lambda_u(F_2)' \cap \mathbb{O}_2$ contains a subalgebra isomorphic to $M_2(\mathbb{C})$ which is invariant under $\text{Ad } u \circ \varphi$ and has $P_1$ and $P_21$ as its minimal projections. Suppose for a moment that this is the case. Then $\text{Ad } u \circ \varphi$ restricts to a non-trivial automorphism of $M_2(\mathbb{C})$, by necessity inner. The implementing unitary matrix $g$ is fixed by $\text{Ad } u \circ \varphi$ and thus belongs to $\lambda_u(\mathbb{O}_2)' \cap \mathbb{O}_2$. Matrix $g$ has both diagonal entries equal to 0. Multiplying $g$ by a suitable scalar of modulus 1, we can find such $g$ that is self-adjoint. Now we see that there is a unitary element $x \in \mathbb{O}_2$ such that

$$g = S_{21} x^* S_1^* + S_1 x S_{21}^* \in \lambda_u(\mathbb{O}_2)' \cap \mathbb{O}_2.$$  

Now, writing $F := S_1 S_2^* + S_2 S_1^*$, we compute

$$\text{Ad} u \circ \varphi(g) = u(S_{11} x S_{12}^* + S_{12} x^* S_{11}^* + S_{21} x S_{21}^* + S_{22} x^* S_{21}^*) u^*$$

$$= S_{212} F x S_{12}^* + S_{12} x^* F S_{212}^* + S_{211} x S_{11}^* + S_{11} x^* S_{211}^*,$$

and hence we get

$$S_1 x S_{21}^* + S_{21} x^* S_1^* = S_{212} F x S_{12}^* + S_{12} x^* F S_{212}^* + S_{211} x S_{11}^* + S_{11} x^* S_{211}^*.$$  

Multiplying by $S_1^*$ from the left-side and by $S_{21}$ from the right-side, we obtain

$$x = S_2 x^* F S_2^* + S_1 x^* S_1^*.$$  

Equation 8 implies $x S_1 = S_1 x^*$ and $S_1^* x = x^* S_1^*$. These two combined then yield $(x + x^*) S_1 = S_1 (x + x^*)$ and $(x - x^*) S_1 = -S_1 (x - x^*)$. By Proposition 4, both $x + x^*$ and $x - x^*$ are scalars, and thus so is $x$. This however contradicts 8.

Thus $\lambda_u(F_2)' \cap \mathbb{O}_2 = \text{span}\{P_1, P_21, P_22\}$ and since $\tau(P_1) \neq \tau(P_21)$, we conclude from Corollary 3.3 that there are no unitaries $w \in \lambda_u(F_2)' \cap \mathbb{O}_2$ and $v \in F_2$ such that $u = w v$. □

**Lemma 3.7.** Let $\psi_T$ be an endomorphism of $C^*(T_1, T_2) \cong \mathbb{O}_2$ such that

$$\psi_T(x) = T_1 x T_1^* + T_2 (\text{Ad } F_T(x)) T_2^*,$$

then $\psi_T$ is an automorphism of $C^*(T_1, T_2)$.
where $F_T := T_2T_1^* + T_1T_2^*$. Then we have
\[
\bigcap_{k=1}^{\infty} (\varphi_T \psi_T)^k(C^*(T_1, T_2)) = \mathbb{C}1.
\]

Proof. We note that
\[
\varphi_T \psi_T(x) = T_{11}x T_{11}^* + T_{21}x T_{21}^* + T_{12}(\text{Ad} F_T(x))T_{12}^* + T_{22}(\text{Ad} F_T(x))T_{22}^*.
\]
Also, we clearly have $F_T T_1 = T_2$ and $F_T T_2 = T_1$. Thus $(\varphi_T \psi_T)^k(x)$ may be written as a finite sum of elements of the form $T_{\mu} X T_{\mu}^*$ with $|\mu| = 2k$. This gives
\[
\bigcap_{k=1}^{\infty} (\varphi_T \psi_T)^k(C^*(T_1, T_2)) \subseteq \mathcal{D}_T \cap C^*(T_1, T_2) = \mathcal{D}_T.
\]
For a positive integer $k$, let
\[
Q_k := \sum_{|\mu| = k-1} T_{\mu} T_{\mu}^*.
\]
Then a straightforward induction on $k$ shows that
\[
Q_{2k}(\varphi_T \psi_T)^k(x) = Q_{2k} \varphi_T^{2k}(x)
\]
for all $x \in C^*(T_1, T_2)$. Take a $d = d^* \in \mathcal{D}_T$ that belongs to $\bigcap_{k=1}^{\infty} (\varphi_T \psi_T)^k(C^*(T_1, T_2))$. Suppose $d$ is not a constant multiple of 1. Then there exist $k \in \mathbb{N}$, $t \in \mathbb{R}$, $\epsilon > 0$ and $\mu, \nu \in W_2^{k-1}$ such that
\[
T_{\mu} T_{\mu}^* d \geq (t + \epsilon) T_{\mu} T_{\mu}^* \quad \text{and} \quad T_{\nu} T_{\nu}^* d \leq (t - \epsilon) T_{\mu} T_{\mu}^*.
\]
Let $x = x^* \in \mathcal{D}_2$ be such that $d = (\varphi_T \psi_T)^k(x)$. Then $Q_{2k} d = Q_{2k} \varphi_T^{2k}(x)$. Since $T_{\mu} T_{\mu}^* \leq Q_{2k}$ and $T_{\nu} T_{\nu}^* \leq Q_{2k}$, we get
\[
T_{\mu} x T_{\mu}^* = T_{\mu} T_{\mu}^* Q_{2k} \varphi_T^{2k}(x) \geq (t + \epsilon) T_{\mu} T_{\mu}^*, \quad \text{and} \quad T_{\nu} x T_{\nu}^* = T_{\nu} T_{\nu}^* Q_{2k} \varphi_T^{2k}(x) \leq (t - \epsilon) T_{\nu} T_{\nu}^*.
\]
This, however, is a contradiction. Indeed, since $T_{\mu} T_{\mu}^*$ and $T_{\nu} T_{\nu}^*$ are isometries, the above two inequalities would imply that both $x \geq (t + \epsilon)$ and $x \leq (t - \epsilon)$. Consequently,
\[
\bigcap_{k=1}^{\infty} (\varphi_T \psi_T)^k(C^*(T_1, T_2)) = \mathbb{C}1,
\]
as required. \qed

References


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