PURE INFINITENESS AND IDEAL STRUCTURE OF $C^*$-ALGEBRAS ASSOCIATED TO FELL BUNDLES

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Abstract. We investigate structural properties of the reduced cross-sectional algebra $C^*_r(B)$ of a Fell bundle $B$ over a discrete group $G$. Conditions allowing one to determine the ideal structure of $C^*_r(B)$ are studied. Notions of aperiodicity, paradoxicality and $B$-infiniteness for the Fell bundle $B$ are introduced and investigated by themselves and in relation to the partial dynamical system dual to $B$. Several criteria of pure infiniteness of $C^*_r(B)$ are given. It is shown that they generalize and unify corresponding results obtained in the context of crossed products, by the following duos: Laca, Spielberg [34]; Jolissaint, Robertson [21]; Sierakowski, Rørdam [47]; Giordano, Sierakowski [18] and Ortega, Pardo [39].

For exact, separable Fell bundles satisfying the residual intersection property primitive ideal space of $C^*_r(B)$ is determined. The results of the paper are shown to be optimal when applied to graph $C^*$-algebras. Applications to a class of Exel-Larsen crossed products are presented.

1. Introduction

Many of $C^*$-algebras studied in literature are equipped with a natural additional structure which can be used to study their properties. This structure can be exhibited by a group co-action (or a group action if the underlying group is abelian) or more generally by a group grading of the $C^*$-algebra. It allows one to investigate the $C^*$-algebra by means of the associated Fell bundle of subspaces determining the grading. Fell bundles over discrete groups proved to be a convenient framework for studying crossed products corresponding to global or partial group actions, and were successfully applied to diverse classes of $C^*$-algebras, [16], [18], [1]. Moreover, the approach based on Fell bundles has recently gained an increased interest in an analysis of $C^*$-algebras associated to generalized graphs [7], Nica-Pimsner algebras [8], and Cuntz-Pimsner algebras [33], [2] associated to product systems over semigroups. We remark that, in contrast to most of applications in [16], [18], [1], in the latter case the core $C^*$-algebra corresponding to the unit in the group, as a rule, is non-commutative. The present paper is devoted to investigations of the ideal structure, pure infiniteness and related features of the reduced cross-sectional algebras $C^*_r(B)$ arising from a Fell bundle $B = \{B_t\}_{t \in G}$ over a discrete group $G$ with the unit fiber $B_e$ being genuinely a non-commutative $C^*$-algebra. One of our primary aims is to give convenient $C^*$-dynamical conditions on $B$ that lead to a coherent treatment unifying various approaches to pure infiniteness of crossed products by group actions [34], [21], [47], [18], and that are applicable to $C^*$-algebras arising from semigroup structures. Actually, for a class of Fell bundles we consider, the $C^*$-algebra $C^*_r(B)$ has the ideal property, and it is known that in the presence of this property pure infiniteness [23], Definition 4.1] is equivalent to strong pure infiniteness [24, Definition 5.1]. Additionally, if $C^*_r(B)$ is separable we provide a description of the primitive spectrum of $C^*_r(B)$. This together with

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known criteria for nuclearity of $C^*_r(B)$, cf. [16, Proposition 25.10], form a full toolkit for producing and analyzing graded $C^*$-algebras that undergo Kirchberg’s classification (up to stable isomorphism) via ideal system equivariant KK-theory [22].

In order to detect pure infiniteness of a non-simple $C^*$-algebra, one needs to understand its ideal structure. The general algebraic necessary and sufficient conditions assuring that the ideals in the ambient algebra are uniquely determined by their intersection with the core are known. These conditions are exactness and the residual intersection property. They were introduced in the context of crossed products in [48], then generalized to partial crossed products in [18] and to cross-sectional algebras in [1]. We give a metric characterisation of the intersection property using a notion of topological grading, and we shed light on the notion of exactness of a Fell bundle $B = \{B_t\}_{t \in G}$ by characterising it in terms of graded and Fourier ideals in $C^*_r(B)$.

An important dynamical condition implying the (residual) intersection property of $B$ is (residual) topological freeness of a dual partial dynamical system $\{(\hat{D}_t)_{t \in G}, (\hat{h}_t)_{t \in G}\}$ defined on the spectrum $\hat{B}_e$ of the core $B_e$. This result is well-known for crossed products, cf. [4]. Recently, it was generalized to cross-sectional algebras of saturated Fell bundles by the authors of the present paper [33], and to general Fell bundles by Beatriz Abadie and Fernando Abadie [11]. The system $\{(\hat{D}_t)_{t \in G}, (\hat{h}_t)_{t \in G}\}$ is very useful in investigation of the ideal structure of $C^*_r(B)$. In particular, it factorizes to a partial dynamical system on the primitive spectrum $\text{Prim}(B_e)$ of $B_e$, and we show that for exact, separable Fell bundles satisfying the residual intersection property the primitive ideal space of $C^*_r(B)$ can be identified with the quasi-orbit space of this dual action on $\text{Prim}(B_e)$. We show below that this result applied to graph $C^*$-algebras $C^*(E)$ with their natural $\mathbb{Z}$-gradings gives a new way of determining primitive ideal space of $C^*(E)$ for an arbitrary graph $E$ satisfying Condition (K). The latter description was originally obtained in [5] by different methods.

In general, the aforementioned dual system is not well suited for determining pure infiniteness of $C^*_r(B)$, as it gives no control on positive elements. Therefore we introduce a concept of aperiodicity for Fell bundles, which is related to the aperiodicity condition for $C^*$-correspondences introduced by Muhly and Solel in [37]. One should note that the origins of this notion go back to the work of Kishimoto [25] and Olesen and Pedersen [38] where the close relationship between this condition and properties of the Connes spectrum were revealed. More recently, similar aperiodicity conditions were investigated in the context of partial actions by Giordano and Sierakowski in [18]. The precise relationship between aperiodicity and topological freeness is not clear, however we prove that, under the additional hypothesis that the primitive ideal space of $B_e$ is Hausdorff, topological freeness of the partial dynamical system on $\text{Prim}(B_e)$ implies aperiodicity of $B$. We show that a Fell bundle associated to a graph $E$ is aperiodic if and only if $E$ satisfies Condition (L).

Exploiting ideas of Rørdam and Sierakowski [17], modulo observations made in [30], we prove that if a Fell bundle $B$ is exact, residually aperiodic, and $B_e$ has the ideal property or contains finitely many $B$-invariant ideal, then $C^*_r(B)$ has the ideal property and pure infiniteness of $C^*_r(B)$ is equivalent to proper infiniteness of every non-zero positive element in $B_e$ (treated as an element in $C^*_r(B)$). If additionally $B_e$ has real rank zero then pure infiniteness of $C^*_r(B)$ is equivalent to proper infiniteness of every non-zero projection in $B_e$. One can find many different dynamical conditions implying proper infiniteness of every non-zero positive element in $B_e$. For instance, in the context of group action this holds for strong boundary

\footnote{In the initially submitted manuscript we considered only the case when $B_e$ has the ideal property
actions [34], \( n \)-filling actions [21], and paradoxical actions [47], [18]. We note that \( n \)-filling actions generalize strong boundary actions and are necessarily minimal and paradoxical actions. However, paradoxical actions considered in [47], [18], are acting on totally disconnected spaces, while actions studied in [34], [21] do not have this restriction. The notion of a paradoxical set can be naturally generalized to the setting of Fell bundles, and we define \( B \)-paradoxical elements for an arbitrary Fell bundle \( B \). However, we found that in general \( B \)-paradoxicality is hard to be checked in practice. Therefore we also introduce a weaker notion of residually \( B \)-infinite elements in \( B_e \). We prove that if, in addition to previously mentioned assumptions on \( B \), every non-zero positive element in \( B_e \) is Cuntz equivalent to a residually infinite element, then \( C^*_e(B) \) is purely infinite. This result can be viewed as a strengthening and unification of all the aforementioned results, as we show that for \( n \)-filling actions considered in [21] every non-zero positive element in \( B_e \) is residually \( B \)-infinite for the corresponding Fell bundle. Moreover, we prove that for a graph \( C^* \)-algebra and the associated Fell bundle our conditions for pure infiniteness are not only sufficient but also necessary.

Apart from already mentioned applications to partial crossed products and graph \( C^* \)-algebras, we use the results of the present paper to study semigroup corner systems \((A, G^+, \alpha, L)\) and their crossed products. These objects are important as they lie on the intersection of various approaches to semigroup crossed products. We explain below that \((A, G^+, \alpha, L)\) can be equivalently treated as an Exel-Larsen system [35], a semigroup of endomorphisms \( \alpha = \{ \alpha_t \}_{t \in G^+} \), a semigroup of retractions (transfer operators) \( L = \{ L_t \}_{t \in G^+} \), or a group interaction \( \mathcal{V} = \{ \nu_g \}_{g \in G} \) in the spirit of [15]. The semigroup \( G^+ \) we consider is a positive cone in a totally ordered abelian group \( G \), and the maps act on an arbitrary \( C^* \)-algebra \( A \). To any corner system \((A, G^+, \alpha, L)\) we associate a Fell bundle \( B \) and define the corresponding crossed product \( A \rtimes_{\alpha, L} G^+ \) to be the cross-sectional algebra \( C^*_e(B) \). Then we identify \( A \rtimes_{\alpha, L} G^+ \) as a universal \( C^* \)-algebra with respect to certain representations. Thus we see that in the unital case, \( A \rtimes_{\alpha, L} G^+ \) coincide with the crossed product constructed, using more direct methods, in [32]. We also conclude that \((A, G^+, \alpha, L)\) coincides with Exel-Larsen crossed product introduced in [35]. We manage to formulate in a natural way the constructions and results for the Fell bundle \( B \) in terms of the systems \( \alpha, L \) and \( \mathcal{V} \). This gives us several completely new results, including description of ideal structure, the primitive ideal space of \( A \rtimes_{\alpha, L} G^+ \), and criteria for pure infiniteness of \( A \rtimes_{\alpha, L} G^+ \). In particular, in the case \( G^+ = \mathbb{N} \), our pure infiniteness criteria imply the result of Ortega and Pardo [39], cf. Remark 8.23 below.

The paper is organized as follows.

After some preliminaries, in Section 3, we discuss notions of exactness, intersection property, and topological freeness for a Fell bundle, recently introduced in the context of bundles in [1]. In particular, we give convenient characterizations of exactness (Proposition 3.7) and the intersection property (Proposition 3.15).

In Section 4 we study the concept of aperiodicity for Fell bundles. We note that aperiodicity of a Fell bundle implies the intersection property (Corollary 4.3) and indicate its relation to topological freeness (Proposition 4.5). In the main result of this section (Theorem 4.10) we give a characterization of pure infiniteness of the reduced cross-sectional algebra \( C^*_e(B) \) of an exact, residually aperiodic Fell bundle \( B \) whose unit fiber \( B_e \) has the ideal property.

Section 5 is devoted to investigation of conditions implying proper infiniteness of elements in the core \( B_e \) of the cross sectional \( C^* \)-algebra \( C^*_e(B) \). We introduce \( B \)-paradoxical elements and closely related residually \( B \)-infinite elements for a Fell bundle \( B \). We clarify the relationship between these notions and other conditions of this type studied in the literature. The main
result of this section (Theorem 5.13) contains a criterion of pure infiniteness of the reduced cross-sectional algebra of a Fell bundle $\mathcal{B}$, phrased in terms of $\mathcal{B}$-infinite elements.

We describe the primitive ideal space of $C^*_r(\mathcal{B})$ in Section 6. More specifically, the main result of this section (Theorem 6.8) identifies the primitive ideal space of the reduced cross-sectional algebra of a separable, exact Fell bundle $\mathcal{B}$ satisfying the residual intersection property both with the space of $\mathcal{B}$-primitive ideals of the trivial fiber (cf. Definition 6.3) and with the quasi-orbit space associated to the partial action dual to $\mathcal{B}$.

In Section 7, we test the results of this paper against graph $C^*$-algebras $C^*_r(E)$ equipped with their natural grading over $\mathbb{Z}$. We show that aperiodicity of the associated Fell bundle is equivalent to Condition (L) on the graph $E$. Likewise, residual aperiodicity of that bundle is equivalent to Condition (K) on the graph. We use our general results to get an alternative way of determining the primitive ideal space of $C^*_r(E)$ for $E$ satisfying Condition (K) (Corollary 7.8). Finally, we show that our criterion of pure infiniteness is optimal in the sense that in the case of graph $C^*$-algebras it is not only sufficient but also necessary (Theorem 7.9).

In Section 8, we present various equivalent points of view on a semigroup corner system $(A, G^+, \alpha, L)$. We associate to $(A, G^+, \alpha, L)$ a Fell bundle $\mathcal{B}$ (Proposition 8.0). This allows us to define the crossed product $A \rtimes_{\alpha, L} G^+$ as a cross sectional algebra of $\mathcal{B}$ (Definition 8.7). We describe $A \rtimes_{\alpha, L} G^+$ as a universal $C^*$-algebra for certain representations of $(A, G^+, \alpha, L)$ (Theorem 8.10) and we show it is isomorphic to Exel-Larsen crossed product (Corollary 8.12). The main structural results on $A \rtimes_{\alpha, L} G^+$ are criteria of faithfulness of representations of $A \rtimes_{\alpha, L} G^+$, description of ideal structure and primitive ideal space (Theorem 8.17), and the criteria for pure infiniteness of $A \rtimes_{\alpha, L} G^+$ (Theorem 8.22).

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2. Preliminaries

2.1. $C^*$-algebras, positive elements and ideals. Let $A$ be a $C^*$-algebra. By 1 we denote the unit in the multiplier $C^*$-algebra $M(A)$. All ideals in $C^*$-algebras are assumed to be closed and two-sided. All homomorphisms between $C^*$-algebras are by definition $*$-preserving. For actions $\gamma: A \times B \to C$ such as multiplications, inner products, etc., we use the notation:

$$\gamma(A, B) = \overline{\text{span}}\{\gamma(a, b) : a \in A, b \in B\}.$$  

The set of positive elements in a $C^*$-algebra $A$ is denoted by $A^+$. In [9], Cuntz introduced a preorder $\preceq$ on $A^+$, which nowadays is called Cuntz comparison, cf., for instance, [23]. Namely, for any $a, b \in A^+$ we write $a \preceq b$ whenever there exists a sequence $\{x_k\}_{k=1}^\infty$ in $A$ with $x_k^* b x_k \to a$. We say two elements $a, b \in A^+$ are Cuntz equivalent if both $a \preceq b$ and $b \preceq a$ hold. We recall, see [23] Definition 3.2], that an element $a \in A^+$ is infinite if there is $b \in A^+ \setminus \{0\}$ such that $a + b \preceq a + 0$ in the matrix algebra $M_2(A)$. An $a \in A^+ \setminus \{0\}$ is properly infinite if $a + a \preceq a + 0$. We have the following simple characterisations of these notions, cf. [23] Proposition 3.3(iv)]

We write $\approx$ to indicate that $\|a - b\| < \varepsilon$, for $a, b \in A$.

Lemma 2.1. If $a \in A^+ \setminus \{0\}$ then

$$a \text{ is infinite } \iff \exists e, a^* x \approx e, x^* y \approx e, x^* y \approx e,$$  

where $x \in A^+, y \in A, a, b \in A, x^* x \approx e a$, $y^* y \approx e b$, $x^* y \approx e 0$.
a is properly infinite ⇐⇒ ∀ε > 0 ∃x,y ∈ A x* x ≈ε a, y* y ≈ε a, x* y ≈ε 0.

Proof. We only show (2). If a is infinite, then there is b ∈ A+\{0} such that for ε > 0 there is a matrix z = \begin{pmatrix} s & t \\ * & * \end{pmatrix} such that z*(a ⊕ 0)z ≈ε a ⊕ b. The last relation implies that s* as ≈ε a, t* at ≈ε b and s* at ≈ε 0. Hence putting x := a^{1/2}s and y := a^{1/2}t we get x, y ∈ A such that x* x ≈ε a, y* y ≈ε b and x* y ≈ε 0. Now assume the condition on the right hand side of (2).

Then there is b ∈ A+\{0} and sequences \{x_n\} ⊆ aA, \{y_n\} ⊆ aA such that x_n x_n → a, y_n y_n → b and x_n y_n → 0. Since \{x_n\}, \{y_n\} ⊆ a^{1/2}A we can find \( s_n \) and \( t_n \) in A with \( \| a^{1/2} s_n - x_n \| → 0 \) and \( \| a^{1/2} t_n - y_n \| → 0 \). Then \( s_n^* a s_n → a, t_n^* a t_n → b \) and \( s_n^* a t_n → 0 \). Put \( z_n = \begin{pmatrix} s_n & t_n \\ 0 & 0 \end{pmatrix} \).

Then \( z_n^*(a ⊕ 0)z_n → a ⊕ b \), showing that \( a ⊕ b ≈ε a ⊕ 0 \). □

We will often exploit \cite{1} Proposition 3.14 which says that \( a ∈ A^+\{0} \) is properly infinite if and only if \( a + I \) in \( A/I \) is either zero or infinite for every ideal \( I \) in \( A \).

In view of \cite{1} Theorem 4.16, pure infiniteness for (not necessarily simple) \( C^* \)-algebras as introduced in \cite{1}, can be expressed as follows: a \( C^* \)-algebra \( A \) is purely infinite if and only if every \( a ∈ A^+\{0} \) is properly infinite. We say that a \( C^* \)-algebra \( A \) has the ideal property \cite{10}, \cite{42}, if every ideal in \( A \) is generated (as an ideal) by its projections. By \cite{42} Proposition 2.14, in the presence of ideal property pure infiniteness is equivalent to strong pure infiniteness \cite{21} Definition 5.1.

Let \( A \) be a \( C^* \)-algebra. We denote by \( \mathcal{I}(A) \) the set of ideals in \( A \) equipped with the Fell topology for which a sub-base of open sets is given by the sets of the form

\[
U_I := \{ J ∈ \mathcal{I}(A) : J \nsubseteq I \}, \quad I ∈ \mathcal{I}(A).
\]

If \( A \) and \( B \) are two \( C^* \)-algebras, then \( h : \mathcal{I}(A) → \mathcal{I}(B) \) is a homeomorphism if and only if it is a bijection which preserves inclusion of ideals.

We denote by \( \text{Irr}(A) \) the set of all irreducible representations of \( A \), and let \( \text{Prim}(A) := \{ \ker \pi : \pi ∈ \text{Irr}(A) \} \) be the set of primitive ideals in \( A \). Fell topology restricted to \( \text{Prim}(A) \) is the usual Jacobson topology. We have a one-to-one correspondence between closed sets in \( \text{Prim}(A) \) and \( \ker \pi \) in \( A \) given by: \( \text{hull}(I) := \{ P ∈ \text{Prim}(A) : P \supseteq I \} \) and \( I = \bigcap_{P \in \text{hull}(I)} P \), for all \( I ∈ \mathcal{I}(A) \). For any ideal \( I ∈ \mathcal{I}(A) \) we have mutually inverse maps \( P_I \mapsto P := \{ a ∈ A : aI ⊆ I \} \) and \( P → P_I := P \cap I \) that allow us to identify \( \text{Prim}(I) \) with the open set \( \text{Prim}(A) \setminus \text{hull}(I) \):

\[
(4) \quad \text{Prim}(I) = \{ P ∈ \text{Prim}(A) : P \nsubseteq I \}.
\]

The above identification extends to hereditary subalgebras. Namely, for any hereditary \( C^* \)-subalgebra \( B \) of \( A \) the map \( P → P \cap B \) allows us to assume the identification \( \text{Prim}(B) = \{ P ∈ \text{Prim}(A) : P \nsubseteq B \} \).

For any \( \pi ∈ \text{Irr}(A) \) we denote by \( [\pi] \) the unitary equivalence class of \( \pi \). Then \( [\pi] → \ker \pi \) is a well defined surjection from the spectrum \( \hat{A} := \{ [\pi] : \pi ∈ \text{Irr}(A) \} \) of \( A \) onto \( \text{Prim}(A) \), which induces Jacobson topology on \( \hat{A} \). Identification \cite{1} lifts to the following identification on the level of spectra:

\[
(5) \quad \hat{I} = \{ [\pi] ∈ \hat{A} : \ker \pi \nsubseteq I \}, \quad I ∈ \mathcal{I}(A).
\]

More generally, for any hereditary \( C^* \)-subalgebra \( B \) of \( A \) identification \( \text{Prim}(B) = \{ P ∈ \text{Prim}(A) : P \nsubseteq B \} \) lifts to the identification \( \hat{B} = \{ [\pi] ∈ \hat{A} : \ker \pi \nsubseteq B \} \).

We recall that \( I ∈ \mathcal{I}(A) \) is a prime ideal if for any pair of \( J_1, J_2 ∈ \mathcal{I}(A) \) with \( J_1 \cap J_2 ⊆ I \) either \( J_1 ⊆ I \) or \( J_2 ⊆ I \). We denote by \( \text{Prime}(A) \) the set of prime ideals in \( A \) and equip it with Fell topology. It is well known that \( \text{Prim}(A) ⊆ \text{Prime}(A) \) and if \( A \) is separable, then
actually $\text{Prim}(A) = \text{Prime}(A)$. Let us note that using the identification \([3]\) the inclusion $\text{Prim}(A) \subseteq \text{Prime}(A)$ actually means that
\[
\overline{I \cap J} = \overline{I} \cap \overline{J}, \quad \text{for all } I, J \in \mathcal{I}(A).
\]

2.2. Hilbert bimodules and induced representations. Let $A$ and $B$ be $C^*$-algebras. Following \([6, 1.8]\), by an $A$-$B$-Hilbert bimodule we mean a linear space $X$ which is both a left Hilbert $A$-module and right Hilbert $B$-module with the corresponding inner products
\[
\langle \cdot, \cdot \rangle_A : X \times X \to A \quad \text{and} \quad \langle \cdot, \cdot \rangle_B : X \times X \to B
\]
satisfying:
\[
x \langle y, z \rangle_B = \langle x, y \rangle_A z, \quad x, y, z \in X.
\]
Note that then $X$ establishes a Morita-Rieffel equivalence between the ideals $A \langle X, X \rangle \in \mathcal{I}(A)$ and $\langle X, X \rangle_B \in \mathcal{I}(B)$. We recall \([16, \text{Definition } 3.1]\), that a Morita-Rieffel (or imprimitivity) $A$-$B$-bimodule is an $A$-$B$-Hilbert bimodule $X$ such that $A \langle X, X \rangle = A$ and $\langle X, X \rangle_B = B$. For the Morita-Rieffel bimodule $X$ the formula
\[
\mathcal{I}(B) \ni I \mapsto A \langle XI, X \rangle \in \mathcal{I}(A)
\]
defines a homeomorphism $\mathcal{I}(B) \cong \mathcal{I}(A)$ called Rieffel correspondence \([15, \text{Proposition } 3.24]\). This correspondence restricts to the homeomorphism $h_X : \text{Prim}(B) \to \text{Prim}(A)$ called Rieffel homeomorphism \([15, \text{Proposition } 3.3]\). The latter has a lift to a homeomorphism $\hat{h}_X : \hat{B} \to \hat{A}$ also called Rieffel homeomorphism.

More specifically, let $X$ be an $A$-$B$-Hilbert bimodule and let $\pi : B \to \mathcal{B}(H_\pi)$ be a representation. We define a Hilbert space $X \otimes_\pi H_\pi$ to be a Hausdorff completion of the tensor product vector space $X \otimes H$ with the semi-inner-product given by
\[
\langle x_1 \otimes_\pi h_1, x_2 \otimes_\pi h_2 \rangle_C = \langle h_1, \pi(\langle x_1, x_2 \rangle A) h_2 \rangle_C.
\]
Then the formula
\[
X - \text{Ind}^A_B(\pi)(a)(x \otimes_\pi h) = (ax) \otimes_\pi h, \quad a \in A,
\]
defines a representation $X - \text{Ind}^A_B(\pi) : A \to \mathcal{B}(H_\pi)$ called induced representation. If $X$ is a Morita-Rieffel bimodule then the formula
\[
\hat{h}_X(\pi) = [X - \text{Ind}^A_B(\pi)], \quad \pi \in \text{Irr}(B),
\]
defines the Rieffel homeomorphism $\hat{h}_X : \hat{B} \to \hat{A}$, see \([15, \text{Corollaries } 3.32, 3.33]\). In particular, we have $h_X(\ker \pi) = \ker (X - \text{Ind}^A_B(\pi))$ for any $\pi \in \text{Irr}(B)$.

Let $X$ be an $A$-$A$-Hilbert bimodule. In this case, we will also call $X$ a Hilbert bimodule over $A$. An ideal $I \in \mathcal{I}(A)$ is said to be $X$-invariant if $IX = XI$. For an $X$-invariant ideal the quotient space $X/XI$ is naturally a Hilbert bimodule over $A/I$. In the sequel we will need the following simple fact, which is probably well-known to experts, but we lack a good reference.

**Lemma 2.2.** Let $X$ be a Hilbert bimodule over a $C^*$-algebra $A$ and let $I$ be an $X$-invariant ideal in $A$. Then for any representation $\pi : A/I \to \mathcal{B}(H_\pi)$ we have the following unitary equivalence of representations of $A$:
\[
((X/XI) - \text{Ind} \pi) \circ q \cong X - \text{Ind}(\pi \circ q)
\]
where $q : A \to A/I$ is the quotient map.
Proof. Note that for any $x_i \in X$ and $h_i \in H_\pi$, $i = 1, \ldots, n$, we have
\[
\left\| \sum_{i=1}^{n} (x_i + XI) \otimes_{\pi} h_i \right\| = \sum_{i,j=1}^{n} \langle h_i, \pi(\langle x_i + XI, x_j + XI \rangle_{A/A}) h_j \rangle = \sum_{i,j=1}^{n} \langle h_i, \pi(q(\langle x_i, x_j \rangle_A)) h_j \rangle = \sum_{i=1}^{n} x_i \otimes_{\pi q} h_i^2.
\]
Accordingly, the mapping $(x + XI) \otimes_{\pi} h \mapsto x \otimes_{\pi q} h$, $x \in X$, $h \in H_\pi$, extends by linearity and continuity to a unitary operator $V : X/XI \otimes_{\pi} H_\pi \to X \otimes_{\pi q} H_\pi$. Unitary $V$ intertwines $((X/XI) - \text{Ind } \pi) \circ q$ and $X - \text{Ind}(\pi \circ q)$ because for any $a \in A$, $x \in X$, $h \in H_\pi$ we have
\[
V((X/XI) - \text{Ind } \pi)(q(a))(x + XI) \otimes_{\pi} h = V(ax + XI) \otimes_{\pi} h = ax \otimes_{\pi q} h = X - \text{Ind}(\pi \circ q)(a)V(x + XI) \otimes_{\pi} h.
\]

2.3. Partial actions. We recall that a partial action of a discrete group $G$ on a C*-algebra $A$ is a pair $\alpha = (\{D_t\}_{t \in G}, \{\alpha_t\}_{t \in G})$, where for each $t \in G$, $\alpha_t : D_{t^{-1}} \to D_t$ is an isomorphism between ideals of $A$ such that
\[
\alpha_e = id_A \quad \text{and} \quad \alpha_{st} \text{ extends } \alpha_s \circ \alpha_t \text{ for } s, t \in G.
\]
The second property above is equivalent to the following relations: $\alpha_t(D_{t^{-1}} \cap D_s) \subseteq D_{ts}$ and $\alpha_s(\alpha_t(a)) = \alpha_{st}(a)$ for $a \in D_{t^{-1}} \cap D_{t^{-1}s^{-1}}$, $s, t \in G$. The triple $(A, G, \alpha)$ is called partial C*-dynamical system. There are two C*-algebras naturally associated to such systems: the full crossed product $A \rtimes_{\alpha} G$ and the reduced crossed product $A \rtimes_{a,r} G$ (they can be defined in terms of C*-algebras associated to Fell bundles, see Example 2.5 below). When $D_t = A$ for every $t \in G$, we talk about global actions and global C*-dynamical systems.

Any partial action $\alpha$ on a commutative C*-algebra $A = C_0(\Omega)$, where $\Omega$ is a locally compact Hausdorff space, is given by
\[
\alpha_t(f)(x) := f(\theta_{t^{-1}}(x)), \quad f \in C_0(\Omega_{t^{-1}}),
\]
where $D_t = C_0(\Omega_t)$, $t \in G$, and $\{\Omega_t\}_{t \in G}$, $\{\theta_t\}_{t \in G}$ is a partial action of $G$ on $\Omega$. In general, a partial action of $G$ on a topological space $\Omega$ is a pair $\theta = (\Omega_t)_{t \in G}$, $\{\theta_t\}_{t \in G}$ where $\Omega_t$’s are open subsets of $\Omega$ and $\theta_t : \Omega_{t^{-1}} \to \Omega_t$ are homeomorphisms such that
\[
\theta_e = id_\Omega \quad \text{and} \quad \theta_{st} \text{ extends } \theta_s \circ \theta_t \text{ for } s, t \in G.
\]
The triple $(\Omega, G, \theta)$ is called partial (topological) dynamical system. In case every $\Omega_t = \Omega$, we say $\theta$ is a global action.

For global actions on commutative C*-algebras, it is a part of C*-folklore, for the extended discussion see, for instance, [1] or [27], that simplicity of the associated reduced partial crossed products is equivalent to minimality and topological freeness of the dual action. This result was adapted to partial actions in [17]. Let us recall the relevant definitions:

Let $\theta = (\{\Omega_t\}_{t \in G}, \{\theta_t\}_{t \in G})$ be a partial action on a (not necessarily Hausdorff) topological space $\Omega$. A subset $V$ of $\Omega$ is $\theta$-invariant if $\theta_t(V \cap \Omega_{t^{-1}}) \subseteq V$ for every $t \in G$ (then we actually have $\theta_t(V \cap \Omega_{t^{-1}}) = V \cap \Omega_t$, for all $t \in G$, and thus $\Omega_t V$ is also $\theta$-invariant). The restriction $\theta_V := (\{\Omega_t \cap V\}_{t \in G}, \{\theta_t|_{\Omega_{t^{-1}} \cap V}\}_{t \in G})$ of $\theta$ to a $\theta$-invariant set $V \subseteq \Omega$ is a again a partial dynamical system, and $\theta$ is called minimal if there are no non-trivial closed $\theta$-invariant subsets of $\Omega$. The partial action $\theta$ is topologically free if for finite set $F \subseteq G \setminus \{e\}$ the set...
\[ \bigcup_{t \in F} \{ x \in \Omega : t(x) = x \} \text{ has empty interior in } \Omega. \] We say, cf. [18, Page 230], [18, Definition 3.4], that \( \theta \) is **residually topologically free** if the restriction of \( \theta \) to any closed \( \theta \)-invariant set is topologically free.

Dynamical conditions implying pure infiniteness of reduced crossed products for global actions on totally disconnected spaces were introduced in [47], and adapted to the case of partial actions in [18]. A crucial notion is that of paradoxical set, cf. [47, Definition 4.2], [18, Definition 4.3]

**Definition 2.3.** Let \( (\{ \Omega_t \}_{t \in G}, \{ \theta_t \}_{t \in G}) \) be a partial action on a topological space \( \Omega \). A non-empty open set \( V \subseteq \Omega \) is called **\( G \)-paradoxical** if there are open sets \( V_1, ..., V_{n+m} \) and elements \( t_1, ..., t_{n+m} \in G \), such that

\begin{enumerate}
  \item \( V = \bigcup_{i=1}^{n+m} V_i \),
  \item \( V_i \subseteq \Omega_{t_i^{-1}} \) and \( \theta_{t_i}(V_i) \subseteq V \) for all \( i = 1, ..., n+m \),
  \item \( \theta_{t_i}(V_i) \cap \theta_{t_j}(V_j) = \emptyset \) for all \( i \neq j \).
\end{enumerate}

The notion of quasi-orbit space adapted to partial actions, cf. [18, Page 5740], is defined as follows:

**Definition 2.4.** Let \( (\{ \Omega_g \}_{g \in G}, \{ \theta_g \}_{g \in G}) \) be a partial action of a group \( G \) on a topological space \( \Omega \). We let the **orbit** of a point \( x \in \Omega \) to be the set

\[ Gx := \bigcup_{t \in G} \{ \theta_t(x) \}. \]

We define the **quasi-orbit** \( \mathcal{O}(x) \) of \( x \) to be the equivalence class of \( x \) under the equivalence relation on \( \Omega \) given by

\[ x \sim y \iff Gx = Gy. \]

We denote by \( \mathcal{O}(\Omega) \) the quasi-orbit space \( \Omega/ \sim \) endowed with the quotient topology.

2.4. **Fell bundles and graded \( \text{C}^* \)-algebras.** Let \( G \) be a discrete group. A **Fell bundle** over \( G \) can be defined as a collection \( \mathcal{B} = \{ B_t \}_{t \in G} \) of closed subspaces of a \( \text{C}^* \)-algebra \( C \) such that \( B_t = B_t^{-1} \) and \( B_sB_t \subseteq B_{st} \) for all \( s, t \in G \) (see [16, Definition 16.1] for the axiomatic description). These relations in particular imply that

\[ B_tB_{t^{-1}}B_t = B_t, \quad t \in G, \]

and \( B_tB_{t^{-1}} \) is an ideal in the **core** \( \text{C}^* \)-algebra \( B_e \), cf. [16, Lemma 16.12]. Moreover, for any \( t \in G \), \( B_t \) is naturally a Hilbert bimodule over \( B_e \) with right and left inner products given by

\[ \langle x, y \rangle_{B_t} := x^*y \text{ and } B_t \langle x, y \rangle := xy^*, \quad x, y \in B_t. \]

If \( \mathcal{B} = \{ B_t \}_{t \in G} \) is a Fell bundle, then the direct sum \( \bigoplus_{t \in G} B_t \) is naturally equipped with the structure of a \( * \)-algebra which admits a \( \text{C}^* \)-norm. In general, there are many \( \text{C}^* \)-norms on \( \bigoplus_{t \in G} B_t \). There is always a maximal such norm, and it satisfies the inequality

\[ \| a_e \| \leq \sum_{t \in G} \| a_t \|, \quad \text{for all } \sum_{t \in G} a_t \in \bigoplus_{t \in G} B_t, \quad a_t \in B_t, \quad t \in G, \]

see [43, Lemma 1.3], [11, Proposition 2.9], or [16, Lemma 17.8]. The completion of \( \bigoplus_{t \in G} B_t \) in the maximal \( \text{C}^* \)-norm is denoted by \( \text{C}^*(\mathcal{B}) \). It is called **cross sectional algebra** of \( \mathcal{B} \). It follows from [11, Theorem 3.3] that there is also a minimal \( \text{C}^* \)-norm on \( \bigoplus_{t \in G} B_t \) satisfying (8) and a completion of \( \bigoplus_{t \in G} B_t \) in this minimal \( \text{C}^* \)-norm is naturally isomorphic to the reduced cross sectional algebra \( C_r^*(\mathcal{B}) \), as introduced in [11, Definition 2.3] or in [43, Definition 3.5] (both definitions are known to be equivalent). A Fell bundle \( \mathcal{B} = \{ B_t \}_{t \in G} \) is said to be **amenable**
Definition 20.1] if the algebras $C^*_p(\mathcal{B})$ and $C^*(\mathcal{B})$ coincide; in other words, if there exists a unique $C^*$-norm on $\bigoplus_{t \in G} B_t$ satisfying (8). It is known that if the group $G$ is amenable (or more generally if $\mathcal{B} = \{B_t\}_{t \in G}$ has the approximation property, see [16, Definition 20.4]) then the Fell bundle $\mathcal{B} = \{B_t\}_{t \in G}$ is automatically amenable, see [16, Theorem 20.7].

Let $\mathcal{B} = \{B_t\}_{t \in G}$ be a Fell bundle. Any $C^*$-algebra $B$ which is a closure of $\bigoplus_{t \in G} B_t$ is called $B$-graded (or simply graded). If additionally the norm in $B$ satisfies (8), then $B$ is called topologically graded [16, Definition 19.2], [11, Definition 3.4]. For any topologically $B$-graded $C^*$-algebra $B$ the canonical projections

$$F_t : \bigoplus_{s \in G} B_s \to B_t, \quad t \in G,$$

extend to contractive linear maps on $B$, cf. [16, Corollary 19.6]. They are called Fourier coefficient operators in [16, page 197]. In particular, $F_e : B \to B_e$ is a conditional expectation onto the core $C^*_r(\mathcal{B})$-algebra $B_e$. This conditional expectation is faithful if and only if $B = C^*_r(\mathcal{B})$, see [11, Proposition 2.12].

Perhaps the most significant example of a Fell bundle is the one coming from partial actions. In particular, every separable Fell bundle whose unit fiber is stable arises in this way, see [16, Theorem 27.11].

Example 2.5 (Fell bundle associated to a partial action). The Fell bundle $\mathcal{B}_\alpha = \{B_t\}_{t \in G}$ associated to a partial action $\alpha = \{(D_t)_{t \in G}, \{\alpha_t\}_{t \in G}\}$ on a $C^*$-algebra $A$ is defined as follows: $B_t := \{a_t \delta_t : a_t \in D_t\}$ is isomorphic as a Banach space to $D_t$ ($\delta_t$ is just an abstract marker), and multiplication and star operation are given by

$$(a_t \delta_t)(a_s \delta_s) = \alpha_t(\alpha_{t^{-1}}(a_t) a_s) \delta_{ts}, \quad (a_t \delta_t)^* = \alpha_{t^{-1}}(a_t^*) \delta_{t^{-1}}.$$

In particular, cf. [16, Proposition 16.28] the full crossed product and the reduced crossed product can be defined as follows

$$A \rtimes_\alpha G := C^*(\mathcal{B}), \quad A \rtimes_{\alpha,r} G := C^*_r(\mathcal{B}).$$

In the sequel, we will identify $B_e$ with $D_e = A$, so we will write $a$ for $a \delta_e$.

2.5. Ideals in Fell bundles and graded $C^*$-algebras. An ideal in a Fell bundle $\mathcal{B} = \{B_t\}_{t \in G}$ is a collection $\mathcal{J} = \{J_t\}_{t \in G}$, consisting of closed subspaces $J_t \subseteq B_t$, such that $B_s J_t \subseteq J_{st}$ and $J_s B_t \subseteq J_{st}$, for all $s, t \in G$, see [12, Definition 2.1]. Then it follows, see [12], that $\mathcal{J}$ is self-adjoint in the sense that $(J_t)^* = J_{t^{-1}}$, so in particular $\mathcal{J}$ is a Fell bundle in its own right (thus our definition agrees with [16, Definition 21.10]). Moreover, the family $\mathcal{B}/\mathcal{J} := \{B_t/J_t\}_{t \in G}$ is equipped with a natural Fell bundle structure and as such is called quotient Fell bundle, cf. [16, Definition 21.14]. In view of [12, Proposition 2.2], see also [16, Proposition 21.15], we have the following natural exact sequence

$$0 \to C^*(\mathcal{J}) \xrightarrow{\iota_r} C^*(\mathcal{B}) \xrightarrow{\kappa_r} C^*(\mathcal{B}/\mathcal{J}) \to 0,$$

which by [12, Lemma 4.2] induces the following (not necessarily exact!) sequence

$$0 \to C^*_r(\mathcal{J}) \xrightarrow{\iota_r} C^*_r(\mathcal{B}) \xrightarrow{\kappa_r} C^*_r(\mathcal{B}/\mathcal{J}) \to 0,$$

where $\iota_r$ is injective and $\kappa_r$ surjective, but in general $\iota_r(C^*_r(\mathcal{J})) \nsubseteq \ker \kappa_r$.

Ideals in Fell bundles and graded algebras are related to each other in the following way. If $J$ is an ideal in a graded $C^*$-algebra $B = \bigoplus_{t \in G} B_t$, then it is easy to see that $\mathcal{J} := \{J \cap B_t\}_{t \in G}$
is an ideal in the Fell bundle $B = \{B_t\}_{t \in G}$. Moreover, by \cite{16} Proposition 23.1] we have the equivalence
\begin{equation}
J \text{ is generated as an ideal by } J \cap B_e \iff J = \bigoplus_{t \in G} J \cap B_t.
\end{equation}
The ideals in $B = \bigoplus_{t \in G} B_t$ satisfying equivalent conditions in (11), are called induced \cite{11} Definition 3.10] or graded \cite{11} Definition 23.2]. In the present general context we prefer the second name.

In topologically graded algebras there is another important class of ideals. Recall that $B = \bigoplus_{t \in G} B_t$ is topologically graded if and only there are Fourier coefficient operators $F_t : B \to B_t \subseteq B$, $t \in G$. In this case an ideal $J$ in $B$ is called Fourier if
$$F_t(J) \subseteq J, \quad \text{for all } t \in G,$$
see \cite{16} Definition 23.8]. The following fundamental relationship between the general, graded and Fourier ideals in topologically graded $C^\ast$-algebras was already established in \cite{11} Theorem 3.9], see also \cite{16} Proposition 23.4].

**Proposition 2.6.** If $J$ is an ideal in a topologically graded $C^\ast$-algebra $B$, then
$$\{b \in B : F_e(b^*b) \in J\} = \{b \in B : F_t(b) \in J, \ t \in G\}$$
and this set is a Fourier ideal in $B$ that contains the graded ideal generated by $J \cap B_e$. In particular, if $J$ is graded, then it is Fourier.

For the sake of discussion let us denote the Fourier and the graded ideal in the above proposition respectively by $J_F$ and $J_G$. Then we have two inclusions $J_G \subseteq J$ and $J_F \subseteq J_G$, and in general this is all we can say. More precisely, $J_G \subseteq J_F$ if and only if $J_F$ is a Fourier ideal which is not graded. There is always such an ideal if $B \neq C^\ast_f(B)$ (consider the kernel of the canonical epimorphism from $B = \bigoplus_{t \in G} B_t$ onto $C^\ast_f(B)$), and even if $B = C^\ast_f(B)$ one can construct such an ideal when the underlying group $G$ is not exact, see \cite{16}, page 199]. On the other hand, considering the Fell bundle arising from the $C^\ast$-dynamical system $(\mathbb{C}, id, \mathbb{Z})$ for any non-trivial ideal $J$ in $\mathbb{C} \rtimes_{id} \mathbb{Z} \cong C(\mathbb{T})$ we get $J \cap B_e = \{0\}$, and therefore $J \neq J_F = J_G = \{0\}$. This indicates that the equality $J_G = J_F$ is related with a notion of ‘exactness’ while inclusion $J \subseteq J_F$ has to do with an ‘intersection property’. We will make these notions precise and study them in more detail in the forthcoming section.

3. **Exactness, the Intersection Property and Topological Freeness**

In this section, we exploit notions of exactness, the intersection property and topological freeness for a Fell bundle $\mathcal{B}$, introduced recently in \cite{1}. As shown in \cite{1}, these properties allow one to parametrize ideals in $C^\ast_f(\mathcal{B})$ by ideals in the core $C^\ast$-algebra $B_e$. The relevant ideals in $B_e$ are defined as follows.

**Definition 3.1** (Definition 3.5 in \cite{1}). Let $B = \{B_t\}_{t \in G}$ be a Fell bundle. We say that an ideal $I$ in $B_e$ is $B$-invariant if $B_tIB_{t^{-1}} \subseteq I$ for every $t \in G$. We denote the set of all $B$-invariant ideals in $B_e$ by $\mathcal{I}^B(B_e)$ and equip it with the Fell topology inherited from $\mathcal{I}(B_e)$.

The relationship between various types of ideals is explained in the following:

**Proposition 3.2.** Let $B = \bigoplus_{t \in G} B_t$ be a graded $C^\ast$-algebra. Relations
$$J = \bigoplus_{t \in G} J_t, \quad J_t = J \cap B_t = B_tI = IB_t, \quad t \in G,$$
establish natural bijective correspondences between the following objects:
In view of \[12\), Proposition 2.2\) a discrete group

Remark 3.5.

Corollary 3.6. If a Fell bundle over \(G\),

Definition 3.4

\[3.1(ii)\]]

notion of exactness for group (partial) actions introduced in \[48\), Definition 1.5] , \[18\), Definition 1.5], \[18\) Definition 3.1(ii)]

Hence the sequence (10) is exact for every ideal \(J\) where \(J\) is graded. In particular, (13) implies that

\(J\) is Fourier, and since \(J_t = J_t J_t^* J_t \subseteq J_t J_t\), for all \(t \in G\), one concludes that \(J^{(1)}\) is an induced ideal. In particular, (13) implies that \(J_t = J^{(i)} \cap B_t = F_t(J^{(i)})\) for all \(t \in G\), \(i = 1, 2\).
Suppose now \( J \) is a Fourier ideal in \( C^*_r(\mathcal{B}) \). For each \( t \in G \), let \( J_t = J \cap B_t \) so that \( \mathcal{J} = \{ J_t \}_{t \in G} \) is an ideal in \( \mathcal{B} \). Consider the ideals \( J^{(1)} \) and \( J^{(2)} \) associated to \( \mathcal{J} = \{ J_t \}_{t \in G} \) as above. By the Fourier property of \( J \), we conclude from (13) that
\[
J^{(1)} \subseteq J \subseteq J^{(2)}.
\]
Thus if (10) is exact then \( J = J^{(1)} = J^{(2)} \) is a graded ideal. If \( J \) is not graded then \( J_1 \neq J \) and (10) is not exact.

Standard arguments show that exactness passes to ideals and quotients.

**Lemma 3.8.** Let \( \mathcal{J} \) be an ideal in an exact Fell bundle \( \mathcal{B} \). Then both \( \mathcal{J} \) and \( \mathcal{B}/\mathcal{J} \) are exact.

**Proof.** If \( \mathcal{I} \) is an ideal in \( \mathcal{J} \), then treating it as an ideal in \( \mathcal{B} \) we get the exact sequence
\[
0 \longrightarrow C^*_r(\mathcal{I}) \xrightarrow{i} C^*_r(\mathcal{B}) \xrightarrow{r} C^*_r(\mathcal{B}/\mathcal{I}) \longrightarrow 0
\]
which restricts to the exact sequence
\[
0 \longrightarrow C^*_r(\mathcal{I}) \xrightarrow{i} C^*_r(\mathcal{J}) \xrightarrow{r} C^*_r(\mathcal{J}/\mathcal{I}) \longrightarrow 0.
\]
Hence \( \mathcal{J} \) is exact. Now suppose that \( \tilde{\mathcal{I}} \) is an ideal in \( \mathcal{B}/\mathcal{J} \) and let \( \mathcal{I} \) be the preimage of \( \tilde{\mathcal{I}} \) under the quotient map. Then \( \mathcal{I} \) is an ideal in \( \mathcal{B} \), containing \( \mathcal{J} \), and we have a natural isomorphism \( (\mathcal{B}/\mathcal{J})/\mathcal{I} \cong \mathcal{B}/\mathcal{I} \). In particular, we have a commutative diagram
\[
\begin{array}{ccc}
0 & \longrightarrow & C^*_r(\mathcal{I}) \xrightarrow{i} C^*_r(\mathcal{B}) \xrightarrow{r} C^*_r(\mathcal{B}/\mathcal{I}) \longrightarrow 0, \\
\downarrow & & \downarrow \cong \\
0 & \longrightarrow & C^*_r(\tilde{\mathcal{I}}) \xrightarrow{i} C^*_r(\mathcal{B}/\mathcal{J}) \xrightarrow{r} C^*_r((\mathcal{B}/\mathcal{J})/\mathcal{I}) \longrightarrow 0
\end{array}
\]
where the upper row is exact and the two left most vertical maps are surjective (they are induced by the quotient map \( \mathcal{B} \rightarrow \mathcal{B}/\mathcal{J} \)). Using this, one readily gets that the lower row is also exact. Hence \( \mathcal{B}/\mathcal{J} \) is exact.

As one would expect, amenability of a Fell bundle implies exactness.

**Lemma 3.9.** Let \( \mathcal{J} \) be an ideal in a Fell bundle \( \mathcal{B} \). Then
\[
\mathcal{B} \text{ is amenable} \iff \mathcal{J} \text{ and } \mathcal{B}/\mathcal{J} \text{ are amenable.}
\]

**Proof.** Denote by \( \Lambda_{\mathcal{B}} : C^*(\mathcal{B}) \rightarrow C^*(\mathcal{B}) \) the canonical epimorphism. By [11, Proposition 3.1] we have \( \ker(\Lambda_{\mathcal{B}}) = \{ a \in C^*(\mathcal{B}) : E_{\mathcal{B}}(a^*a) = 0 \} \) where \( E_{\mathcal{B}} : C^*(\mathcal{B}) \rightarrow B_v \) is the canonical conditional expectation. Note that, for any Fell bundles \( \mathcal{B} \) and \( \mathcal{B} \) and any homomorphism \( \Phi : C^*(\mathcal{B}) \rightarrow C^*(\mathcal{B}') \) that preserves the gradings we have \( \Phi(E_{\mathcal{B}}(a^*a)) = E_{\mathcal{B}'}(\Phi(a)^*\Phi(a)) \), \( a \in C^*(\mathcal{B}) \), and therefore \( \Phi(\ker(\Lambda_{\mathcal{B}})) \subseteq \ker(\Lambda_{\mathcal{B}'}) \). Thus the exact sequence (9) restricts to the sequence
\[
(14) \quad 0 \longrightarrow \ker(\Lambda_{\mathcal{J}}) \xrightarrow{i} \ker(\Lambda_{\mathcal{B}}) \xrightarrow{\kappa} \ker(\Lambda_{\mathcal{B}/\mathcal{J}}) \longrightarrow 0.
\]
It is not hard to see that (14) is also exact. Indeed, \( i \) is injective and clearly we have \( i(\ker(\Lambda_{\mathcal{J}})) = \ker(\Lambda_{\mathcal{B}}) \cap i(\ker(\mathcal{J})) \). Hence the isomorphism \( C^*(\mathcal{B})/i(\ker(\mathcal{J})) \cong C^*(\mathcal{B}/\mathcal{J}) \), induced by the epimorphism \( \kappa : C^*(\mathcal{B}) \rightarrow C^*(\mathcal{B}/\mathcal{J}) \), ‘restricts’ to the isomorphism \( \ker(\Lambda_{\mathcal{B}})/i(\ker(\Lambda_{\mathcal{J}})) \cong \ker(\Lambda_{\mathcal{B}/\mathcal{J}}) \). Thus (14) is exact.

Now, since \( \mathcal{B} \) is amenable if and only if \( \ker(\Lambda_{\mathcal{B}}) = \{ 0 \} \), the assertion follows from exactness of the sequence (14).

**Corollary 3.10.** Every amenable Fell bundle is exact.
Proof. In view of Lemma 3.9 if \( \mathcal{B} \) is amenable, the sequences (9) and (10) coincide. \( \square \)

The notion of (residual) intersection property, in the context of crossed products was introduced in [18] Definition 1.9, see also [18] Definition 3.1(iii)].

**Definition 3.11** (Definition 3.14 in [1]). We say that a Fell bundle \( \mathcal{B} = \{B_t\}_{t \in G} \) has the intersection property if every non-zero ideal in \( C_r^* (\mathcal{B}) \) has a non-zero intersection with \( B_e. \) We say that a Fell bundle \( \mathcal{B} = \{B_t\}_{t \in G} \) has the residual intersection property if \( \mathcal{B}/J \) has the intersection property for every ideal \( J \) in \( \mathcal{B}. \)

The following theorem is essence reformulation of [1, Theorem 3.19]. It is a generalization of [18, Theorem 3.2], cf. also [48, Theorem 1.13].

**Theorem 3.12.** Let \( \mathcal{B} = \{B_t\}_{t \in G} \) be a Fell bundle. The following statements are equivalent:

(i) The map (12) establishes a homeomorphism \( \mathcal{I}(C_r^*(\mathcal{B})) \cong \mathcal{I}(B_e). \)

(ii) All ideals in \( C_r^*(\mathcal{B}) \) are graded.

(iii) \( \mathcal{B} \) is exact and has the residual intersection property.

**Proof.** Equivalence (i)\( \iff \) (ii) is clear. Equivalence (i)\( \iff \) (iii) follows from [1] Theorem 3.19 (note that for every ideal \( J \in \mathcal{I}(B_e) \) the \( C^*\)-algebra \( C_r^*(J) \) embeds into \( C_r^*(\mathcal{B}) \) as a graded ideal). \( \square \)

**Corollary 3.13.** Let \( \mathcal{B} = \{B_t\}_{t \in G} \) be a Fell bundle with the intersection property. Then \( C_r^*(\mathcal{B}) \) is simple if and only if there are no non-trivial \( \mathcal{B} \)-invariant ideals in \( B_e. \)

**Proof.** The ‘only if’ part is clear. For the ‘if’ part note that if there are no non-trivial \( \mathcal{B} \)-invariant ideals in \( B_e, \) then \( \mathcal{B} \) is trivially an exact Fell bundle. Thus it suffices to apply Theorem 3.12. \( \square \)

**Corollary 3.14.** Let \( \mathcal{B} = \{B_t\}_{t \in G} \) be an exact Fell bundle with the residual intersection property. If \( B_e \) has the ideal property then \( C_r^*(\mathcal{B}) \) has the ideal property.

**Proof.** By Theorem 3.12 every ideal \( J \in \mathcal{I}(C_r^*(\mathcal{B})) \) is generated by \( I = B_e \cap J. \) Denoting by \( P(J) \) and \( P(I) \) respectively the sets of projections in \( J \) and \( I, \) we see that the ideal generated by \( P(J) \) in \( C_r^*(\mathcal{B}) \) contains

\[
C_r^*(\mathcal{B}) P(I) C_r^*(\mathcal{B}) = C_r^*(\mathcal{B}) B_e P(I) B_e C_r^*(\mathcal{B}) = C_r^*(\mathcal{B}) I C_r^*(\mathcal{B}) = J.
\]

Hence \( C_r^*(\mathcal{B}) P(J) C_r^*(\mathcal{B}) = J, \) which shows the ideal property for \( C_r^*(\mathcal{B}). \) \( \square \)

We have the following characterization of the intersection property in terms of graded \( C^\ast \)-algebras.

**Proposition 3.15.** Let \( \mathcal{B} = \{B_t\}_{t \in G} \) be a Fell bundle. The following conditions are equivalent:

(i) \( \mathcal{B} \) has the intersection property,

(ii) every graded \( C^\ast \)-algebra \( B = \bigoplus_{t \in G} B_t \) is automatically topologically graded,

(iii) for every graded \( B = \bigoplus_{t \in G} B_t \) and any positive element \( b = \bigoplus_{t \in G} b_t \in \bigoplus_{t \in G} B_t \) we have

\[
\|b_e\| \leq \|b\|. \]

**Proof.** (i)\( \Rightarrow \) (ii) Suppose \( \mathcal{B} = \{B_t\}_{t \in G} \) has the intersection property and \( B = \bigoplus_{t \in G} B_t \) is a graded \( C^\ast \)-algebra. We have two surjective homomorphisms \( \Psi : C^\ast(\mathcal{B}) \to B \) and \( \Lambda_B : C^\ast(\mathcal{B}) \to C_r^*(\mathcal{B}) \) which are identities on \( \bigoplus_{t \in G} B_t. \) The image of \( J := \ker \Psi \) under \( \Lambda_B \) is an ideal in \( C_r^*(\mathcal{B}) \) whose intersection with \( B_e \) is zero. Thus (by the intersection property)
\[ J \subseteq \ker \Lambda_B. \] Therefore \( \Lambda_B \) factors through to the epimorphism \( \Phi \) from \( B \), identified with \( C^*(B)/J \), onto \( C^*_r(B) \) which is injective on \( \bigoplus_{t \in G} B_t \). Since \( C^*_r(B) \) is topologically graded it follows that \( B = \bigoplus_{t \in G} B_t \) is topologically graded. Indeed, if \( b = \bigoplus_{t \in G} b_t \) is in \( \bigoplus_{t \in G} B_t \subseteq B \), then \( \|b_e\| = \|\Phi(b_e)\| \leq \|\Phi(b)\| \leq \|b\| \).

(ii) \( \Rightarrow \) (i) is trivial. To show (iii) \( \Rightarrow \) (i), assume on the contrary that there is a non-zero ideal \( J \) in \( C^*_r(B) \) such that \( J \cap B_e = \{0\} \). Then \( J \) has trivial intersection with all the spaces \( B_t, t \in G \). Hence \( B := C^*_r(B)/J \) is graded by the Fell bundle \( \{B_t\}_{t \in G} \) and the quotient map \( q : C^*_r(B) \rightarrow B = C^*_r(B)/J \) is injective on \( \bigoplus_{t \in G} B_t \). Since the conditional expectation \( E : C^*_r(B) \rightarrow B_e \) is faithful and \( J \neq \{0\} \), there is a positive element \( a \in J \) with \( \|E(a)\| = 1 \). As \( \bigoplus_{t \in G} B_t \) is a dense \(*\)-algebra in \( C^*_r(B) \), we may find a positive element \( b = \bigoplus_{t \in G} b_t \) in \( \bigoplus_{t \in G} B_t \) such that \( \|a - b\|_{C^*_r(B)} < 1/3 \). Then \( \|b_e - E(a)\|_{C^*_r(B)} = \|E(b - a)\|_{C^*_r(B)} < 1/3 \), which implies \( \|b_e\| > 2/3 \), and \( \|q(b)\|_B = \|q(b - a)\|_B < 1/3 \). Thus if we assume (iii), we get

\[ \frac{2}{3} < \|b_e\| = \|q(b_e)\| \leq \|q(b)\|_B < 1/3, \]

a contradiction. \( \square \)

A useful condition that implies the intersection property is topological freeness of a dual system. A dynamical system dual to a saturated Fell bundle was considered in [33, Corollary 6.5] (it is a special case of a dual semigroup associated to a product system in [33, Definition 4.8]). A partial dynamical system dual to a semi-saturated Fell bundle over \( \mathbb{Z} \) was studied in [27]. A partial dynamical system dual to an arbitrary Fell bundle over a discrete group was defined in [1, Definition 2.3]. Let us now recall the relevant constructions and facts.

Let \( B = \{B_t\}_{t \in G} \) be a Fell bundle over a discrete group \( G \), and put

\[ D_t := B_t B_{t^{-1}} \quad \text{for all } t \in G. \]  

Then \( D_t \) is an ideal in \( B_e \) and we may treat \( B_t \) as a Morita-Rieffel \( D_t \)-\( D_{t^{-1}} \)-bimodule. Thus we get a partial homeomorphism \( \hat{h}_t : \hat{D}_{t^{-1}} \rightarrow \hat{D}_t \) of \( \hat{B}_e \), where

\[ \hat{h}_t := [B_t - \text{Ind}^{D_{t^{-1}}}_{D_t}] \quad \text{for all } t \in G. \]

Recall that \( \hat{h}_t \) is a lift of a partial homeomorphism \( h_t : \text{Prim}(D_{t^{-1}}) \rightarrow \text{Prim}(D_t) \) of \( \text{Prim}(B_e) \), and the latter is a restriction of the Rieffel homeomorphism \( h_t : \mathcal{I}(D_{t^{-1}}) \rightarrow \mathcal{I}(D_t) \), which in this case is given by the formula

\[ \mathcal{I}(D_{t^{-1}}) \ni I \mapsto h_t(I) = B_t B_{t^{-1}} \in \mathcal{I}(D_t), \]

cf. also [27, Remark 2.3].

**Proposition 3.16.** Formulas (15) and (16) define a partial action \( \{(\hat{D}_t)_{t \in G}, \{(\hat{h}_t)_{t \in G}\} \) of \( G \) on \( \hat{B}_e \). It is a lift of a partial action \( \{\text{Prim}(D_t)\}_{t \in G}, \{h_t\}_{t \in G}\) of \( G \) on \( \text{Prim}(B_e) \).

**Proof.** The first part follows from [1, Proposition 2.2]. Now, since the maps \( \hat{h}_t : \hat{D}_{t^{-1}} \rightarrow \hat{D}_t \) and \( h_t : \text{Prim}(D_{t^{-1}}) \rightarrow \text{Prim}(D_t) \) are intertwined by the surjection \( \hat{B}_e \ni [\pi] \mapsto \ker \pi \in \text{Prim}(B_e) \), one readily concludes that \( \{(\text{Prim}(D_t))_{t \in G}, \{(h_t)_{t \in G}\}\} \) is also a partial action. \( \square \)

**Remark 3.17.** If \( B \) is the Fell bundle of a partial action \( \alpha = \{(D_t)_{t \in G}, \{\alpha_t \}_{t \in G}\} \) on a \( C^* \)-algebra \( A \) then the ideals \( D_t, t \in G \), coincide with those given by (15). Modifying slightly the proof of [33, Lemma 6.7] or [26, Proposition 2.18], see also the proof of Lemma 8.13 below, one sees that the dual partial dynamical system \( \{(\hat{D}_t)_{t \in G}, \{(\hat{h}_t)_{t \in G}\} \) is given by the formula

\[ \hat{h}_t([\pi]) = [\pi \circ \alpha_{t^{-1}}], \quad [\pi] \in \hat{D}_{t^{-1}}, \quad t \in G. \]
The Rieffel homeomorphism is given by $h_t(I) = \alpha_t(I) \in \mathcal{I}(D_t)$ for $I \in \mathcal{I}(D_{t^{-1}})$. In particular, if $A = C_0(\Omega)$ is commutative and $(\{\Omega_g\}_{g \in G}, \{\theta_g\}_{g \in G})$ is the system that determines $\alpha$ by (6), then $(\{\Omega_g\}_{g \in G}, \{\theta_g\}_{g \in G})$ can be identified with $(\{D_g\}_{g \in G}, \{h_g\}_{g \in G})$, cf. [1] Subsection 4.1.

Definition 3.18. We call both of the systems $(\{\hat{D}_t\}_{t \in G}, \{\hat{h}_t\}_{t \in G})$ and $(\{\text{Prim}(D_t)\}_{t \in G}, \{h_t\}_{t \in G})$ described above partial dynamical systems dual to the Fell bundle $\mathcal{B}$.

Remark 3.19. The authors of [1] consider only the system $(\{\hat{D}_t\}_{t \in G}, \{\hat{h}_t\}_{t \in G})$ and call it, [1] Definition 2.3], the partial action associated to $B$.

The following theorem can be proved by a straightforward adaptation of the argument leading to the main result of [27]. In a slightly different way, it was proved in [1].

Theorem 3.20 (Corollary 3.4.(ii) in [1]). If the partial dynamical system $(\{\hat{D}_t\}_{t \in G}, \{\hat{h}_t\}_{t \in G})$ dual to a Fell bundle $\mathcal{B}$ is topologically free, then $\mathcal{B}$ has the intersection property.

Remark 3.21. In view of Remark 3.17 and Proposition 3.15, the above theorem is a generalization of similar results obtained earlier for saturated Fell bundles [33, Corollary 6.5(i)], classical crossed products [1, Theorem 1], partial crossed products [36, Theorem 2.4] and semigroup crossed products of corner endomorphisms [28, Theorem 6.5], cf. Section 8. Actually, Theorem 3.20 is much stronger than the last mentioned two results which concern topological freeness of the system $(\{\text{Prim}(D_t)\}_{t \in G}, \{h_t\}_{t \in G})$. Clearly, the latter implies topological freeness of $(\{\hat{D}_t\}_{t \in G}, \{\hat{h}_t\}_{t \in G})$, but the converse fails drastically already for the Cuntz algebra $\mathcal{O}_\alpha$, cf. example after [26, Proposition 3.16], or Proposition 7.3 below.

In order to get a description of all ideals in $C^*_r(\mathcal{B})$, we need the following lemma.

Lemma 3.22. If $I$ is an ideal in $B_e$, then the following conditions are equivalent:

(i) $I$ is $\mathcal{B}$-invariant,
(ii) $\hat{I} \subseteq \hat{B}_e$ is invariant under the partial action $(\{\hat{D}_t\}_{t \in G}, \{\hat{h}_t\}_{t \in G})$ dual to $\mathcal{B}$,
(iii) hull$(I)$ is invariant under the partial action $(\{\text{Prim}(D_t)\}_{t \in G}, \{h_t\}_{t \in G})$.

Proof. The equivalence (i)$\iff$(ii) was proved in [1] Proposition 3.10. Since $\hat{I}$ is invariant if and only if its complement $\hat{B}_e \setminus \hat{I}$ is invariant and $(\{\hat{D}_t\}_{t \in G}, \{\hat{h}_t\}_{t \in G})$ is a lift of $(\{\text{Prim}(D_t)\}_{t \in G}, \{h_t\}_{t \in G})$, we conclude that hull$(I) = \{\ker \pi : [\pi] \in \hat{B}_e \setminus \hat{I}\}$ is invariant if and only if $\hat{I}$ is invariant. Hence (ii)$\iff$(iii).

Corollary 3.23. Suppose that the partial dynamical system $(\{\hat{D}_t\}_{t \in G}, \{\hat{h}_t\}_{t \in G})$ dual to $\mathcal{B}$ is residually topologically free. Then $\mathcal{B}$ has the residual intersection property. In particular,

(i) if $\mathcal{B}$ is exact then $J \to \hat{J} \cap \hat{B}_e$ is a lattice isomorphism from $\mathcal{I}(C^*_r(\mathcal{B}))$ onto the set of all open invariant subsets in $\hat{B}_e$,
(ii) $C^*_r(\mathcal{B})$ is simple if and only if $(\{\hat{D}_t\}_{t \in G}, \{\hat{h}_t\}_{t \in G})$ is minimal (and then $\mathcal{B}$ is exact).

Proof. Let $J$ be an ideal in $\mathcal{B}$. Let $t \in G$ and treat $B_t$ as a Hilbert bimodule over $B_e$. By Proposition 3.22, we see that $I := J_e$ is a $B_t$-invariant ideal and $B_t/J_t = B_t/B_tI$ is the quotient Hilbert bimodule. By Lemma 3.22 the closed set $Y := \hat{\hat{A}} \setminus \hat{I}$ is invariant under $(\{\hat{D}_t\}_{t \in G}, \{\hat{h}_t\}_{t \in G})$. Using Lemma 2.2, we conclude that the restricted partial dynamical system $(\{\hat{D}_g \cap Y\}_{g \in G}, \{\hat{h}_g|Y\}_{g \in G})$ can be naturally identified with the partial dynamical system dual to the quotient bundle $\mathcal{B}/J$. Thus by Theorem 5.20 $\mathcal{B}/J$ has the intersection property. Accordingly, $\mathcal{B}$ has the residual intersection property. Now, part (i) follows from Theorem 3.12 and Lemma 3.22. Part (ii) is a consequence of Corollary 3.13.
Remark 3.24. Items (i) and (ii) in Corollary 3.23 are in essence a content of the second part of [1] Corollary 3.20 and of [1] Corollary 3.12, respectively. The first part of the assertion in Corollary 3.23 was stated without a proof in [1] Corollary 3.16.

4. Aperiodicity for Fell bundles and criteria of pure infiniteness

Muhly and Solel introduced a notion of aperiodicity for $C^*$-correspondences, [37] Definition 5.1], which was in turn inspired by the results of Kishimoto, see [25] Lemma 1.1], and Olesen and Pedersen, see [35] Theorems 6.6 and 10.4]. In the context of partial group actions, a similar condition was exploited in [18]. We formulate it for Fell bundles as follows.

Definition 4.1. A Fell bundle $\mathcal{B} = \{B_g\}_{g \in G}$ is aperiodic if for each $t \in G \setminus \{e\}$, each $b_t \in B_t$ and every non-zero hereditary subalgebra $D$ of $B_e$,

$$\inf \{ \|ab_t a\| : a \in D^+, \|a\| = 1 \} = 0.$$  \hfill (18)

In other words, $\mathcal{B}$ is aperiodic if for each $t \in G \setminus \{e\}$ the Hilbert $B_t$-bimodule $B_t$ is aperiodic in the sense of [37], Definition 5.1]. In particular, reinterpreting [37], Lemma 5.2] we get the following lemma.

Lemma 4.2. Let $\mathcal{B} = \{B_g\}_{g \in G}$ be a Fell bundle and $\mathcal{B} = \bigoplus_{g \in G} B_g$ a $\mathcal{B}$-graded $C^*$-algebra. The Fell bundle $\mathcal{B}$ is aperiodic if and only if for every element $b = \bigoplus_{g \in G} b_g$ in $\bigoplus_{g \in G} B_g$, with $b_e > 0$, and every $\varepsilon > 0$ there exists $a$ in the hereditary subalgebra $b_e B_e b_e$ of $B_e$, with $a \geq 0$ and $\|a\| = 1$, such that

$$\|ab_t a - a\|_B < \varepsilon, \quad \|ab_t a\| > \|b_e\| - \varepsilon.$$  

Proof. By Lemma 5.2 of [37], or more precisely very straightforward generalization of its proof, $\mathcal{B} = \{B_g\}_{g \in G}$ is aperiodic if and only if for every element $b = \bigoplus_{g \in G} b_g$ in $\bigoplus_{g \in G} B_g$, with $b_e > 0$, and every $\varepsilon > 0$ there is $a \in b_e B_e b_e$, with $a \geq 0$ and $\|a\| = 1$, such that

$$\|ab_t a\| > \|b_e\| - \varepsilon \quad \text{and} \quad \|ab_t a\| < \varepsilon \quad \text{for} \ t \in G \setminus \{e\}.$$  

The assertion follows from the above if one puts $\epsilon = \varepsilon/n$ where $n$ is the number of elements in the set $\{t \in G : b_t \neq 0\}$ (if $n = 0$ the assertion is trivial).

Corollary 4.3. If the Fell bundle $\mathcal{B}$ is aperiodic then it has the intersection property.

Proof. By Lemma 4.2 condition (iii) in Proposition 3.15 is satisfied.

Corollary 4.4. If the Fell bundle $\mathcal{B}$ is aperiodic then for any $b \in C^*_r(\mathcal{B})^+ \setminus \{0\}$ there is $a \in C^*_r(\mathcal{B})^+ \setminus \{0\}$ such that $a \leq b$.

Proof. Let $b \in C^*_r(\mathcal{B})^+ \setminus \{0\}$. Lemma 4.2 implies that there exists a positive contraction $h$ in $B_e$ such that

$$\|h E(b) h - h b h\| \leq 1/4, \quad \|h E(b) h\| \geq \|E(b)\| - 1/4 = 3/4,$$

where $E$ is the conditional expectation from $C^*_r(\mathcal{B})$ onto $B_e$. Putting $a := (h E(b) h - 1/2)_+ \in B_e^+$ we see that $a \neq 0$. We conclude that $a \leq b$, exactly as in the proof of [37], Lemma 3.2].

Corollary 4.3 and Theorem 3.20 indicate that notions of aperiodicity and topological freeness are closely related. The general relationship is rather mysterious, cf. Remark 4.6] below. Nevertheless, we have the following:

Proposition 4.5. Suppose that the unit fiber $B_e$ in the Fell bundle $\mathcal{B} = \{B_g\}_{g \in G}$ has a Hausdorff primitive ideal space. If the partial action $\{\{\text{Prim}(D_t)\}_{t \in G}, \{h_t\}_{t \in G}\}$ dual to $\mathcal{B}$ is topologically free then $\mathcal{B}$ is aperiodic.

The assertion follows from the above if one puts $\epsilon = \varepsilon/n$ where $n$ is the number of elements in the set $\{t \in G : b_t \neq 0\}$ (if $n = 0$ the assertion is trivial).

Corollary 4.3. If the Fell bundle $\mathcal{B}$ is aperiodic then it has the intersection property.

Proof. By Lemma 4.2 condition (iii) in Proposition 3.15 is satisfied.

Corollary 4.4. If the Fell bundle $\mathcal{B}$ is aperiodic then for any $b \in C^*_r(\mathcal{B})^+ \setminus \{0\}$ there is $a \in C^*_r(\mathcal{B})^+ \setminus \{0\}$ such that $a \leq b$.

Proof. Let $b \in C^*_r(\mathcal{B})^+ \setminus \{0\}$. Lemma 4.2 implies that there exists a positive contraction $h$ in $B_e$ such that

$$\|h E(b) h - h b h\| \leq 1/4, \quad \|h E(b) h\| \geq \|E(b)\| - 1/4 = 3/4,$$

where $E$ is the conditional expectation from $C^*_r(\mathcal{B})$ onto $B_e$. Putting $a := (h E(b) h - 1/2)_+ \in B_e^+$ we see that $a \neq 0$. We conclude that $a \leq b$, exactly as in the proof of [37], Lemma 3.2].

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The assertion follows from the above if one puts $\epsilon = \varepsilon/n$ where $n$ is the number of elements in the set $\{t \in G : b_t \neq 0\}$ (if $n = 0$ the assertion is trivial).
Proof. Take any \( b_t \in B_t \), where \( t \in G \setminus \{ e \} \), and any hereditary subalgebra \( D \) of \( B_e \). Let \( U = \{ x \in \text{Prim}(B_e) : x \notin D \} \) be the open subset of \( \text{Prim}(B_e) \) corresponding to \( D \). If \( \text{Prim}(B_e) \) is aperiodic for any ideal \( D \), and any hereditary subalgebra \( D \) of \( B_e \). Let \( U = \{ x \in \text{Prim}(B_e) : x \notin D \} \) be the open subset of \( \text{Prim}(B_e) \) corresponding to \( D \). If \( D \cap D_{t^{-1}} = \{ 0 \} \) then \( DD_{t^{-1}} = \{ 0 \} \) and \( D_t \). Hence we may assume that \( D \cap D_{t^{-1}} \neq \{ 0 \} \). Then \( U \cap \text{Prim}(D_{t^{-1}}) \) is a non-empty open subset of \( \text{Prim}(B_e) \). By topological freeness there exists \( x \in U \cap \text{Prim}(D_{t^{-1}}) \) such that \( h_t(x) \neq x \). Since \( \text{Prim}(B_e) \) is Hausdorff we can actually find an open set \( V \subseteq U \cap \text{Prim}(D_{t^{-1}}) \) such that \( h_t(V) \cap V = \emptyset \).

Now we exploit the \( 'C_0(X)' \)-picture’ of \( B_e \). For each \( x \in \text{Prim}(B_e) \) and \( a \in B_e \) we denote by \( a(x) \) the image of \( a \) in the quotient \( A/x \). It is a consequence of the Dauns-Hofmann theorem, see for instance [45, Theorem A.34], that the formula \( (f \cdot a) = f(x)a(x) \) defines a module action of \( C_0(\text{Prim}(B_e)) \) on \( B_e \) via central elements in \( M(B_e) \). In particular, since \( D \) is hereditary, we have \( f \cdot a \in D \) for any \( a \in D \) and \( f \in C_0(\text{Prim}(B_e)) \). Using this fact, we may find an element \( a \in D^+ \), \( \|a\| = 1 \), such that \( a(x) = 0 \) if \( x \notin V \). The latter property means that \( a \in \bigcap_{x \in V} x \). Thus we have

\[
b_t a b_t^* \in b_t \left( \bigcap_{x \notin V} x \right) b_t^* \subseteq \bigcap_{x \in V, x \notin \text{Prim}(D_t)} h_t(x) \subseteq \bigcap_{x \notin h_t(V)} x.
\]

Since \( h_t(V) \) and \( V \) are disjoint, we get \( \bigcap_{x \notin V} x \cap \bigcap_{x \notin h_t(V)} x = \{ 0 \} \). Therefore

\[
\|ab_t a\|^2 = \|(ab_t a)(ab_t a)^*\| = \|a(b_t a b_t^*)a\| = 0.
\]

\[\square\]

Remark 4.6. If \( B \) is the Fell bundle associated to a partial action \( \alpha = (\{D_t\}_{t \in G}, \{\alpha_t\}_{t \in G}) \) on a commutative \( C^* \)-algebra \( A \), then both aperiodicity of \( B \) and topological freeness of the dual action are equivalent to the intersection property, see [18, Proposition A.7]. If additionally \( \alpha \) is a global action, these notions are also known to be equivalent to pointwise proper outerness or pointwise spectral non-triviality, see respectively [18, Proposition A.7] and [11, Lemma 1.8]. For global actions on a separable (not necessarily commutative) \( C^* \)-algebra \( A \), [38, Theorem 6.6 and Lemma 7.1] imply that topological freeness of the dual system on \( \hat{A} \) is equivalent to aperiodicity of the associated bundle, and also to pointwise proper outerness. In particular, all the aforementioned notions are closely related to the Connes spectrum and the Borcher’s spectrum, cf. [38, 11].

Before we proceed we need the following definition.

Definition 4.7. We say that a Fell bundle \( B = \{B_g\}_{g \in G} \) is residually aperiodic if \( B/\mathcal{J} \) is aperiodic for any ideal \( \mathcal{J} \) in \( B \).

Corollary 4.8. If \( B = \{B_g\}_{g \in G} \) is residually aperiodic then it has the residual intersection property, and thus if additionally \( B \) is exact then \( J \to J \cap B_e \) is a homeomorphism from \( \mathcal{I}(C^*_r(B)) \) onto \( \mathcal{I}^B(B_e) \).

Proof. Apply Corollary [1.3] and Theorem [3.12] \[\square\]

The following theorem is a generalization of [17, Theorem 3.3], [18, Theorem 4.2], and [30, Proposition 2.46], proved respectively for crossed products by group actions, partial actions, and single endomorphisms (see Section 8 below, for more information on latter crossed products). In order to prove it we need a lemma which is interesting in its own right.

Lemma 4.9. Suppose that \( A \) is a purely infinite \( C^* \)-algebra with finitely many ideals. Then \( A \) has the ideal property.
Theorem 4.10. Suppose that $\mathcal{B} = \{B_g\}_{g \in G}$ is an exact, residually aperiodic Fell bundle. Assume that either $B_e$ has the ideal property or that $B_e$ contains finitely many $\mathcal{B}$-invariant ideals. Then the following statements are equivalent:

(i) Every non-zero positive element in $B_e$ is properly infinite in $C^*_r(\mathcal{B})$.
(ii) $C^*_r(\mathcal{B})$ is purely infinite.
(iii) $C^*_r(\mathcal{B})$ is purely infinite and has the ideal property.
(iv) Every non-zero hereditary $C^*$-subalgebra in any quotient $C^*_r(\mathcal{B})$ contains an infinite projection.

If $B_e$ is of real rank zero, then each of the above conditions is equivalent to

(i') Every non-zero projection in $B_e$ is properly infinite in $C^*_r(\mathcal{B})$.

Proof. Implications (iv)⇔(iii)⇔(ii)⇒(i) are general facts, see respectively [42] Propositions 2.11, [23] Proposition 4.7 and [23] Theorem 4.16. If $A$ is if real rank zero the equivalence (i)⇔(i') is ensured by [30] Lemma 2.44. Thus it suffices to show that (i) implies (iii) or (iv). Let us then assume that every element in $B_e^+ \setminus \{0\}$ is properly infinite in $C^*_r(\mathcal{B})$. We note that, in view of Corollary 3.3, the equivalent conditions in Theorem 3.12 hold.

Suppose first that $B_e$ has the ideal property. We will show (iv). Let $J$ be an ideal in $C^*_r(\mathcal{B})$ and $D$ be a non-zero hereditary $C^*$-subalgebra in the quotient $C^*_r(\mathcal{B})/J$. We need to show that $D$ contains an infinite projection. By Corollary 3.6, we have $C^*_r(\mathcal{B})/J \cong C^*_r(\mathcal{B}/J)$, where $J = \{J_t\}$, $J_t = J \cap B_t$, $t \in G$. Fix a non-zero positive element $b$ in $D$. By Corollary 4.4 there exists a non-zero positive element $a$ in $B_e/J_e$ such that $a \preceq b$ relative to $C^*_r(\mathcal{B}/J)$. Note that $a$ is properly infinite in $C^*_r(\mathcal{B}/J)$ as a non-zero homomorphic image of a properly infinite positive element in $C^*_r(\mathcal{B})$, by the assumption in (i). Since $B_e$ has the ideal property we can find a projection $q \in B_e$ that belongs to the ideal in $B_e$ generated by the preimage of $a$ in $B_e$ but not to $J_e$. Then $q + J_e$ is a non-zero projection that belongs to the ideal in $B_e/J_e$ generated by $a$, whence $q + J_e \preceq a \preceq b$, by [23] Proposition 3.5(ii)]. From the comment after [23] Proposition 2.6] we can find $z \in C^*_r(\mathcal{B}/J)$ such that $q + J_e = z^*bz$. With $v := b^{1/2}z$ it follows that $v^*v = q + J_e$, whence $p := vv^* = b^{1/2}zz^*b^{1/2}$ is a projection in $B$, which is equivalent to $q$. By the assumption in (i), $q$ and hence also $p$ is properly infinite.

Suppose now that $B_e$ has finitely many, say $n$, $\mathcal{B}$-invariant ideals. Since they are in one-to-one correspondence with ideals in $C^*_r(\mathcal{B})$ (recall Corollary 4.3, Lemma 4.9) implies that the conditions (ii) and (iii) are equivalent. We will prove (ii). The proof goes by induction on $n$.

Assume first that $n = 2$ so that $C^*_r(\mathcal{B})$ is simple. Take any $b \in C^*_r(\mathcal{B})/\lambda \setminus \{0\}$. By Corollary 3.6 there is $a \in B_e^+ \setminus \{0\}$ such that $a \preceq b$. Then $b \in C^*_r(\mathcal{B})aC^*_r(\mathcal{B}) \subseteq C^*_r(\mathcal{B})$ and as $a$ is properly
infinite we get $b \preceq a$ by [23] Proposition 3.5]. Hence $b$ is properly infinite as it is Cuntz equivalent to $a$. Thus $C^*_r(B)$ purely infinite.

Now suppose that our claim holds for any $k < n$. Let $J$ be any non-trivial ideal in $C^*_r(B)$. By Corollary 3.6 we have $J \cong C^*_r(J)$ and $C^*_r(B)/J \cong C^*_r(B/J)$, where $J = \{J_t\}$, $J_t = J \cap B_t$, $t \in G$. By Lemma 3.8 exactness passes to ideals and quotients, and clearly the same holds for residual aperiodicity. Thus both $J$ and $B/J$ satisfy the assumptions of the assertion and the corresponding unit fibers have less than $n$ invariant ideals. Moreover, both $J$ and $B/J$ satisfy condition (i). Indeed, for $B/J$ it is clear, as proper infiniteness passes to quotients, and for $J$ it follows from the fact that proper infiniteness of $a \in J_c^+ \setminus \{0\}$ in $C^*_r(B)$ imply proper infiniteness of $a$ in $J$, by [23] Proposition 3.3. Concluding, by induction hypotheses, both $J$ and $C^*_r(B)/J$ are purely infinite, and since pure infiniteness is closed under extensions [23] Theorem 4.19] we get that $C^*_r(B)$ is purely infinite. 

\[ \square \]

**Remark 4.11.** We recall, see [42] Propositions 2.11, 2.14], that in the presence of the ideal property pure infiniteness of a $C^*$-algebra is equivalent to strong pure infiniteness, weak pure infiniteness, and many other notions of infiniteness appearing in the literature. Thus the list of equivalent conditions in Theorem 4.10 could be considerably extended.

**5. Paradoxicality, residual infiniteness and pure infiniteness**

Now, we give and study a noncommutative, algebraic version of the notion of paradoxical sets, cf. Definition 2.3 phrased in terms of Fell bundles. We also introduce a notion of residually $B$-infinite elements, which we think is a good alternative for $B$-paradoxical elements. In particular, the former elements seem to be more convenient to work with in practice, cf. Remark 7.10 below.

**Definition 5.1.** Let $B = \{B_g\}_{g \in G}$ be a Fell bundle and let $a \in B^+_c \setminus \{0\}$. We say that:

(i) $a$ is $B$-paradoxical if for every $\varepsilon > 0$ there are elements $a_i \in aB_{t_i}$, where $t_i \in G$ for $i = 1, \ldots, n + m$, such that

(19) \[ a \approx_{\varepsilon} \sum_{i=1}^{n} a_i^* a_i, \quad a \approx_{\varepsilon} \sum_{i=n+1}^{n+m} a_i^* a_i, \quad \text{and} \quad \|a_i^* a_j\| < \frac{\varepsilon}{\max\{n^2, m^2\}} \quad \text{for } i \neq j. \]

If the elements $a_i$, $i = 1, \ldots, n + m$, above can be chosen so that

(20) \[ a = \sum_{i=1}^{n} a_i^* a_i = \sum_{i=n+1}^{n+m} a_i^* a_i \quad \text{and} \quad a_i^* a_j = 0 \quad \text{for } i \neq j, \]

we call $a$ a strictly $B$-paradoxical.

(ii) $a$ is $B$-infinite if there is $b \in B^+_c \setminus \{0\}$ such that for every $\varepsilon > 0$ there are elements $a_i \in aB_{t_i}$, where $t_i \in G$ for $i = 1, \ldots, n + m$, such that

(21) \[ a \approx_{\varepsilon} \sum_{i=1}^{n} a_i^* a_i, \quad b \approx_{\varepsilon} \sum_{i=n+1}^{n+m} a_i^* a_i, \quad \text{and} \quad \|a_i^* a_j\| < \frac{\varepsilon}{\max\{n^2, m^2\}} \quad \text{for } i \neq j. \]

We say $a$ is strictly $B$-infinite if there is a non-zero positive element $b \in aB_a$ and elements $a_i \in aB_{t_i}$ where $t_i \in G$ for $i = 1, \ldots, n$, such that

(21) \[ a = \sum_{i=1}^{n} a_i^* a_i, \quad a_i^* a_j = 0 \quad \text{for } i \neq j, \quad \text{and} \quad a_i^* b = 0 \quad \text{for } i = 1, \ldots, n. \]
Corollary 5.5. Let Proposition 3.14 and Proposition 5.3(iii).

\[ B \]

Electronic equivalent to properly infinite in \( \mathcal{B} \). We say that \( a \) is \emph{residually \( \mathcal{B} \)-infinite} if for every ideal \( \mathcal{J} = \{ J_g \}_{g \in G} \) in \( \mathcal{B} \) the element \( a + J_e \) is either zero in \( B/e/J_e \) or it is \( \mathcal{B}/\mathcal{J} \)-finite. We say that \( a \) is \emph{residually strictly \( \mathcal{B} \)-infinite} if for every ideal \( \mathcal{J} = \{ J_g \}_{g \in G} \) in \( \mathcal{B} \) the element \( a + J_e \) is either zero in \( B/e/J_e \) or it is strictly \( \mathcal{B}/\mathcal{J} \)-finite.

Remark 5.2. If \( a \) is strictly \( \mathcal{B} \)-infinite it is \( \mathcal{B} \)-infinite; take \( m = 1 \), \( t_{n+1} = e \) and \( a_{n+1} = \sqrt{b} \). Moreover, if there are elements \( a_i \in aB_{t_i} \) where \( t_i \in G \) for \( i = 1, \ldots, n + m \), such that

\[ a = \sum_{i=1}^{n} a_i^*a_i, \quad \sum_{i=n+1}^{n+m} a_i^*a_i \neq 0 \quad \text{and} \quad a_i^*a_j = 0 \quad \text{for} \quad i \neq j, \]

then putting \( b = \sum_{i=n+1}^{n+m} a_i a_i^* \), we see that \( a \) is strictly \( \mathcal{B} \)-infinite. In particular, it follows that every strictly \( \mathcal{B} \)-paradoxical element is strictly \( \mathcal{B} \)-infinite and actually residually strictly \( \mathcal{B} \)-infinite. Also it is readily seen that every \( \mathcal{B} \)-paradoxical element is residually \( \mathcal{B} \)-infinite. Whether the converse holds is an open problem.

The following Proposition 5.3 provides a motivation for this definition. It also implies that both paradoxicality and residual \( \mathcal{B} \)-infiniteness can be viewed as generalizations of proper infiniteness.

Proposition 5.3. Let \( B = \bigoplus_{g \in G} B_g \) be a \( \mathcal{B} \)-graded \( C^* \)-algebra and let \( a \in B_e^+ \setminus \{ 0 \} \).

(i) If \( a \) is \( \mathcal{B} \)-paradoxical then \( a \) is properly infinite in \( B \).

(ii) If \( a \) is \( \mathcal{B} \)-infinite then \( a \) is infinite in \( B \).

(iii) If \( a \) is residually \( \mathcal{B} \)-infinite then for any graded ideal \( J \) in \( B \) the image of \( a \) in \( B/J \) is either zero or infinite.

Proof. (i). Let \( \varepsilon > 0 \) and choose elements \( a_i \in aB_{t_i} \), \( t_i \in G \), \( i = 1, \ldots, n + m \), witnessing \( \varepsilon \)-paradoxicality of \( a \), that is assume \( [19] \) holds. Putting \( x = \sum_{i=1}^{n} a_i \) and \( y = \sum_{i=n+1}^{n+m} a_i \) we immediately get that \( x, y \in aB \). Using that \( \| a_i^*a_j \| < \varepsilon / \max\{ n^2, m^2 \} \) for \( i \neq j \) we get

\[
\left\| \sum_{i,j=1 \atop i \neq j}^{n} a_i^*a_j \right\| < \varepsilon, \quad \left\| \sum_{i=n+1}^{n+m} a_i^*a_j \right\| < \varepsilon, \quad \left\| \sum_{i=1}^{n} \sum_{j=n+1}^{n+m} a_i^*a_j \right\| < \varepsilon.
\]

The above inequalities imply respectively that \( x^*x \approx \varepsilon a \), \( y^*y \approx \varepsilon a \) and \( x^*y \approx \varepsilon 0 \). Hence \( a \) is properly infinite in \( B \) by \( [3] \).


(iii). By Proposition 5.2 graded ideals \( J \) in \( B \) are in one-to-one correspondence with ideals \( \mathcal{J} = \{ J_g \}_{g \in G} \) in \( \mathcal{B} \). For any such pair the \( C^* \)-algebra \( B/J \) is \( \mathcal{B}/\mathcal{J} \)-graded. Hence the assertion follows from part (ii).

\[
\square
\]

Corollary 5.4. Suppose that \( B = \{ B_g \}_{g \in G} \) is an exact Fell bundle with the residual intersection property. Then any residually \( \mathcal{B} \)-infinite \( a \in B_e^+ \setminus \{ 0 \} \) is properly infinite in \( C_r^*(\mathcal{B}) \).

Proof. By Theorem 3.12 every ideal in \( C_r^*(\mathcal{B}) \) is graded. Hence the assertion follows from \( [23] \) Proposition 3.14 and Proposition 5.3(iii).

\[
\square
\]

Corollary 5.5. Let \( A \) be a \( C^* \)-algebra and let \( a \in A^+ \setminus \{ 0 \} \). The following statements are equivalent

(i) \( a \) is properly infinite in \( A \).

(ii) \( a \) is \( \mathcal{B} \)-paradoxical for every Fell bundle \( B = \{ B_g \}_{g \in G} \) with \( B_e = A \).

(iii) \( a \) is residually \( \mathcal{B} \)-infinite for every Fell bundle \( B = \{ B_g \}_{g \in G} \) with \( B_e = A \).
Proof. (i)⇒(ii). Let $a \in A^+ \setminus \{0\}$ be properly infinite. By (8) for any $\varepsilon > 0$ there are $x, y \in aAa$ with $x^*x \approx_\varepsilon a$, $y^*y \approx_\varepsilon a$, and $x^*y \approx_\varepsilon 0$. Thus for any Fell bundle $\mathcal{B} = \{B_g\}_{g \in G}$ with $B_e = A$ condition (19) is satisfied with $n = m = 1$, $t_1 = t_2 = e$ and $a_1 = x$, $a_2 = y$.

(ii)⇒(iii). It is clear, since $\mathcal{B}$-paradoxicality implies residual $\mathcal{B}$-infiniteness.

(iii)⇒(i). Apply Corollary 5.4 to the Fell bundle $\mathcal{B} = \{B_g\}_{g \in G}$ with $B_e = A$ and $B_g = \{0\}$ for $g \in G \setminus \{e\}$.

A set-theoretic counterpart of the notion of a $\mathcal{B}$-infinite element is defined as follows.

Definition 5.6. Let $\{\Omega_t\}_{t \in G}$, $\{\theta_t\}_{t \in G}$ be a partial action on a topological space $\Omega$. An open set $V \subseteq \Omega$ is called $G$-infinite if there are open sets $V_1, ..., V_n$ and elements $t_1, ..., t_n \in G$, $n \geq 1$, such that

1. $V = \bigcup_{i=1}^n V_i$,
2. $V_i \subseteq \Omega_{t_i^{-1}}$ for all $i = 1, ..., n$ and $\bigcup_{i=1}^n \theta_{t_i}(V_i) \subseteq V$,
3. $\theta_{t_i}(V_i) \cap \theta_{t_j}(V_j) = \emptyset$ for all $i \neq j$.

A non-empty open set $V \subseteq \Omega$ is called residually $G$-infinite if for every closed invariant subset $Y \subseteq \Omega$ the set $V \cap Y$ is either empty or $G$-infinite for the partial action $\{(\Omega_t \cap Y)\}_{t \in G}$, $\{\theta_t|_{\Omega_t \cap Y}\}_{t \in G}$.

Remark 5.7. Clearly, for strong boundary actions of discrete groups on compact spaces considered by Laca and Spielberg in [34] every open set is residually $G$-infinite (note that strong boundary actions are necessarily minimal). Actually, we will show below a much more general fact. We will prove that for any $n$-filling action, notion introduced in [21], on a unital $C^*$-algebra $A$ without finite-dimensional corners, every element in $A^+ \setminus \{0\}$ is residually $\mathcal{B}$-infinite, for the corresponding Fell bundle $\mathcal{B}$.

The following two propositions shed more light on the relationship between Definitions 2.6, 2.3 and 5.1.

Proposition 5.8. Let $\mathcal{B} = \{B_g\}_{g \in G}$ be the Fell bundle of a partial action $\alpha$ on $C_0(\Omega)$ which is induced by a partial action $\{(\Omega_g \cap Y)\}_{g \in G}$ of $G$ on a locally compact Hausdorff space $\Omega$, see (4). Let $a \in B_+ = C_0(\Omega)^+$ be a non-zero element and put $V := \{x \in \Omega : a(x) > 0\}$.

(i) $V$ is $G$-paradoxical if and only if $a$ is strictly $\mathcal{B}$-paradoxical.

(ii) $V$ is $G$-infinite if and only if $a$ is strictly $\mathcal{B}$-infinite.

Proof. ‘Only if’ parts. Let $V_1, ..., V_n$ be open sets and $t_1, ..., t_n \in G$ be such that $V = \bigcup_{i=1}^n V_i$, $V_i \subseteq \Omega_{t_i^{-1}}$ and $\theta_{t_i}(V_i) \subseteq V$ for all $i = 1, ..., n$, and $\theta_{t_i}(V_i) \cap \theta_{t_j}(V_j) = \emptyset$ for all $i \neq j$. Let $\{h_i\}_{i=1}^n$ be a partition of unity for $V$ relative to the open cover $\{V_i\}_{i=1}^n$. Let $i$ be fixed for a while. Since $ah_i \in C_0(\Omega)$ is supported on $V_i \subseteq \Omega_{t_i^{-1}}$ we can treat $(ah_i) \circ \theta_{t_i^{-1}}$ as a continuous function on $\Omega$ supported on $\theta_{t_i}(V_i) \subseteq V$. By definition of $V$ we actually get $(ah_i) \circ \theta_{t_i^{-1}} \in aC_0(\Omega)$. Hence

$$a_i := \left((ah_i) \circ \theta_{t_i^{-1}}\right)^{1/2} \delta_{t_i}$$

is an element of $a_i \in aB_{t_i}$, and we claim that

$$a = \sum_{i=1}^n a_i^* a_i \quad \text{and} \quad a_i^* a_j = 0 \quad \text{for} \quad i \neq j.$$  

Indeed, for any $i, j$ we have $a_i^* a_j = \left((ah_i) \circ \theta_{t_i^{-1}} \cdot (ah_j) \circ \theta_{t_j}^{-1}\right)^{1/2} \circ \theta_{t_i} \delta_{t_i^{-1} t_j^{-1}}$. For $i \neq j$ the functions $h_i \circ \theta_{t_i}^{-1}$ and $h_j \circ \theta_{t_j}^{-1}$ are supported on disjoint sets $\theta_{t_i}(V_i)$ and $\theta_{t_j}(V_j)$. Hence $a_i^* a_j = 0$. On the other hand, $a_i^* a_i = ah_i$ and consequently $\sum_{i=1}^n a_i^* a_i = \sum_{i=1}^n ah_i = a$. 


Now, if \( \bigcup_{i=1}^{n} \theta_{t_i}(V_i) \subset V \), that is if \( V \) is \( G \)-infinite, then taking \( b \in B_e \) to be any non-zero positive function supported on the non-empty open set \( V \setminus \bigcup_{i=1}^{n} \theta_{t_i}(V_i) \) we infer that \( a \) is strictly \( B \)-infinite.

If \( V_{n+1}, \ldots, V_{n+m} \) are open sets and \( t_{n+1}, \ldots, t_{n+m} \in G \) are such that \( V = \bigcup_{i=m+1}^{n+m} V_i \), \( V_i \subseteq \Omega_{t_i} \), and \( \theta_{t_i}(V_i) \subseteq V \) for all \( i = n + 1, \ldots, n + m \), and \( \theta_{t_i}(V_i) \cap \theta_{t_j}(V_j) = \emptyset \) for all \( i \neq j \), \( i, j = 1, \ldots, n + m \), then constructing elements \( a_i \in aB_t \) for \( i = n + 1, \ldots, n + m \) exactly as we did for \( i = 1, \ldots, n \) we get relations \( \circ \). Hence if \( V \) is \( G \)-paradoxical, then \( a \) is strictly \( B \)-paradoxical.

If parts. Let \( a_i \in aB_t, t_i \in G, i = 1, \ldots, n \), be such that \( \circ \) holds. For any \( i = 1, \ldots, n \) we have \( a_i = b_i \delta_{t_i} \) where \( b_i \in aD_{t_i} = C_0(\Omega_{t_i} \cap V) \). We put

\[
V_i := \{ x \in \Omega : a_i^* a_i(x) > 0 \} = \{ x \in \Omega : |b_i|^2(\theta_{t_i}(x)) \neq 0 \} = \theta_{t_i}^{-1}(\{ x \in \Omega : b_i(x) \neq 0 \}).
\]

Thus \( V_i \) is an open subset of \( \Omega_{t_i} \) and \( \theta_{t_i}(V_i) \subseteq V \). Moreover,

\[
V = \{ x \in \Omega : a^* a(x) > 0 \} = \{ x \in \Omega : \sum_{i=1}^{n} a_i^* a_i(x) > 0 \} = \bigcup_{i=1}^{n} V_i,
\]

and

\[
(a_i^* a_j = 0) \iff (b_i b_j = 0) \implies (\theta_{t_i}(V_i) \cap \theta_{t_j}(V_j) = \emptyset).
\]

Using this one readily sees that if \( a \) is strictly \( B \)-paradoxical then \( V \) is \( G \)-paradoxical. Moreover, if there is a non-zero positive \( b \in aB_t \) such that \( a_i^* b = 0 \) for \( i = 1, \ldots, n \), then \( W := \{ x \in \Omega : b(x) > 0 \} \) is a non-empty open subset of \( V \) and

\[
(a_i^* b = 0) \iff (b_i b = 0) \implies (\theta_{t_i}(V_i) \cap W = \emptyset).
\]

Hence \( \bigcup_{i=1}^{n} \theta_{t_i}(V_i) \subseteq V \) and \( V \) is \( G \)-infinite. \qed

It is not clear whether a version of the following proposition for residually infinite elements hold.

**Proposition 5.9.** Retain the assumptions of Proposition 5.8. If \( \Omega \) is totally disconnected and every open compact subset of \( \Omega \) is \( G \)-paradoxical, then every element in \( B_e^+ \setminus \{ 0 \} \) is \( B \)-paradoxical.

**Proof.** Let \( a \in C_0(\Omega)^+ \setminus \{ 0 \} = B_e^+ \setminus \{ 0 \} \). For any \( \varepsilon > 0 \) there exists an open compact set \( V \) such that

\[
\{ x \in \Omega : a(x) \geq \varepsilon \} \subseteq V \subseteq \{ x \in \Omega : a(x) > 0 \}.
\]

Let \( a_V \in C_0(\Omega) \) denote the function such that \( a_V(x) = a(x) \) for \( x \in V \) and \( a_V(x) = 0 \) outside \( V \). Using paradoxicality of \( V \), by Proposition 5.8 there are elements \( a_i \in aV B_{t_i} \subseteq aB_{t_i}, t_i \in G, i = 1, \ldots, n + m \), such that \( a_V = \sum_{i=1}^{n} a_i^* a_i = \sum_{i=n+1}^{n+m} a_i^* a_i \) and \( a_i^* a_j = 0 \) for \( i \neq j \).

Since \( a_V \approx a \) we see that relations \( \circ \) hold. \qed

We now turn to the promised relationship between residual \( B \)-infiniteness and \( n \)-filling actions introduced by Jolissaint and Robertson [21].

**Definition 5.10** (Definition 0.1 in [21]). A global action \( \alpha = (A, \{ \alpha_t \}_{t \in G}) \) on a unital \( C^* \)-algebra \( A \) is called \( n \)-filling, for \( n \geq 2 \), if, for all elements \( b_1, \ldots, b_n \in A^+ \) of norm one, and for all \( \varepsilon > 0 \), there exist \( g_1, \ldots, g_n \in G \) such that \( \sum_{i=1}^{n} \alpha_{g_i}(b_i) \geq 1 - \varepsilon \).

**Remark 5.11.** A 2-filling action on a commutative \( C^* \)-algebra is equivalent to what is called a strong boundary action in [34], see [21].
Lemma 5.12. Suppose that $\alpha$ is an $n$-filling action on a unital $C^*$-algebra $A$ such that for every nonzero projection $e \in A$ the algebra $eAe$ is infinite dimensional. Let $B$ be the Fell bundle corresponding to $\alpha$. Then the action $\alpha$ is minimal and any element $a \in A^+ \setminus \{0\}$ is strictly (residually) $B$-infinite.

Proof. That $\alpha$ does not admit non-trivial invariant ideals is clear. Let $a \in A^+ \setminus \{0\}$ and $0 < \varepsilon < 1$ be smaller than $1$. We may suppose that $\|a\| = 1$. Arguing in a similar way as in the proof of [21, Lemma 1.4], we see that there are normalized positive elements $b_1, ..., b_{n+1} \in aA$ such that $b_ib_j = 0$ for $i \neq j$. By $n$-filling property, there are elements $g_1, ..., g_n \in G$ such that $h := \sum_{i=1}^n b_i g_i(b_i) \geq 1 - \varepsilon$. Hence $h$ is positive and invertible. Put

$$a_i := \sqrt{b_i \alpha_{g_i^{-1}}(\sqrt{h^{-1}}) \alpha_{g_i^{-1}}(\sqrt{a})} u_{g_i}, \quad i = 1, ..., n,$$

and $b := b_n + 1$.

Then clearly $a_i a_j = 0$ for $i \neq j$ and $a_i a_i^* b = 0$ for all $i, j = 1, ..., n$. Moreover,

$$\sum_{i=1}^n a_i a_i^* = \sum_{i=1}^n \sqrt{a} \sqrt{h^{-1}} \alpha_{g_i}(b_i) \sqrt{h^{-1}} \sqrt{a} = \sqrt{a} \sqrt{h^{-1}} \sqrt{h^{-1}} \sqrt{a} = a.$$ 

Hence $a \in A^+ \setminus \{0\}$ is strictly $B$-infinite, and as $\alpha$ is minimal, actually strictly residually $B$-infinite. \hspace{1cm} \Box

We state the main result of this section using the notion of residual $B$-infiniteness. As noted above this is a weaker condition than dynamical conditions implying pure infiniteness that appear in [34], [21], [47], or [18].

Theorem 5.13. Suppose that $B = \{B_g\}_{g \in G}$ is an exact, residually aperiodic Fell bundle and one of the following conditions holds

(i) $B_e$ contains finitely many $B$-invariant ideals and every element in $B^+_e \setminus \{0\}$ is Cuntz equivalent to a residually $B$-infinite element,

(ii) $B_e$ has the ideal property and every element in $B^+_e \setminus \{0\}$ is Cuntz equivalent to a residually $B$-infinite element,

(iii) $B_e$ is of real rank zero and every non-zero projection in $B_e$ is Cuntz equivalent to a residually $B$-infinite element.

Then $C^*_r(B)$ has an ideal property and is purely infinite.

Proof. Note that in each of the cases (i)-(iii) Theorem 4.10 applies. By Corollary 5.4 every residually $B$-infinite element in $B^+_e \setminus \{0\}$ is properly infinite in $C^*_r(B)$. Since an element equivalent to a properly infinite one is properly infinite, the assertion holds. \hspace{1cm} \Box

As a consequence of Theorem 5.13 we get the following strengthening and unification of the following results [34, Theorem 5], [21, Theorem 1.2](in the commutative case), [47, Corollary 4.4] and [18, Theorem 4.4].

Corollary 5.14. Suppose that $\alpha$ is an exact partial action on $C_0(\Omega)$ induced by residually topologically free partial action $\{\Omega_g\}_{g \in G}$ of $G$ on a locally compact space $\Omega$. Assume that one of the following conditions holds

(i) $\Omega$ contains finitely many $\theta$-invariant closed sets and and every non-empty open set is residually $G$-infinite,

(ii) $\Omega$ is totally disconnected space and every non-empty compact and open set is residually $G$-infinite.

Then $A \rtimes_{\alpha,r} G$ has the ideal property and is purely infinite.
Proof. Apply Theorem 5.13 and Proposition 5.8(ii). \qed

In connection with the Theorem 5.13, it is useful to make the following observation.

**Lemma 5.15.** A sum of orthogonal (residually) strictly $\mathcal{B}$-infinite elements is (residually) strictly $\mathcal{B}$-infinite. Any projection in $B_e$ which is Murray-von Neumann equivalent to a (residually) strictly $\mathcal{B}$-infinite projection in $B_e$ is (residually) strictly $\mathcal{B}$-infinite.

**Proof.** Suppose that $a^{(1)}$, $a^{(2)}$ are strictly $\mathcal{B}$-infinite elements and $a^{(1)}a^{(2)} = 0$. For each $j = 1, 2$, let $b^{(j)} \in a^{(j)}B_ea^{(j)} \setminus \{0\}$ and $a^{(j)}_i \in a^{(j)}B_{t_i}$, $t_i \in G$, $i = 1, \ldots, n_j$, be elements satisfying the counterpart of (21). Putting $a = \sum_{j=1}^2 a^{(j)}$, $b = \sum_{j=1}^2 b^{(j)}$, $a_i := a^{(1)}_i$ for $i = 1, \ldots, n_1$, and $a_{n_1+i} := a^{(2)}_i$ for $i = 1, \ldots, n_2$ we see that $b \in aB_ea$ and $a_i \in aB_{t_i}$ for $i = 1, \ldots, n := n_1 + n_2$ satisfy (21). Hence $a$ is strictly $\mathcal{B}$-infinite. Since for any quotient map $q$, $q(a) = 0$ if and only if $q(a^{(1)}) = q(a^{(2)}) = 0$, we see that if $a^{(1)}$, $a^{(2)}$ are residually strictly $\mathcal{B}$-infinite then $a$ is residually strictly $\mathcal{B}$-infinite.

Suppose that $p = w^*w$ and $p' = ww^*$ are projections in $B_e$, $w \in B_e$, and let $b \in (pB_ep) \setminus \{0\}$ and $a_i \in pB_{t_i}$, $t_i \in G$, $i = 1, \ldots, n$, satisfy (21), with $p$ in place of $a$. Putting $b' := wbw^*$ and $a'_i := wa_iw^*$, $i = 1, \ldots, n$ we get $b' \in (p'B_{p'}e) \setminus \{0\}$ and $a'_i \in p'B_{t_i}$ such that $p' = \sum_{i=1}^n (a'_i)^*a'_i$, $(a'_i)^*a'_{i'} = 0$ for $i \neq j$, and $(a'_i)^*b' = 0$ for $i = 1, \ldots, n$. Hence $p'$ is strictly $\mathcal{B}$-infinite. Since for any quotient map $q$, $q(p) = 0$ if and only if $q(p') = 0$, it follows that $p$ is residually strictly $\mathcal{B}$-infinite if and only if $p'$ is residually strictly $\mathcal{B}$-infinite. \qed

6. Primitive ideal space of the reduced cross-sectional algebra

In this section, we describe the space of prime ideals in $C^*_r(\mathcal{B})$, which in the separable case will lead us to a description of the primitive spectrum of $C^*_r(\mathcal{B})$, via a quasi-orbit space for the dual partial action introduced in Definition 3.18.

Throughout we fix a Fell bundle $\mathcal{B} = \{B_t\}_{t \in G}$. The following notions and results are generalizations of classical facts for crossed products [10], see also [11].

**Definition 6.1.** We say that a $\mathcal{B}$-invariant ideal $I \in \mathcal{T}^\mathcal{B}(B_e)$ is $\mathcal{B}$-prime if for any pair of $\mathcal{B}$-invariant ideals $J_1, J_2 \in \mathcal{T}^\mathcal{B}(B_e)$ with $J_1 \cap J_2 \subseteq I$ we have either $J_1 \subseteq I$ or $J_2 \subseteq I$. We equip the set of $\mathcal{B}$-prime ideals with Fell topology and denote it by $\text{Prime}^\mathcal{B}(B_e)$.

We get the following generalization of [11] Proposition 2.3.

**Proposition 6.2.** Let $\mathcal{B}$ be an exact Fell bundle. Suppose that $\mathcal{B}$ has the residual intersection property, which holds for instance if one of the following conditions is satisfied:

(i) the system $\{(\hat{D}_g)_{g \in G}, \{\hat{h}_g\}_{g \in G}\}$ dual to $\mathcal{B}$ is residually topologically free,

(ii) $\mathcal{B}$ is residually aperiodic.

Then the map $\mathcal{I}(C^*_r(\mathcal{B})) \ni J \mapsto J \cap B_e \in \mathcal{T}^\mathcal{B}(B_e)$ restricts to a homeomorphism $\text{Prime}(C^*_r(\mathcal{B})) \cong \text{Prime}^\mathcal{B}(B_e)$.

**Proof.** By Theorem 3.12 see also Corollaries 3.23 and 4.8 the map $\mathcal{I}(C^*_r(\mathcal{B})) \ni J \mapsto J \cap B_e \in \mathcal{T}^\mathcal{B}(B_e)$ is a homeomorphism. It restricts to a homeomorphism from $\text{Prime}(C^*_r(\mathcal{B}))$ onto $\text{Prime}^\mathcal{B}(B_e)$.

For any ideal $J \in \mathcal{I}(B_e)$ we have a natural continuous embedding of $\mathcal{I}(J)$ in $\mathcal{I}(B_e)$ given by $\mathcal{I}(J) \ni I \mapsto I_J := \{a \in B_e : a \cdot J \subseteq I\} \in \mathcal{I}(B_e)$.

This map has a left inverse given by $\mathcal{I}(B_e) \ni K \mapsto I := K \cap J \in \mathcal{I}(J)$. Note that we use the mapping $\text{Prim}(D_t) \ni P \mapsto P_{D_t} \in \text{Prim}(B_e)$ to identify the space $\text{Prim}(D_t)$ with an
open subset of Prim($B_e$), $t \in G$. Thus formally, the dual partial action on Prim($B_e$), given by restriction of the Rieffel homeomorphism \cite{17}, should be defined by $h_t(P_{D_{t-1}}) := h_t(P)_{D_t}$ where $P \in$ Prim($D_{t-1}$), $t \in G$.

**Definition 6.3.** For any $P \in$ Prim($B_e$) we put

$$P_B := \bigcap_{t \in G} h_t(P \cap D_{t-1})_{D_t} = \bigcap_{t \in G} \{ a \in B_e : aD_t \subseteq B_tPB_{t-1} \}$$

and call it a $\mathcal{B}$-primitive ideal in $B_e$. We denote the space of $\mathcal{B}$-primitive ideals by

$$\text{Prim}^B(B_e) := \{ P_B : P \in \text{Prim}(B_e) \}$$

and endow it with the Fell-topology inherited from $\mathcal{I}(B_e)$.

**Lemma 6.4.** For any $P \in$ Prim($B_e$), $P_B$ is the largest $\mathcal{B}$-invariant ideal contained in $P$.

**Proof.** Let $GP := \bigcup_{t \in G, P \in \text{Prim}(D_{t-1})} h_t(P)_{D_t}$ be the orbit of $P \in$ Prim($B_e$) under the partial action $\{ \{ \text{Prim}(D_t) \}_{t \in G}, \{ h_t \}_{t \in G} \}$. Since $h_t(D_{t-1})_{D_t} = A$ we see that

$$P_B = \bigcap_{t \in G} h_t(P \cap D_{t-1})_{D_t} = \bigcap_{t \in G} \bigcap_{P \notin D_{t-1}} h_t(P \cap D_{t-1})_{D_t} = \bigcap_{Q \in GP} Q.$$  

Thus

$$B_tP_BB_{t-1} = h_t(P_B \cap D_{t-1}) \subseteq \bigcap_{Q \in GP} h_t(Q \cap D_{t-1}) \subseteq \bigcap_{Q \in GP} Q = P_B,$$

where in the last inclusion we used the fact that $h_t$ extends $h_t \circ h_s$ (as elements of the partial action on Prim($B_e$)). Hence $P_B$ is $\mathcal{B}$-invariant and clearly $P_B \subseteq P$. Now suppose that $I$ is $\mathcal{B}$-invariant ideal contained in $P$. Then

$$ID_t = D_tID_t = B_tB_{t-1}IB_tB_{t-1} \subseteq B_tIB_{t-1} \subseteq B_tPB_{t-1}, \quad t \in G.$$

Thus $I \subseteq P_B$. \hfill $\square$

**Lemma 6.5.** The mapping Prim($B_e$) $\ni P \mapsto P_B \in$ Prim$^B(B_e)$ is continuous and open.

**Proof.** Let $J$ be an ideal in $B_e$. Since intersection of $\mathcal{B}$-invariant ideals is a $\mathcal{B}$-invariant ideal, there exists the smallest $\mathcal{B}$-invariant ideal in $B_e$ containing $J$. We denote it by $J^B$. Using Lemma 6.4 twice we get

$$P_B \nmid J \iff P_B \nmid J^B \iff P \nmid J^B,$$

for any $P \in$ Prim($B_e$). Thus any open set $U \subseteq$ Prim$^B(B_e)$ is of the form

$$U = \{ P_B : P_B \nmid J, P \in \text{Prim}(B_e) \} = \{ P_B : P \nmid J^B, P \in \text{Prim}(B_e) \},$$

where $J \in \mathcal{I}(B_e)$. It follows that the map $P \mapsto P_B$ is continuous. To verify openness, choose an open set $V = \{ P \in$ Prim($B_e$) : $P \nmid J \}$, $J \in \mathcal{I}(B_e)$, and let $U = \{ P_B : P_B \nmid J, P \in$ Prim($B_e$) $\}$. Clearly, if $P \in V$ then $P_B \in U$. On the other hand, if $P_B \in U$ then $h_t(P \cap D_{t-1})_{D_t} \nmid J$ for a certain $t \in G$ such that $P \nmid D_{t-1}$. Since $h_t(P \cap D_{t-1})_{D_t} \in V$ and $P_B = (h_t(P \cap D_{t-1})_{D_t})_B$, cf. (23), it follows that the image of $V$ is equal to $U$. \hfill $\square$

We can use the above lemma to identify the space Prim$^B(B_e)$ with the quasi-orbit space for the partial action $\{ \{ \text{Prim}(D_g) \}_{g \in G}, \{ h_g \}_{g \in G} \}$ on Prim($B_e$) dual to $\mathcal{B}$.
Lemma 6.6. Let $\mathcal{O}(\text{Prim}(B_e))$ be the quasi-orbit space associated to the partial action $\{\text{Prim}(D_t)\}_{t \in G}$, $\{h_t\}_{t \in G}$) of $G$ on $\text{Prim}(B_e)$, cf. Proposition 3.10. Then the map
\begin{equation}
\mathcal{O}(\text{Prim}(B_e)) \ni \mathcal{O}(P) \mapsto P_B \in \text{Prim}^B(B_e), \quad P \in \text{Prim}(B_e),
\end{equation}
is a homeomorphism. In particular, the closure of a point in $\mathcal{O}(\text{Prim}B_e)$ is given by
\[
\{\mathcal{O}(P)\} = \{\mathcal{O}(Q) : Q \supseteq P_B\}.
\]

Proof. Let $P, Q \in \text{Prim}(B_e)$. By (23), we get $\mathcal{O}(P) = \{K \in \text{Prim}(B_e) : K \supseteq P_B\}$. Therefore
\[
\mathcal{O}(P) = \mathcal{O}(Q) \iff P_B = Q_B.
\]
Thus (26) is a well-defined bijection, and we see that the quasi-orbit space $\mathcal{O}(\text{Prim}(B_e))$ may be identified with the quotient space $\text{Prim}(B_e)/\sim$ arising from the open continuous map $\text{Prim}(B_e) \ni P \mapsto P_B \in \text{Prim}^B(B_e)$, see Lemma 6.5. Hence (26) is a homeomorphism. For the last part of the assertion, note that the closure of $\{P_B\}$ in $\text{Prim}^B(B_e)$ is $\{Q_B : Q_B \supseteq P_B\}$.

Lemma 6.7. Let $\mathcal{B} = \{B_t\}_{t \in G}$ be a Fell bundle with $B_e$ separable. Then
\[
\text{Prim}^B(B_e) = \text{Prim}^B(B_e) \simeq \mathcal{O}(\text{Prim}(B_e)),
\]
where $\mathcal{O}(\text{Prim}(B_e))$ is the quasi-orbit space for $\{\text{Prim}(D_t)\}_{t \in G}$, $\{h_t\}_{t \in G}$.

Proof. Since $B_e$ is separable we have $\text{Prim}(B_e) = \text{Prim}^B(B_e)$. Let $P \in \text{Prim}(B_e)$. If $J_1, J_2 \in T^B(B_e)$ are such that $J_1 \cap J_2 \subset P_B$, then $J_1 \cap J_2 \subset P$, and hence $J_i \subset P$ for some $i \in \{1, 2\}$. But then $J_i \subset P_B$ by Lemma 6.4. Hence $P_B$ is $\mathcal{B}$-prime.

Conversely, let $I \in \text{Prim}^B(B_e)$. We need to show that there is a $P \in \text{Prim}(B_e)$ with $I = P_B$. For this it suffices to show that $\text{hull}(I) = \{Q \in \text{Prim}(B_e) : Q \supseteq I\}$ coincides with the closure $\mathcal{O}(I) = \{Q \in \text{Prim}(B_e) : Q \supseteq P_B\}$ of the orbit of some $P \in \text{Prim}(B_e)$. To this end, note that $\text{hull}(I)$ is closed and invariant, cf. Lemma 3.22. Thus $P \in \text{hull}(I)$ implies $\mathcal{O}(P) \subseteq \text{hull}(I)$. Accordingly, the equality $\mathcal{O}(P) = \text{hull}(I)$ is satisfied if and only if the set
\[
C := \{Q \in \mathcal{O}(Q) : Q \supseteq \text{hull}(I)\}
\]
is equal to the closure $\{\mathcal{O}(Q) : Q \supseteq P_B\}$ of the point $\mathcal{O}(P)$ in $\mathcal{O}(\text{Prim}(B_e))$. It is known that $\text{Prim}(B_e)$ is a totally Baire space and that an image of a totally Baire space under an open map is a totally Baire space, cf. discussion preceding [19] Lemma on page 222. Thus $\mathcal{O}(\text{Prim}(B_e))$ is a totally Baire space by Lemma 6.5. Also it is second countable because $B_e$ is separable. Thus, by virtue of [19] Lemma on page 222, to conclude that $\mathcal{O}(P) = \text{hull}(I)$ it suffices to show that $C$ is not a union of two proper closed subsets. So assume that $C_1, C_2$ are closed subsets of $\mathcal{O}(\text{Prim}(B_e))$ with $C = C_1 \cup C_2$. Let $F_1, F_2$ denote their inverse images in $\text{Prim}(B_e)$ and let $J_i := \bigcap_{Q \in F_i} Q$ for $i = 1, 2$. Then $\text{hull}(I) = F_1 \cup F_2$ which implies $J_1 \cap J_2 = I$. Since $I$ is $\mathcal{B}$-prime we have $J_i \subset I$ for some $i \in \{1, 2\}$. This implies that $F_i = \text{hull}(J_i) = \{Q \in \text{Prim}(B_e) : Q \supseteq J_i\}$ contains $\text{hull}(I)$, and this completes the proof.

We recall [16] Definition 27.5] that a Fell bundle $\mathcal{B} = \{B_t\}_{t \in G}$ is separable if $G$ is countable and each space $B_t$, $t \in G$, is separable. Of course, $\mathcal{B}$ is separable if and only if $C^*_a(\mathcal{B})$ (or any other $\mathcal{B}$-graded $C^*$-algebra) is separable. The following result can be viewed as a generalization of [10] Corollaries 2.6 and 2.7 and [18] Corollary 3.8 proved respectively for global actions on $C^*$-algebras and for partial actions on topological spaces.

Theorem 6.8. Retain the assumptions of Proposition 6.2 and additionally suppose that $\mathcal{B}$ is separable. Then we have natural homeomorphisms
\[
\text{Prim}(C^*_a(\mathcal{B})) \simeq \text{Prim}^B(B_e) = \text{Prim}^B(B_e) \simeq \mathcal{O}(\text{Prim}B_e).
\]
Proof. Since \( B \) is separable, \( \mathcal{C}^*(B) \) is separable and \( \text{Prim}(\mathcal{C}^*(B)) = \text{Prime}(\mathcal{C}^r(B)) \). Now it suffices to combine Proposition 6.2 and Lemmas 6.6 and 6.7. □

It is natural to treat the space \( \text{Prim}^B(B_e) \) (or equivalently \( \mathcal{O}(\text{Prim} B_e) \), see Lemma 6.6) as a ‟primitive spectrum” of the Fell bundle \( B \). For instance, by Proposition 3.2, ideals in \( B \) correspond to elements in \( \mathcal{T}^B(B_e) \) and we have the following simple fact.

Lemma 6.9. For every \( I \in \mathcal{T}^B(B_e) \) we have \( I = \bigcap_{P \in \text{Prim}^B(B_e)} P \).

Proof. Since \( I = \bigcap_{P \in \text{Prim}(B_e)} P \), it suffices to note, using Lemma 6.4, that \( I \subseteq P \) implies \( I \subseteq P_B \) for every \( I \in \mathcal{T}^B(B_e) \) and \( P \in \text{Prim}(B_e) \). □

7. Graph algebras

Throughout this section, we fix a directed graph \( E = (E^0, E^1, r, s) \). Hence \( E^0 \) and \( E^1 \) are countable sets and \( r, s : E^1 \to E^0 \) are range and source maps. For graphs and their \( C^* \)-algebras we use the notation and conventions of [14]. Thus \( E^\infty \) is the set of infinite paths and \( E^n, n > 0 \), stands for the set of finite paths \( \mu = \mu_1...\mu_n \), satisfying \( s(\mu_i) = r(\mu_{i+1}) \) for all \( i \), where \( |\mu| = n \) is the length of \( \mu \). If additionally \( r(\mu_1) = s(\mu_n) \) we say that \( \mu \) is a cycle. The cycle \( \mu \) have an entrance if there is an edge \( e \) such that \( r(e) = r(\mu_k) \) and \( e \neq \mu_k \), for some \( k = 1,...,n \). We say that \( E \) satisfies Condition (L) if every cycle in \( E \) has an entrance. A graph is said to satisfy Condition (K) if for every vertex \( v \in E^0 \), either there are no cycles based at \( v \), or there are two distinct cycles \( \alpha \) and \( \mu \) such that \( v = s(\alpha) = s(\mu) \) and neither \( v \) nor \( \mu \) is an initial subpath of the other.

The \( C^* \)-algebra \( \mathcal{C}^*(E) \) is generated by a universal Cuntz-Krieger \( E \)-family consisting of partial isometries \( \{s_e : e \in E^1\} \) and mutually orthogonal projections \( \{p_v : v \in E^0\} \) such that \( s^*_es_e = p_{s(e)} \), \( s_es^*_e \leq p_{r(e)} \) and \( p_v = \sum_{r(e)=v} s_es^*_e \) whenever the sum is finite (i.e. whenever \( v \) is a finite receiver). It follows that

\[
\mathcal{C}^*(E) = \overline{\text{span}}\{s_\mu s^*_\nu : \mu, \nu \in E^*\},
\]

where \( E^* = \bigcup_{n \in \mathbb{N}} E^n \), \( s_\mu = s_{\mu_1}s_{\mu_2}...s_{\mu_n} \) for \( \mu = \mu_1...\mu_n \in E^n, n > 0 \), and \( s_\mu = p_\mu \) for \( \mu \in E^0 \). We extend the maps \( r \) and \( s \) onto \( E^* \) in the obvious way. The algebra \( \mathcal{C}^*(E) \) is graded by the subspaces

\[
B_n := \overline{\text{span}}\{s_\mu s^*_\nu : \mu, \nu \in E^*, |\nu| - |\mu| = n\}, \quad n \in \mathbb{Z}.
\]

The only Fell bundle considered in this section will be \( B = \{B_n\}_{n \in \mathbb{Z}} \) defined above. The gauge-uniqueness theorem readily implies that \( \mathcal{C}^*(E) = \mathcal{C}^*(B) \). In particular, graded ideals in \( \mathcal{C}^*(E) \) coincide with gauge-invariant ideals in \( \mathcal{C}^*(B) \) and their description is known. We now use it to describe the corresponding \( B \)-invariant ideals in \( B_0 \). cf. Proposition 3.2.

For every \( v, w \in E^0 \) we write \( v \geq w \) if there is a path from \( w \) to \( v \). A subset \( H \) of \( E^0 \) is hereditary if \( v \in H \) and \( v \geq w \) imply \( w \in H \). A subset \( H \) of \( E^0 \) is saturated if every vertex \( v \) which satisfies \( 0 < |r^{-1}(v)| < \infty \) and \( r(e) = v \Rightarrow s(e) \in H \) itself belongs to \( H \). Given a saturated hereditary subset \( H \subseteq E^0 \), define

\[
H^\text{fin}_\infty := \{v \in E^0 \setminus H : |r^{-1}(v)| = \infty \text{ and } 0 < |r^{-1}(v) \cap s^{-1}(E^0 \setminus H)| < \infty\}.
\]

If \( v \in H^\text{fin}_\infty \) we write

\[
p_{v,H} := \sum_{r(e)=v, s(e) \notin H} s_es^*_e.
\]
For any $B \subseteq H^\text{fin}_x$ we put
\[ I_{H,B} := \text{span} \{ s_\alpha s_\beta^*: v \in H, w \in B, \alpha, \beta, \mu, \nu \in E^n, n \in \mathbb{N} \}. \]
Clearly, $I_{H,B}$ is an ideal in $B_0$ (in fact, it is a $\mathcal{B}$-invariant ideal). Let us consider the set
\[ \mathcal{H}_E := \{(H, B): H \text{ is saturated and hereditary, } B \subseteq H^\text{fin}_x \}. \]
We equip it with a partial order (actually a lattice) structure by writing
\[ (H, B) \preceq (H', B') \iff H \subseteq H' \text{ and } B \subseteq H' \cup B'. \]
The following description follows readily from the analysis in [5].

**Proposition 7.1.** The mapping $\mathcal{H}_E \ni (H, B) \mapsto I_{H,B} \in \mathcal{I}(\mathcal{B}_e)$ is a well-defined order preserving bijection (hence a lattice isomorphism).

**Proof.** In view of [5] Theorem 3.6 we see that the mapping under consideration is a bijection. Using [5] Corollary 3.10 we get that inclusion $I_{H,B} \subseteq I_{H',B'}$ holds if and only if $(H, B) \preceq (H', B')$. \[\square\]

**Remark 7.2.** It follows that $\mathcal{H}_E$ is a lattice and the meet operation is given by the formula
\[ (H, B) \land (H', B') = (H \land H', (H \land B') \cup (B \land H') \cup (B \land B')), \]
cf. [5] Proposition 3.9].

Having the above description of $\mathcal{B}$-invariant ideals in hand, we can show that the conditions introduced in the present paper are natural, can be verified in practice, and in general cannot be weakened. Let us start with the notion of residual aperiodicity.

**Proposition 7.3.** The following statements are equivalent:

(1) $\mathcal{B}$ is aperiodic.

(2) $\mathcal{B}$ has the intersection property.

(3) $E$ satisfies Condition $(L)$.

If $E$ is finite, i.e. both $E^0$ and $E^1$ are finite, then the above conditions are equivalent to

(4) the partial dynamical system $(\{\hat{D}_n\}_{n \in \mathbb{Z}}, \{\hat{h}_n\}_{n \in \mathbb{Z}})$ on $\hat{B}_0$ dual to $\mathcal{B}$ is topologically free.

**Proof.** By Corollary [13] we have (i) $\implies$ (ii). Implication (ii) $\implies$ (iii) belongs to the folklore in the field, see, for instance, the proof of [20] Lemma 2.1.

(iii) $\implies$ (i). Let $b = \sum_{\alpha, \beta \in E^*} \lambda_{\alpha, \beta} s_\alpha s_\beta^* \in B_n$, where $n \neq 0$, and let $A$ be a hereditary $C^*$-subalgebra of $B_0$. A moment of thought yields that to show condition (18) it suffices to consider the case when $n > 0$, the number of non-zero coefficients $\lambda_{\alpha, \beta} \in \mathbb{C}$ is finite and $A$ is a corner of the form
\[ (s_\mu s_\mu^*)_B(s_\mu s_\mu^*) = \text{span} \{ s_{\mu \nu} s_{\mu \nu}^*: \eta, \nu \in E^*, |\eta| = |\nu| \}, \]
where $\mu \in E^n$. In particular, we may further assume that all $\alpha$’s and $\beta$’s appearing in the sum
\[ b = \sum_{\alpha, \beta \in E^*} \lambda_{\alpha, \beta} s_\alpha s_\beta^* \]
start with the path $\mu$, that is there is a finite set
\[ F \subseteq \{(\alpha, \beta) \in \bigcup_{k \geq |\mu|} E^k \times E^{k+n}: s(\alpha) = s(\beta), \alpha = \mu \alpha', \beta = \mu \beta' \} \]
such that $b = \sum_{(\alpha, \beta) \in F} \lambda_{\alpha, \beta} s_\alpha s_\beta^*$. Let $(\alpha_0, \beta_0) \in F$ be such that $(\alpha, \beta) \in F$ implies $|\beta| \leq |\beta_0|$. If the vertex $s(\alpha_0) = s(\beta_0)$ does not lie on a cycle (of length $n$) then
\[ s_{\beta_0}^* s_\alpha = 0 \quad \text{for all } (\alpha, \beta) \in F. \]
Thus \(ab = 0\) with \(a = s_{\beta_0}s^*_\beta_0 \in A\). If \(s(\alpha_0) = s(\beta_0)\) lies on a cycle, then using the fact that this cycle has an entrance one can construct a path \(\eta = \eta_1\ldots\eta_|\eta|\) with \(|\eta| \geq |\beta_0|\) such that 
\[
\eta_1\ldots\eta|\beta_0| = \beta_0 \text{ and }
\eta|\beta_0| + 1\ldots\eta|\eta| \neq \eta|\alpha_0| + 1\ldots\eta|\eta| - n.
\]
Then for all \((\alpha, \beta) \in F\) we see that \(s^*_\eta s_\alpha s^*_\beta s_\eta\) is either zero, when \(\alpha \neq \eta_1\ldots\eta|\eta|\) or \(\beta \neq \eta_1\ldots\eta|\beta|\), or it is equal to \(s^*_\eta s_{|\eta|+1}\ldots\eta_{|\eta|+1}\ldots\eta|\eta|\), which is also zero by (27). Thus \(aba = 0\) with \(a = s_\eta s^*_\eta \in A\).

If \(E\) is finite then we have (iv) \(\Leftrightarrow\) (ii) by [26, Theorem 3.19(I)] modulo [26, Propositions 2.19 and 3.2] and the fact that the partial dynamical system \(((\hat{D}_n)_{n \in \mathbb{Z}}, \{\hat{h}_n\}_{n \in \mathbb{Z}})\) is generated by the single partial homeomorphism \(h_1\), cf. [27].

**Remark 7.4.** By Theorem 3.20 the implication (iv) \(\Rightarrow\) [(i) \(\Leftrightarrow\) (ii) \(\Leftrightarrow\) (iii)] in the above proposition is valid for an arbitrary graph \(E\). We suspect that the converse implication is also true and the proof would in essence require generalizing [26, Theorem 3.19], however this is beyond the scope of the present paper.

**Corollary 7.5.** The following statements are equivalent:

(i) \(B\) is residually aperiodic.

(ii) \(B\) has the residual intersection property.

(iii) \(E\) satisfies Condition (K).

If \(E\) is finite, that is both \(E^0\) and \(E^1\) are finite, then the above conditions are equivalent to

(iv) the partial dynamical system \(((\hat{D}_n)_{n \in \mathbb{Z}}, \{\hat{h}_n\}_{n \in \mathbb{Z}})\) dual to \(B\) is residually topologically free.

**Proof.** For any \((H, B) \in \mathcal{H}_E\) let \(J_{H,B}\) denote the ideal in \(C^*(E^*)\) generated by \(I_{H,B}\). By Proposition 7.1, every graded ideal in \(C^*(E^*)\) is of this form. By [5, Corollary 3.6] the quotient \(C^*(E^*)/J_{H,B}\) is naturally isomorphic to the graph \(C^*\)-algebra \(C^*((E/H)\backslash \beta(B))\) of a certain graph \((E/H)\backslash \beta(B))\), see [5]. Moreover, it is well known, and follows, for instance, from the proof of [5, Corollary 3.8], that \(E\) satisfies Condition (K) if and only if every graph \((E/H)\backslash \beta(B)\) satisfies Condition (L). Now the assertion follows from Proposition 7.3.

By Lemma 6.7 we have \(\text{Prime}^B(B_0) = \text{Prim}^B(B_0)\). We now turn to a description of this set. This will allow us, in the presence of condition (K), to deduce from Theorem 6.8 description of \(\text{Prim}(C^*(E))\) originally obtained using a different approach in [5, Corollary 4.8].

**Lemma 7.6.** Suppose that \(I_{H,B} \in T^E(B_0)\), \((H, B) \in \mathcal{H}_E\), belongs to \(\text{Prime}^B(B_0)\). Then \(M := E^0 \backslash H\) satisfies:

(a) If \(v \in E^0\), \(w \in M\), and \(v \geq w\) in \(E\), then \(v \in M\),

(b) If \(v \in M\) and \(0 < |r^{-1}(v)| < \infty\), then there exists \(e \in E^1\) with \(r(e) = v\) and \(s(e) \in M\),

(c) For every \(v, w \in M\) there exists \(y \in M\) such that \(v \geq y\) and \(w \geq y\).

**Proof.** Conditions (a), (b), say that \(H\) is hereditary and saturated. In the proof of [5, Lemma 4.1] it was shown that if condition (c) is not satisfied then \(I_{H,B} \notin \text{Prime}^B(B_0)\).

A non-empty subset \(M\) of \(E^0\) satisfying conditions (a), (b), (c) of Lemma 7.6 is called a maximal tail in \(E\). For any non-empty subset \(X\) of \(E^0\) we write

\[
\Omega(X) := \{w \in E^0 \backslash X : w \geq v \text{ for all } v \in X\}.
\]

We also put \(\Omega(v) := \Omega(\{v\})\). Note that \(\Omega(M) = E^0 \backslash M\) for any maximal tail \(M\). Moreover, for any \(v \in E^0\) that receives infinitely many edges, the set \(\Omega(v)\) is hereditary and saturated;
actually $\Omega(v)$ is a complement of a maximal tail. Any such vertex with the property that $v \in \Omega(v)_f$ is called a breaking vertex. In other words, $v \in E^0$ is a breaking vertex if and only if $|r^{-1}(v)| = \infty$ and $0 < |r^{-1}(v) \cap s^{-1}(\Omega(v))| < \infty$.

**Proposition 7.7.** The set $\text{Prime}^B(B_0) = \text{Prim}^B(B_0)$ consists of ideals $I_{\Omega(M),\Omega(M)}|_v$ associated to maximal tails $M$, and ideals $I_{\Omega(v),\Omega(v)|_v}$ associated to breaking vertices $v$.

**Proof.** Suppose that $I_{H,B}$ is in $\text{Prime}^B(B_0)$. By Lemma 7.6, we have $H = \Omega(M)$ where $M$ is a maximal tail. Note that the set $\Omega(M)|_v \setminus B$ cannot contain more than one vertex. Indeed, if we had two distinct vertices $v_1, v_2 \in \Omega(M)|_v \setminus B$, then we would get two $B$-invariant ideals $I_i := I_{\Omega(M),B \cup \{v_i\}}$, $i = 1, 2$, such that $I_1 \cap I_2 \subseteq I_{\Omega(M),B}$ and $I_i \not\subseteq I_{\Omega(M),B}$ for $i = 1, 2$. Suppose then that $B = \Omega(M)|_v \setminus \{v\}$ for a vertex $v \in \Omega(M)|_v$. If $v$ is not a breaking vertex then we get two $B$-invariant ideals $I_2 := I_{\Omega(v),\Omega(v)|_v}$ and $I_2 := I_{\Omega(M),\Omega(M)|_v}$ such that $I_1 \cap I_2 \subseteq I_{\Omega(M),B}$ and $I_i \not\subseteq I_{\Omega(M),B}$ for $i = 1, 2$, a contradiction. Hence $v$ must be a breaking vertex.

Let us now consider an ideal $I_{\Omega(M),B}$ where $M$ is a maximal tail and $B = \Omega(M)|_v$ or $B = \Omega(M)|_v \setminus \{v\}$ where $v$ is a breaking vertex such that $M = \Omega(v)$. We need to show that $I_{\Omega(M),B}$ is $B$-prime. Suppose that $I_{H_1,B_1}, I_{H_2,B_2} \in \mathcal{T}^B(B_0)$ are such that $I_{H_1,B_1} \cap I_{H_2,B_2} \subseteq I_{\Omega(M),B}$. By Proposition 7.1 and Remark 7.2 this is equivalent to saying that $H_1 \cap H_2 \subseteq \Omega(M) = E^0 \setminus M$ and

$$H_1 \cap B_2 \cup B_1 \cap H_2 \cup B_1 \cap B_2 \subseteq \Omega(M) \cup B.$$  

We claim that either $H_1 \subseteq \Omega(M)$ or $H_2 \subseteq \Omega(M)$. Indeed, if we assume on the contrary that there is $v \in H_1 \cap M$ and $w \in H_2 \cap M$, then taking $y \in M$ such that $v \geq y$ and $w \geq y$ we get $y \in H_1 \cap H_2 \cap M$, a contradiction.

Thus we may assume that $H_1 \subseteq \Omega(M)$. If $(H_1,B_1) \subseteq (\Omega(M),B)$, then $I_{H_1,B_1} \subseteq I_{\Omega(M),B}$. Suppose then that $(H_1,B_1) \not\subseteq (\Omega(M),B)$, and consider two cases:

1. Let $B = \Omega(M)|_v$. Since $H_1 \subseteq \Omega(M)$, $(H_1,B_1) \not\subseteq (\Omega(M),B)$ implies that there is $v \in B_1 \cap M$ such that $|r^{-1}(v) \cap s^{-1}(M)| = 0$. By properties (a) and (c) in Lemma 7.6 we see that $M = \{w \in E^0 : w \geq v\}$ (since $v$ is a `source’ in $M$, $v$ is a unique vertex with this property). In particular, if $H_2 \cap M \neq \emptyset$ then $v \in H_2$. However, in view of (28) we see that $v$ cannot belong neither to $H_2$ nor to $B_2$. Hence $H_2 \subseteq \Omega(M)$ and since $v \notin B_2$ we must have $(H_2,B_2) \leq (\Omega(M),B)$, that is $I_{H_2,B_2} \subseteq I_{\Omega(M),B}$.  

2. Let $B = \Omega(M)|_v \setminus \{v\}$, where $v$ is a breaking vertex and $\Omega(M) = \Omega(v)$. The last relation implies that $M = \{w \in E^0 : w \geq v\}$. Note that $B_1$ must contain $v$. Indeed, if $v \notin B_1$ then arguing as in case 1) we see that there is a unique $v' \in B_1 \cap M$ with the property that $M = \{w \in E^0 : w \geq v'\}$. This leads to a contradiction as $v \neq v'$. Thus $v \in B_1$. It follows from (28) that $v \notin H_2$ and $v \notin B_2$. Since $M = \{w \in E^0 : w \geq v'\}$ and $H_2$ is hereditary, we conclude that $H_2 \subseteq \Omega(M)$. Combining the last inclusion with $v \notin B_2$ we get that $(H_2,B_2) \leq (\Omega(M),B)$. Hence $I_{H_2,B_2} \subseteq I_{\Omega(M),B}$. \qed

**Corollary 7.8.** If the equivalent conditions in Corollary 7.2 hold, then the elements in the primitive spectrum $\text{Prim}(C^*(E))$ of the graph algebra $C^*(E)$ are in a bijective correspondence with maximal tails and breaking vertices in the graph $E$.

**Proof.** Combine Proposition 7.7 with Theorem 6.8 \qed

Now, we show that Theorem 5.13 is optimal in the sense that when applied to graph $C^*$-algebras our conditions implying pure infiniteness are not only sufficient but also necessary.
Theorem 7.9. The C*-algebra $C^*(E)$ of a directed graph $E$ is purely infinite if and only if the associated Fell bundle $\mathcal{B} = \{B_n\}_{n \in \mathbb{Z}}$ is residually aperiodic and every non-zero projection in is residually strictly $\mathcal{B}$-infinite.

Proof. The ‘if’ part follows from Theorem 5.13(ii). To show the ‘only if’ part, suppose that $C^*(E)$ is purely infinite. Every projection in $B_0$ is Murray-von Neumann equivalent to a projection of the form $\sum_{\alpha, \beta \in F} \lambda_{\alpha, \beta} s_\alpha s_\beta^*$ where $F \subseteq E^*$ is finite. The latter is in turn a finite sum of mutually orthogonal projections each of which is Murray-von Neumann equivalent to a projection of the form $s_\mu s_\mu^*$ with $\mu \in E^*$. Thus, by Lemma 5.15, it suffices to show that $a := s_\mu s_\mu^*$ is residually strictly $\mathcal{B}$-infinite. Let then $I_{H,B}$, $(H, B) \in H_E$, be a $\mathcal{B}$-invariant ideal in $B_0$. Suppose that $a \notin I_{H,B}$. Then $v := s(\mu) \in E^0 \setminus H$. Since $I_{H,B}$ is an intersection of $\mathcal{B}$-prime ideals, see Lemma 6.9, we see that $E^0 \setminus H$ is a sum of maximal tails, cf. Remark 7.2. Thus $v \in M$ for a certain maximal tail $M$ contained in $E^0 \setminus H$. By [20] Theorem 2.3, there is a path $\alpha$ in $M$ which connects a loop $\gamma$ in $M$ to $v$ and $\gamma$ has an entrance in $M$. This implies that $s_\mu s_\mu^* - s_\mu \gamma s_\mu^* s_\mu \gamma \notin I_{H,B}$. Moreover, putting

$$a_1 = s_{\mu \alpha} s_{\mu}^*, \quad a_2 = s_{\mu}^* - s_{\mu \alpha} s_{\mu}^*, \quad b = s_{\mu \alpha} s_{\mu}^* - s_{\mu \gamma} s_{\mu \gamma}^*,$$

one readily sees that $b \in aB_0a$, $a_1 \in aB_{-|\gamma|}$, $a_2 \in aB_0$ and

$$s_\mu s_\mu^* = a_1^* a_1 + a_2^* a_2, \quad a_1^* a_2 = a_2^* b = a_2^* b = 0.$$

Hence the image of $a$ in the quotient $B_0/I_{H,B}$ is $\mathcal{B}/\mathcal{J}$-infinite where $\mathcal{J} := \{B_n I_{H,B}\}_{n \in \mathbb{Z}}$. \qed

Remark 7.10. One could conjecture that if $C^*(E)$ is purely infinite then every projection in $B_e$ is $\mathcal{B}$-paradoxical (or at least is equivalent to a $\mathcal{B}$-paradoxical one). However, even for finite graphs this conjecture is very hard to verify. In particular, the above theorem illustrates the practical advantage of residual $\mathcal{B}$-infiniteness over $\mathcal{B}$-paradoxicality.

8. Crossed products by a semigroup of corner systems

In this section, we consider systems $(A, G^+, \alpha, L)$ studied in [32] when $A$ is a unital C*-algebra. We will generalize concepts of [32] to non-unital case. Combining them with the findings of [31] we reveal their internal structure and connections with other construction. Then we apply the results of the present paper to C*-algebras associated to $(A, G^+, \alpha, L)$.

8.1. Various pictures of semigroup corner systems. Throughout this subsection, we let $G^+$ be the positive cone of a totally ordered abelian group $G$ with the identity 0, that is we have:

$$G^+ \cap (-G^+) = \{0\}, \quad G = G^+ \cup (-G^+),$$

and we write $g \geq h$ if $g - h \in G^+$, $g, h \in G$. We fix a C*-algebra $A$. We denote by $\text{End}(A)$ the set of endomorphisms of $A$ and by $\text{Pos}(A)$ the space of positive maps on $A$. Composition of mappings yields a structure of a unital semigroup on $\text{End}(A)$ and $\text{Pos}(A)$, where the unit is the identity map. We always assume that a homomorphism between two semigroups with unit preserves the units. We recall that a multiplicative domain of $\varrho \in \text{Pos}(A)$ is a C*-algebra given by

$$\text{MD}(\varrho) = \{a \in A : \varrho(ba) = \varrho(b) \varrho(a), \quad \varrho(ab) = \varrho(a) \varrho(b), \quad \text{for all } b \in A\},$$

cf., for instance, [31] and references therein. We say that $\varrho : A \to A$ is extendible if it extends to a strictly continuous map $\overline{\varrho} : M(A) \to M(A)$.

An Exel system, originally defined in [13], is a triple $(A, \alpha, L)$ where $\alpha \in \text{End}(A)$ and $L \in \text{Pos}(A)$ are such that $L(\alpha(a)b) = aL(b)$, for all $a, b \in A$. Then $L$ is called a transfer...
operator for $\alpha$, and as shown in [31, Proposition 4.2], $L$ is automatically extendible. In [31, Definition 4.19], an Exel system $(A, \alpha, L)$ is called a corner system if $E := \alpha \circ L$ is a conditional expectation onto a hereditary $C^*$-subalgebra $\alpha(A)$ of $A$. By [31, Lemma 4.20], an Exel system $(A, \alpha, L)$ is a corner system if and only if $\alpha$ is extendible and
\[
\alpha(L(a)) = \overline{\alpha}(1)a\overline{\alpha}(1), \quad \text{for all } a \in A.
\]

Note that then both maps $\alpha$ and $L$ are extendible and $(M(A), \overline{\alpha}, \overline{L})$ is again a corner system.

In unital case, systems $(A, \alpha, L)$ satisfying (29) were called complete in [3]. One readily sees that an iteration $(A, \alpha^n, L^n)$, $n \in \mathbb{N}$, of an Exel system $(A, \alpha, L)$ is an Exel system, and if $(A, \alpha, L)$ is a corner system then $(A, \alpha^n, L^n)$ is a corner system (one can check (29) inductively). Thus these systems can be treated as semigroup dynamical systems with the underlying semigroup $\mathbb{N}$. We will consider systems over the more general semigroup $G^+$.

We will use [31, Lemma 4.20] to show that the following three ‘corner systems’ are actually different sides of the same ‘coin’. They form a subclass of Exel-Larsen systems introduced in [35], and they may be viewed as generalizations to the non-unital case of (finely representable) $C^*$-dynamical systems considered in [32].

Recall that a $C^*$-subalgebra $B$ of a $C^*$-algebra $A$ is a corner in $A$ if there is a projection $p \in M(A)$ such that $B = pAp$. In particular, an ideal $I$ in a $C^*$-algebra is a corner in $A$ if and only if it is complemented in $A$, that is if $A$ is a direct sum of $I$ and $I^\perp$.

**Definition 8.1.** Consider two semigroup homomorphisms $\alpha : G^+ \to \text{End}(A)$ and $L : G^+ \to \text{Pos}(A)$. We say that:

(i) $(A, G^+, \alpha, L)$ is a semigroup corner (Exel) system if for each $t \in G^+$, $(A, \alpha_t, L_t)$ is a corner system.

(ii) $\alpha$ is a semigroup of corner endomorphisms if each $\alpha_t$, $t \in G^+$, has a corner range and a complemented kernel (note that then $\alpha_t$ is necessarily extendible and $\alpha_t(A) = \overline{\alpha_t}(1)A\overline{\alpha_t}(1)$).

(iii) $L$ is a semigroup of corner retractions if for every $t \in G^+$, $L_t(A)$ is a complemented ideal in $A$ and the annihilator $(\ker L_t|_{MD(L_t)})^\perp$ of the kernel of $L_t : MD(L_t) \to L_t(A)$ is mapped by $L_t$ onto $L_t(A)$.

**Remark 8.2.** Suppose that $(A, G^+, \alpha, L)$ is a semigroup corner system. Clearly $t \mapsto \overline{\alpha}_t$ and $t \mapsto \overline{L}_t$ define semigroup homomorphisms $\overline{\alpha} : G^+ \to \text{End}(M(A))$ and $\overline{L} : G^+ \to \text{Pos}(M(A))$. Thus $(M(A), G^+, \overline{\alpha}, \overline{L})$ is a semigroup corner system. In particular, $(A, G^+, \alpha, L)$ is a $C^*$-dynamical system in the sense of [35] and $(M(A), G^+, \overline{\alpha}, \overline{L})$ is a dynamical system in the sense of [32]. Note also, cf. [32, 2.2], that
\[
\{\overline{\alpha}_t(1)\}_{t \in G^+} \subseteq M(A) \quad \text{and} \quad \{\overline{L}_t(1)\}_{t \in G^+} \subseteq Z(M(A))
\]
are non-increasing families of projections (the latter are central).

**Proposition 8.3.** Let $\alpha : G^+ \to \text{End}(A)$ and $L : G^+ \to \text{Pos}(A)$ be two maps. The following statements are equivalent:

(i) $(A, G^+, \alpha, L)$ is a semigroup corner system;

(ii) $\alpha$ is a semigroup of corner endomorphisms, and for every $t \in G^+$
\[
L_t(a) = \alpha_t^{-1}(\overline{\alpha}_t(1)a\overline{\alpha}_t(1)), \quad a \in A,
\]
where $\alpha_t^{-1}$ is the inverse to the isomorphism $\alpha_t : (\ker \alpha_t)^\perp \to \overline{\alpha_t}(1)A\overline{\alpha_t}(1)$;

(iii) $L$ is semigroup of corner retractions, and for every $t \in G^+$
\[
\alpha|_{L_t(A)^\perp} \equiv 0 \quad \text{and} \quad \alpha_t(a) = L_t^{-1}(a), \quad \text{for } a \in L(A),
\]

where $\alpha_t^{-1}$ is the inverse to the isomorphism $\alpha_t : (\ker \alpha_t)^\perp \to \overline{\alpha_t}(1)A\overline{\alpha_t}(1)$.
Moreover, since the above equivalent conditions hold, then \( \alpha \) and \( L \) determine each other uniquely, and
\[
(33) \quad \alpha_t(A) = \overline{\alpha_t(1)} A \overline{\alpha_t(1)} = (\ker L_t|_{MD(L_t)})^1, \quad L_t(A) = \overline{L_t(1)} A = (\ker \alpha_t)^1,
\]
for each \( t \in G^+ \).

**Proof.** By [31, Lemma 4.20], for each \( t \in G^+ \) the following statements are equivalent:
- \((A, \alpha_t, L_t)\) is a corner system;
- \( \alpha_t \) has a corner range, complemented kernel and \( L_t \) is given by (31);
- \( L_t(A) \) is a complemented ideal in \( A \), \( L_t((\ker L_t|_{MD(L_t)})^1) = L_t(A) \) and \( \alpha_t \) is given by (32);

and if these conditions are satisfied then (33) holds. Thus item (i) implies (ii) and (iii). To show (ii) \( \Rightarrow \) (i), we need to check that \( L_t \)'s given by (31) satisfy the semigroup law. To this end, note that \( \{ \overline{\alpha_t(1)} \}_{t \in G^+} \) form a semigroup of endomorphisms of \( M(A) \), and in particular \( \{ \overline{\alpha_t(1)} \}_{t \in G^+} \) is a non-increasing family of projections, cf. [32, Page 405]. Thus for any \( t, s \in G^+ \) and \( a \in A \) we have
\[
L_s(L_t(a)) = \alpha_s^{-1} (\overline{\alpha_t(1)} L_t(a) \overline{\alpha_s(1)}) = \alpha_s^{-1} (L_t(\overline{\alpha_t(1)} a \overline{\alpha_s(1)})) = \alpha_s^{-1} (\alpha_s^{-1}(\alpha_{s+t}(1))a \alpha_s(1))) = \alpha_s^{-1}(\alpha_{s+t}(1)a \alpha_s(1)) = L_{t+s}(a).
\]

Similarly, to show (iii) \( \Rightarrow \) (i) we need to check that \( \alpha_t 's \) given by (32) satisfy the semigroup law. To this end, note that \( \overline{\alpha_t(1)} \) is a central projection in \( M(A) \) such that \( \overline{\alpha_t(1)} A = L_t(A) \). In particular, \( \alpha_t \) is given by the formula
\[
\alpha_t(a) = L_t^{-1}(\overline{L_t(1)} a), \quad a \in A.
\]
Moreover, since \( L_{s+t}(A) = L_t(L_s(A)) \subseteq L_t(A) \) we conclude that the family \( \{ \overline{\alpha_t(1)} \}_{t \in G^+} \) is non-increasing. Now, using (29), for any \( t, s \in G^+ \) and \( a \in A \) we get
\[
\overline{\alpha_t(\overline{\alpha_s(1)})} = \overline{\alpha_t(\overline{\alpha_s(1)} \overline{\alpha_t(1)})} = \overline{\alpha_t(1)} \overline{\alpha_t(1)} = \overline{\alpha_t(1)} \overline{\alpha_t(1)} = \overline{\alpha_t(1)} \overline{\alpha_s(1)}.
\]
Hence
\[
\alpha_s(\alpha_t(a)) = L_s^{-1}(\overline{\alpha_t(1)} a) = L_s^{-1}(\overline{\alpha_t(1)} \overline{\alpha_t(1)} a) = L_s^{-1}(\overline{\alpha_t(1)} a) = L_s^{-1}(L_t^{-1}(\overline{\alpha_t(1)} a)) = L_s^{-1}(L_t^{-1}(\overline{\alpha_t(1)} a)) = L_s^{-1}(L_t^{-1}(\overline{\alpha_t(1)} a)) = \alpha_{s+t}(a).
\]

We also reveal a connection between semigroup corner systems \((A, G^+, \alpha, L)\) and the concept of an interaction group introduced in [15]. We emphasize that Exel in [15] considered arbitrary discrete groups but he assumed that the algebra \( A \) and the maps involved in an interaction are unital. Since the latter requirements applied to \((A, G^+, \alpha, L)\) would force \( \alpha \) and \( L \) to act by automorphisms, we consider a version of [15, Definition 3.1] where we drop the assumptions on units. We formulate it using our abelian group \( G \).

**Definition 8.4** (cf. Definition 3.1 of [15]). An interaction group is a triple \((A, G, \mathcal{V})\) where \( \mathcal{V} : G \to \text{Pos}(A) \) is a partial representation, that is, \( \mathcal{V}_0 = \text{id} \), and
\[
(34) \quad \mathcal{V}_g \mathcal{V}_h \mathcal{V}_{-h} = \mathcal{V}_{g+h} \mathcal{V}_{-h}, \quad \mathcal{V}_{-g} \mathcal{V}_g \mathcal{V}_h = \mathcal{V}_{-g} \mathcal{V}_{g+h}, \quad g, h \in G;
\]
and for each \( g \in G \) the pair \((V_g, V_{-g})\) is an interaction in the sense of [31], that is in view of [31] it suffices to require that

\[
V_g(A) \subseteq MD(V_{-g}), \quad \text{and} \quad V_{-g}(A) \subseteq MD(V_g),
\]

cf. [31] Remark 3.15.

**Proposition 8.5.** Suppose that \((A, G^+, \alpha, L)\) is a semigroup corner system. Then putting

\[
(35) \quad V_g := \begin{cases} 
\alpha_g & \text{if } g \geq 0, \\
L_{-g} & \text{if } g \leq 0,
\end{cases} \quad g \in G,
\]

we get an interaction group \( \mathcal{V} \).

**Proof.** It is tempting to apply Proposition 13.4 from [15]. However, its proof relies in an essential way on the assumption that the maps preserve units, which we dropped in our setting. Thus we need to prove it ad hoc. In particular, by [31] Proposition 4.13 we know that for each \( g \in G \), \((V_g, V_{-g})\) is an interaction. Hence it suffices to check (31). To this end, note that for \( t \in G^+ \) and \( a \in A \) we have

\[
(V_t V_{-t})(a) = \overline{\alpha}_t(1)a\overline{\alpha}_t(1), \quad (V_{-t} V_t)(a) = \overline{T}_t(1)a.
\]

We recall that the families of projections (31) are non-increasing. We proceed with a case by case proof:

1) If \( g, h \geq 0 \) or \( g, h \leq 0 \) then relations (31) hold by semigroup laws for \( \alpha \) and \( L \).

2) If \( g \leq 0 \leq h \) and \( h \leq -g \), then

\[
(V_g V_h V_{-h})(a) = L_{-g}(\overline{\alpha}_h(1)a\overline{\alpha}_h(1)) = L_{-g}(a) = L_{-g-h}(L_h(a)) = (V_{g+h} V_h)(a),
\]

\[
(V_{-g} V_g V_h)(a) = \overline{\alpha}_g(-g)(a)\overline{\alpha}_g(-g) = \overline{\alpha}_h(\overline{\alpha}_g(-g-h)(1))
\]

\[
= \alpha_h(\alpha_{g-h}(L_{-g-h}(a))) = \alpha_{-g}(L_{-g-h}(a)) = (V_{-g} V_{g+h})(a).
\]

3) If \( g \leq 0 \leq h \) and \( h \geq -g \), then

\[
(V_g V_h V_{-h})(a) = L_{-g}(\overline{\alpha}_h(1)a\overline{\alpha}_h(1)) = \overline{\alpha}_g+h(1)L_{-g}(a)\overline{\alpha}_g+h(1)
\]

\[
= \alpha_{g+h}(L_{g+h}(L_{-g}(a))) = \alpha_{g+h}(L_h(a)) = (V_{g+h} V_h)(a),
\]

\[
(V_{-g} V_g V_h)(a) = \overline{\alpha}_g(-g)(a)\overline{\alpha}_g(-g) = \alpha_{-g}(\alpha_{g+h}(a)) = (V_{-g} V_{g+h})(a).
\]

4) If \( h \leq 0 \leq g \) and \( g \geq -h \), then

\[
(V_g V_h V_{-h})(a) = \alpha_g(\overline{T}_{-h}(1)a) = \alpha_g(a) = \alpha_{g+h}(\alpha_{-h}(a)) = (V_{g+h} V_h)(a),
\]

\[
(V_{-g} V_g V_h)(a) = \overline{T}_g(1)L_{-h}(a) = L_{-h}(\overline{\pi}_{-h}(\overline{T}_g(1))a)
\]

\[
= L_{-h}(\overline{\pi}_{-h}(1)L_{g+h}(1)\overline{\pi}_{-h}(1)a) = L_{-h}(\overline{L}_{g+h}(1)a)
\]

\[
= L_{-h}(L_{g+h}(\alpha_{g+h}(a))) = L_{g}(\alpha_{g+h}(a)) = (V_{-g} V_{g+h})(a).
\]

5) If \( h \leq 0 \leq g \) and \( g \leq -h \), then

\[
(V_g V_h V_{-h})(a) = \alpha_g(\overline{T}_{-h}(1)a) = \overline{\alpha}_g(\overline{L}_{g-h}(1)a)\alpha_g(a)
\]

\[
= \overline{\alpha}_g(1)\overline{L}_{g-h}(1)\overline{\alpha}_g(1)\alpha_g(a) = \overline{\alpha}_{g-h}(1)\alpha_g(a)
\]

\[
= \alpha_{g-h}(\alpha_{-h}(a)) = (V_{g+h} V_h)(a),
\]

\[
(V_{-g} V_g V_h)(a) = \overline{T}_g(1)L_{-h}(a) = L_{-h}(a) = L_{g}(L_{g-h}(a)) = (V_{-g} V_{g+h})(a).
\]

□
8.2. Fell bundles and C*-algebras associated to semigroup corner systems. Let \((A, G^+, \alpha, L)\) be a semigroup corner system. The authors of [32] (who considered the case with \(A\) unital) associated to \((A, G^+, \alpha, L)\) a Banach *-algebra \(\ell^1(G^+, \alpha, A)\) and then constructed a faithful representation of \(\ell^1(G^+, \alpha, A)\) on a Hilbert space. This regular representation induces a pre-C*-norm on \(\ell^1(G^+, \alpha, A)\). Completion of \(\ell^1(G^+, \alpha, A)\) in this norm yields the crossed product \(A \rtimes_{\alpha} G^+\), see [32] Subsection 6.5]. In this subsection we generalize this construction to non-unital case, by making explicit the Fell bundle structure underlying the *-Banach algebra \(\ell^1(G^+, \alpha, A)\).

We associate to \((A, G^+, \alpha, L)\) a Fell bundle \(\mathcal{B} = \{B_g\}_{g \in G}\) defined as follows. For any \(t \in G^+\), we denote by \(\delta_t\) and \(\delta_{-t}\) abstract markers and consider Banach spaces
\[
B_t := \{a\delta_t : a \in A\overline{\alpha}(1)\}, \quad B_{-t} := \{a\delta_{-t} : a \in \overline{\alpha}(1)A\}
\]
naturally isomorphic to \(A\overline{\alpha}(1)\) and \(\overline{\alpha}(1)A\), respectively. We define the ‘star’ and the ‘multiplication’ operations by the formulas:
\[
(37) \quad (a\delta_g)^* := a^*\delta_{-g},
\]
\[
(38) \quad a\delta_g \cdot b\delta_h := \begin{cases} 
\alpha \alpha_g(b)\delta_{g+h}, & g, h \geq 0, \\
L_{-g}(ab)\delta_{g+h}, & g < 0, h \geq 0, g + h \geq 0, \\
L_h(ab)\delta_{g+h}, & g < 0, h \geq 0, g + h < 0, \\
\alpha_{g+h}(a)\delta_{g+h}, & h < 0, g \geq 0, g + h \geq 0, \\
\alpha_{g-h}(a)\delta_{g+h}, & h < 0, g \geq 0, g + h < 0, \\
\alpha_h(a)\delta_{g+h} & g, h < 0.
\end{cases}
\]

To understand where these relation come from, see Remark 8.11 below. The well-definiteness of (37) is clear, and with a little more effort, cf. the first part of proof of [32] Proposition 4.2], it can also be seen for (38).

Proposition 8.6. The family \(\mathcal{B} = \{B_g\}_{g \in G}\) of Banach spaces (36) with operations (37), (38) forms a Fell bundle.

Proof. The only not obvious axioms of a Fell bundle that we need to check, see, for instance [16] Definition 16.1], are:
\[
(39) \quad (a\delta_g \cdot b\delta_h)^* = (b\delta_h)^* \cdot (a\delta_g)^*, \quad a\delta_g \cdot (b\delta_h \cdot c\delta_f) = (a\delta_g \cdot b\delta_h) \cdot c\delta_f,
\]
for all \(a\delta_g \in B_g, b\delta_h \in B_h, c\delta_f \in B_f, g, h, f \in G^+\). The first relation in (39) follows from the first part of the proof of [32] Theorem 4.3], but can also be inferred directly from (38). The second relation in (39) was checked in the second part of the proof of [32] Theorem 4.3] in the case when \(g + h + f \geq 0\). The case \(g + h + f \leq 0\) can be covered by passing to adjoints. □

Definition 8.7. We call \(\mathcal{B} = \{B_g\}_{g \in G}\) constructed above the Fell bundle associated to the corner system \((A, G^+, \alpha, L)\). We define the crossed product for \((A, G^+, \alpha, L)\) to be
\[
A \rtimes_{\alpha,L} G^+ := C^*(\mathcal{B}),
\]
the cross sectional C*-algebra of \(\mathcal{B}\). We will identify \(A\) with \(B_0 \subseteq A \rtimes_{\alpha,L} G^+\).

The crossed product \(A \rtimes_{\alpha,L} G^+\) can be viewed as a crossed product for the semigroup \(\alpha\), for the semigroup \(L\), or as a crossed product for the group interaction \(\mathcal{V}\) given by (35). To show this we use the following lemma, which is of its own interest. It is related to the problem of extension of a representation of a semigroup to a partial representation of a group, studied for instance in [16], cf. [16] Definition 31.19].
Lemma 8.8. Any semigroup homomorphism $U : G^+ \to B$ to a multiplicative subsemigroup of a $C^*$-algebra $B$ consisting of partial isometries extends to a $*$-partial representation $V : G \to B$ of $G$.

Proof. Assume that $U : G^+ \to B$ is a semigroup homomorphism attaining values in partial isometries. We only need to show, cf. [16, 9.2], that putting $V_t := U_t$ and $V_{-t} := U_t^*$, for $t \in G^+$, we have

$$V_gV_hV_{-h} = V_{g+h}V_{-h}, \quad g, h \in G.$$ 

Since the product of two partial isometries is a partial isometry if and only if the corresponding initial and final projections commute, see [16, Proposition 12.8], we conclude that the projections $U_tU_t^*, U_sU_s^*$ commute for all $s, t \in G^+$. We have the following cases:

1) If $g, h \geq 0$ or $g, h \leq 0$ then $V_gV_hV_{-h} = V_{g+h}V_{-h}$ holds because $V$ is a semigroup homomorphism when restricted to $G^+$ or to $-G^+$.

2) If $g \leq 0 \leq h$ and $h \leq -g$, then

$$V_gV_hV_{-h} = (U_{-g}U_{-h}^*)U_hU_h^* = U_{g-h}U_{g-h}^* = V_{g+h}V_{-h}.$$ 

3) If $g \leq 0 \leq h$ and $h \geq -g$, then

$$V_gV_hV_{-h} = U_{-g}U_{-h}^*U_hU_h^* = (U_{-g}U_{-h})U_{g+h}U_{g+h}^*U_{-h} = (U_{g+h}U_{g+h}^*)U_{g+h}U_{g+h}^*U_{-h} = V_{g+h}V_{-h}.$$ 

4) If $h \leq 0 \leq g$ and $g \geq -h$, then

$$V_gV_hV_{-h} = U_gU_{g-h}^*U_{-h}^*U_{-h} = U_{g+h}U_{g+h}^*U_{g+h}^*U_{-h} = V_{g+h}V_{-h}.$$ 

5) If $h \leq 0 \leq g$ and $g \leq -h$, then

$$V_gV_hV_{-h} = U_gU_{g-h}^*U_{-h}^*U_{-h} = U_g(U_{g}U_{g-h})U_{g} = (U_{g}U_{g}^*)(U_{g-h}U_{g-h})U_{g} = (U_{g-h}U_{g-h}^*U_{g})U_{g} = U_{g-h}U_{g-h}^*U_{g} = V_{g-h}V_{-h}.$$ 

□

Proposition 8.9. Suppose $(A, G^+, \alpha, L)$ is a corner system. Let $\pi : A \to B(H)$ be a non-degenerate representation and let $U : G^+ \to B(H)$ be a mapping. The following statements are equivalent:

(i) $U$ is a semigroup homomorphism, and

$$U_t\pi(a)U_t^* = \pi(\alpha_t(a)), \quad U_t^*\pi(a)U_t = \pi(L_t(a)), \quad \text{for all } a \in A, t \in G^+.$$ 

(ii) $U$ is a semigroup homomorphism, and

$$U_t\pi(a)U_t^* = \pi(\alpha_t(a)), \quad \pi((\ker \alpha_t)^\perp) \subseteq U_t^*U_t\pi(A), \quad \text{for all } a \in A, t \in G^+.$$ 

(iii) $U$ is a semigroup homomorphism, and

$$U_t^*\pi(a)U_t = \pi(L_t(a)), \quad \pi((\ker L_t|_{MD(L_t)})^\perp) \subseteq U_t\pi(A)U_t^*, \quad \text{for all } a \in A, t \in G^+.$$ 

(iv) $U$ extends to a $*$-partial representation $V : G \to B(H)$ of $G$ such that

$$V_g\pi(a)V_{-g}^* = \pi(V_g(a)), \quad \text{for all } a \in A, g \in G,$n

where $V$ is the group interaction given by (35).
Proof. (i)⇔(ii). This follows from Proposition 4.2.

(i)⇒(iii). This is clear because \( \ker L_t|_{MD(L_t)} = \alpha_t(A) \), see (53).

(iii)⇒(i). Let \( a \in A \) and \( t \in G^+ \). Denote by \( \pi : M(A) \to B(H) \) the unique extension of \( \pi \) to a representation of \( M(A) \). Note that \( \pi(T_t(1)) = U^*_tU_t \), so in particular \( U_t \) is a partial isometry. Thus in view of (32) and (33) we get

\[
\pi(\alpha_t(a)) = U_tU^*_t\pi(\alpha_t(a))U_tU^*_t = \pi(L_t(\alpha_t(a))) = U_t\pi(T_t(1)a)U^*_t = U_t\pi(a)U^*_t.
\]

(i)⇒(iv). This follows from Lemma 8.8 since \( \pi(T_t(1)) = U^*_tU_t \) is a projection and hence \( U_t \) is a partial isometry, for every \( t \in G^+ \).

(iv)⇒(i). For \( t \in G^+ \), \( U_t \) is a partial isometry and we necessarily have \( V_t = U^*_t \). Moreover, \( \pi(T_t(1)) = U^*_tU_t \) and hence \( \{U^*_tU_t\}_{t \in G^+} \) forms a non-decreasing family of projections. Using this, for any \( t, s \in G^+ \), we get

\[
U_tU_s = U_tU_sU^*_sU_s = U_{t+s}U^*_sU_s = U_{t+s}.
\]

This proves the equivalence of conditions (i)-(iv).

\[\square\]

Theorem 8.10. For any corner system \((A, G^+, \alpha, L)\) there is a semigroup homomorphism \( a : G^+ \to M(A \rtimes_{\alpha, L} G^+) \) taking values in partial isometries such that we have

\[
u_tau_t^* = \alpha_t(a), \quad u_t^*au_t = L_t(a), \quad a \in A, \ t \in G^+, \quad (40)
\]

and the elements of the form

\[
a = \sum_{t \in F} u_t^*a_{-t} + a_0 + \sum_{t \in F} a_tu_t, \quad F \subseteq G^+ \setminus \{0\}, \quad |F| < \infty,
\]

where \( a_t \in A\bar{T}_t(1), \ a_{-t} \in \bar{T}_t(1)A, \) form a dense \( * \)-subalgebra of \( A \rtimes_{\alpha, L} G^+ \).

The crossed product \( A \rtimes_{\alpha, L} G^+ \) is a universal \( C^* \)-algebra for pairs \((\pi, U)\) satisfying the equivalent conditions (i)-(iv) in Proposition 8.9, in the sense that for any such pair the assignments

\[
(\pi \times U)(a_tu_t) \mapsto \pi(a_t)U_t, \quad (\pi \times U)(u_t^*a_{-t}) \mapsto U^*_t\pi(a_{-t}), \quad t \in G^+, \quad (41)
\]

extend to a non-degenerate representation \( \pi \times U \) of \( A \rtimes_{\alpha, L} G^+ \) and every non-degenerate representation of \( A \rtimes_{\alpha, L} G^+ \) arises this way.

Proof. Recall that we identify \( B_0 = A\delta_0 \) with \( A \). In particular, \( A \) is a non-degenerate \( C^* \)-subalgebra of \( A \rtimes_{\alpha, L} G^+ \), that is \( A(A \rtimes_{\alpha, L} G^+) = A \rtimes_{\alpha, L} G^+ \). Moreover, if we let \( t \in G^+ \) and \( \{\mu_\lambda\} \) be an approximate unit in \( A \), then using (38) we get

\[
\lim_{\lambda} \mu_\lambda \delta_t \delta_0 = \mu_\lambda \alpha_t(a) \delta_t = \alpha_t(a) \delta_t \quad \text{and} \quad \lim_{\lambda} (\mu_\lambda \delta_t)^* \delta_0 = \bar{T}_t(1) \mu_\lambda a \delta_{-t} = \bar{T}_t(1) a \delta_{-t}.
\]

Therefore, we conclude that \( \lim_{\lambda} \mu_\lambda \delta_t \) converges strictly to an element of \( M(A \rtimes_{\alpha, L} G^+) \). Let us denote it by \( u_t \), and note that we have \( u_t \delta_0 = \alpha_t(a) \delta_t \) and \( u_t^* \delta_0 = \bar{T}_t(1) a \delta_{-t} \). For any \( t, s \in G^+ \) we have

\[
u_tu_s(\delta_0) = u_t\alpha_s(\delta_0) = \lim_{\lambda} \mu_\lambda \delta_t \alpha_s(\delta_0) = \lim_{\lambda} \mu_\lambda \alpha_t(\alpha_s(a)) \delta_{t+s} \quad \alpha_t(a) \delta_t = \lim_{\lambda} \mu_\lambda \alpha_t(a) \delta_{t+s} = u_{t+s}(a \delta_0).
\]

It follows that \( G^+ \ni t \mapsto u_t \in M(A \rtimes_{\alpha, L} G^+) \) is a semigroup homomorphism. We have

\[
u_t \delta_0 u_t^* = \alpha_t(a) \delta_0, \quad u_t^* \delta_0 u_t = L_t(a) \delta_0, \quad a \in A, \ t \in G^+,
\]
which follows from the following calculations, where $b \in A$ is arbitrary:
\[
\begin{align*}
u_t(a\delta_b)u_t^*(b\delta_0) &= \alpha_t(a)\delta_t \cdot \overline{\alpha_t}(1)b\delta_{-t} = \alpha_t(a)\overline{\alpha_t}(1)b\delta_0 = \alpha_t(a)\delta_t \cdot b\delta_0, \\
u_t^*(a\delta_b)u_t(b\delta_0) &= \overline{\alpha_t}(1)a\delta_{-t} \cdot \alpha_t(b)\delta_t = L_t(\overline{\alpha_t}(1)a\alpha_t(b))\delta_0 = L_t(a)\delta_0 = L_t(a)\delta_0 \cdot b\delta_0.
\end{align*}
\]
This proves the first part of the assertion, because we clearly have $a_0\delta_t = a_tu_t$ and $a_{-t}\delta_{-t} = u_{-t}a_{-t}$ for any $a_0\delta_t \in B_t$, $a_{-t}\delta_{-t} \in B_{-t}$, $t \in G^+$. Assume now that the pair $(\pi, U)$ satisfies equivalent conditions (i)-(iv). In view of Definition \ref{def:representation}, we see that the maps in \eqref{eq:representation} extend to a representation of $A \rtimes_{\alpha, L} G^+$ if it suffices to check that
\[
(\pi \times U)(a\delta_g)^* = (\pi \times U)((a\delta_g)^*), \quad (\pi \times U)(a\delta_g \cdot b\delta_h) = (\pi \times U)(a\delta_g)(\pi \times U)(a\delta_h),
\]
for all $a\delta_g \in B_g$ and $b\delta_h \in B_h$, $g, h \in G$. The first of the above relations is clear and the second one is shown in the proof of \cite[Proposition 5.3]{kwaśniewski2018}*.

If $\sigma : A \rtimes_{\alpha, L} G^+ \rightarrow B(H)$ is an arbitrary non-degenerate representation it extends uniquely to a representation $\overline{\sigma} : M(A \rtimes_{\alpha, L} G^+) \rightarrow B(H)$. Clearly, putting
\[
\pi(a) := \sigma(a\delta_0), \quad a \in A, \quad U_t := \overline{\sigma}(u_t), \quad t \in G^+,
\]
we get a pair $(\pi, U)$ satisfying equivalent conditions (i)-(iv) and such that $\sigma = \pi \times U$. \hfill \Box

**Remark 8.11.** Theorem \ref{thm:structure} allows us to assume the following ‘dictionary’ determining the structure of $A \rtimes_{\alpha, L} G^+$: we have
\[
\alpha_0(1)\delta_t = au_t, \quad (\overline{\alpha_t}(1)a)\delta_{-t} = u_t^*a, \quad t \in G^+, \ a \in A,
\]
and then multiplication of these spanning elements is determined by relations \eqref{eq:commutation_relations}. In particular, these relations imply the following commutation relations:
\[
au_t^* = u_t^*\alpha_t(a), \quad u_t a = \alpha_t(a)u_t, \quad a \in A, \ t \in G^+.
\]
Note also that the above description makes explicit the fact that the $C^*$-algebra $A \rtimes_{\alpha, L} G^+$ coincides with the crossed product studied in \cite[when $A$ is unital, see \cite[Theorem 5.4]{kwaśniewski2018}].

**Corollary 8.12.** Let $(A, G^+, \alpha, L)$ be a corner system. The crossed product $A \rtimes_{\alpha, L} G^+$ is naturally isomorphic to Exel-Larsen crossed product introduced in \cite[Definition 2.2]{exel2016}.

**Proof.** By \cite[Proposition 4.3]{exel2016}, we may identify Exel-Larsen crossed product for $(A, G^+, \alpha, L)$ with the quotient $C^*$-algebra $\mathcal{T}_X / \mathcal{I}_K$ where $\mathcal{T}_X$ is the Toeplitz algebra of a product system $X = \bigcup_{t \in G^+} X_t$ naturally associated to $(A, G^+, \alpha, L)$, see \cite[Proposition 3.1]{exel2016}, and $\mathcal{I}_K$ is an ideal in $\mathcal{T}_X$ generated by differences
\[
i(t) - i_X(t)(\phi_t(a)), \quad \text{for } a \in K_t := A_0(A)A \cap \phi_t^{-1}(\mathcal{K}(X_t)) \text{ and } t \in G^+,
\]
where $i_X : X \rightarrow \mathcal{T}_X$ is a universal representation of $X$ in $\mathcal{T}_X$, $i_X^{(t)} : \mathcal{K}(X_t) \rightarrow \mathcal{T}_X$ is the associated homomorphism, and $\phi_t : A \rightarrow L(X_t)$ is the left action of $A$ on $X_t$, $t \in G^+$. For more details see \cite[Proposition 4.3]{exel2016}. As shown in \cite[Proposition 4.3]{exel2016}, there exists a semigroup homomorphism $i_{G^+} : G^+ \rightarrow M(\mathcal{T}_X)$ such that putting $i_{A} := i_{G^+}|_{\{0\}} = i_X|_{A}$ we get that
\[
\mathcal{T}_X = C^*(\bigcup_{t \in G^+} i_A(A)i_{G^+}(t)), \quad \text{and} \quad i_{G^+}(t)^*i_A(a)i_{G^+}(t) = i_A(L_t(a)), \quad a \in A, \ t \in G^+.
\]
Moreover, the latter picture of $\mathcal{T}_X$ is universal (note that the author of \cite{exel2016} considers also additional relations $i_{G^+}(t)i_A(a) = i_A(\alpha_t(a))i_{G^+}(t), \ a \in A, \ t \in G^+$, but by \cite[Proposition 4.3]{kwaśniewski2018} they are automatic). In particular, the assignments
\[
i_A(a)i_{G^+}(t) \mapsto au_t, \quad a \in A, \ t \in G^+,
\]
Lemma 3.12], shows that \( \varphi \colon A \cong \).

Accordingly, \( a \) and \( b \) are a semigroup corner system. For each \( t \in G^+ \), let now \( \pi \colon \mathcal{T}_X/\mathcal{I}_K \to A \times_{\alpha,L} G^+ \). By the universality of \( A \times_{\alpha,L} G^+ \), \( \Psi \) admits an inverse. Hence \( \mathcal{T}_X/\mathcal{I}_K \cong A \times_{\alpha,L} G^+ \). \( \square \)

8.3. Partial dynamical systems dual to semigroup corner systems. Let \( (A, G^+, \alpha, L) \) be a semigroup corner system. For each \( t \in G^+ \), \( L_t(A) \) is an ideal in \( A \) and \( \alpha_t(A) \) is a hereditary subalgebra of \( A \). Thus we may treat \( L_t(A) \) and \( \alpha_t(A) \) as open subsets of \( \hat{A} \). Then the mutually inverse isomorphisms \( \alpha_t : L_t(A) \to \alpha_t(A) \) and \( L_t : \alpha_t(A) \to L_t(A) \) give rise to partial homeomorphisms of \( \hat{A} \):

\[
\hat{\alpha}_t([\pi]) := [\pi \circ \alpha_t], \quad \hat{L}_t([\pi]) := [\pi \circ L_t],
\]

cf. [29] Section 4.5 for a detailed description of these maps. Using the group interaction we can express it in a more symmetric way. Namely, we have homeomorphisms

\[
\hat{V}_g : \widetilde{V}_g(A) \to \mathbb{V}_g(A) \quad \text{where} \quad \hat{V}_g([\pi]) = [\pi \circ V_g], \quad g \in G,
\]

and we assume the identification \( \widetilde{V}_g(A) = \{[\pi] \in \hat{A} : \pi(V_g(A)) \neq 0\} \).

Lemma 8.13. The family \( \{\{V_g(A)\}_{g \in G}, \{\hat{V}_g\}_{g \in G}\} \) is a partial action on \( \hat{A} \) which coincides with the opposite to the partial action dual to the Fell bundle \( \mathcal{B} = \{B_g\}_{g \in G} \) associated to \( (A, G^+, \alpha, L) \).

In particular, \( \{\{\mathbb{V}_g(A)\}_{g \in G}, \{\hat{V}_g\}_{g \in G}\} \) is a lift of a partial action \( \{\{\text{Prim}(V_g(A))\}_{g \in G}, \{\mathbb{V}_g\}_{g \in G}\} \) on \( \text{Prim}(A) \) given by

\[
\hat{V}_g(P) = A\mathbb{V}_g(P)A, \quad g \in G;
\]

in other words, \( \hat{V}_{-g}(P) = A\alpha_t(P)A \) and \( \hat{V}_t(P) = L_t(P), \) for \( t \in G^+ \).

Proof. For any \( t \in G^+ \), we have \( A\alpha_t(A)A = A\overline{\alpha_t}(1)A = B_t \cdot B_{-t} = D_t \) and \( L_t(A) = L_t(\overline{\alpha_t}(1)A\alpha_t(1)) = B_{-t} \cdot B_t = D_{-t} \). Thus, with our identifications, we have \( \hat{L}_t(A) = \hat{D}_{-t} \) and

\[
\overline{\alpha_t}(A) = \{[\pi] \in \hat{A} : \pi(\alpha_t(A)) \neq 0\} = \{[\pi] \in \hat{A} : \pi(A\alpha_t(A)A) \neq 0\} = \hat{D}_t.
\]

Let now \( \pi : A \to B(H) \) be an irreducible representation such that \( \pi(\alpha_t(A)) \neq 0 \). Then \( \hat{\alpha}_t([\pi]) \) is the equivalence class of the representation \( \pi \circ \alpha_t : A \to B(\pi(\alpha_t(A))H) \). For any \( a_i\delta_{-i} \in B_{-i} = \overline{\alpha_t}(1)A\delta_{-i}, \) for \( t \in H, \) \( i = 1, \ldots, n \), we have

\[
\| \sum_i a_i\delta_{-i} \otimes_\pi h_i \|^2 = \sum_{i,j} \langle h, \pi(a_i^*a_j)h \rangle_p = \sum_i \| \pi(a_i)h_i \|^2.
\]
Since $\pi(\alpha_t(A))H = \pi(\overline{\pi(1)A})H$, we see that $a\delta_t \otimes h \mapsto \pi(a)h$ yields a unitary operator $U : B_{-t} \otimes H \rightarrow \pi(\alpha_t(A))H$. Furthermore, for $a \in A$, $b \in \alpha_t(A)$ and $h \in H$ we have

$$
\left( B_{-t} - \text{Ind}_{D_{-t}}^{D_t}(\pi)(a)U^* \right) \pi(b)h = B_{-t} - \text{Ind}_{D_{-t}}^{D_t}(\pi)(a)(b\delta_{-t} \otimes h) = (\alpha_t(b)a) \otimes h
$$

Hence $U$ intertwines $B_{-t} - \text{Ind}_{D_{-t}}^{D_t}$ and $\pi \circ \alpha_t$. Accordingly, $h_{-t} = \hat{\alpha}_t$ and $\hat{h}_t = \hat{\alpha}_t^{-1} = \hat{\alpha}_t$. This proves the first part of the assertion. To show the second part, we use the ‘dictionary’ from Remark 8.11. Then it is immediate that the corresponding Rieffel homeomorphisms, cf. (17), are given by

$$
h_t(I) = (Au_t)I(u^*_t A) = A\alpha_t(I)A,
$$

$$
h_{-t}(I) = (u^*_t A)I(Au_t) = L_t(I) = AL_t(I)A,
$$

for any $I \in \mathcal{I}(A)$ and $t \in G^+$. \hfill \qed

**Definition 8.14.** We call $(\{\widehat{V}_g(A)\}_{g \in G}, \{\widehat{\nu}_g\}_{g \in G})$ and $(\{\text{Prim}(V_g(A))\}_{g \in G}, \{\widehat{\nu}_g\}_{g \in G})$ described above partial dynamical systems dual to the interaction $\mathcal{V}$.

Before we state the main result of this subsection we need a lemma and a definition.

**Lemma 8.15.** If $I \in \mathcal{I}(A)$, then the following conditions are equivalent:

(i) $\alpha_t(I) = \pi_t(1)/\pi_t(1)$ for every $t \in G^+$,

(ii) $L_t(I) = \mathcal{T}_t(1)I$ for every $t \in G^+$,

(iii) $\mathcal{V}_g(I) \subseteq I$ for every $g \in G$, where $\mathcal{V}$ is the group interaction

(iv) $\mathcal{T}$ is invariant under the partial action $(\{\overline{\mathcal{V}_g(A)}\}_{g \in G}, \{\widehat{\nu}_g\}_{g \in G})$.

In particular, if the above equivalent conditions hold, then we have a quotient semigroup corner system $(A/I, G^+, \alpha^I, L^I)$ where $\alpha^I_t(a) = a + I$ and $L^I_t(a) = a + I$, and its associated group interaction is given by $\mathcal{V}_g(a) = a + I$, for $a \in A$, $t \in G^+$, $g \in G$.

**Proof.** The equivalence of conditions (i)-(iv) follows from [26, Lemma 2.22], which was proved for unital $A$ but the proof carries over to the general case. The second part of the assertion is now clear. \hfill \qed

**Definition 8.16.** If $I \in \mathcal{I}(A)$ satisfies the equivalent conditions (i)-(iv) in Lemma 8.15 we call $I$ an invariant ideal for $(A, G^+, \alpha, L)$.

**Theorem 8.17.** Let $(A, G^+, \alpha, L)$ be a semigroup corner system, $\{\mathcal{V}_g\}_{g \in G}$ its associated group interaction, and $\mathcal{V} = (\{\overline{\mathcal{V}_g(A)}\}_{g \in G}, \{\widehat{\nu}_g\}_{g \in G})$ and $\mathcal{V} = (\{\text{Prim}(V_g(A))\}_{g \in G}, \{\widehat{\nu}_g\}_{g \in G})$ the dual partial actions. Then

(i) If $\mathcal{V}$ is topologically free, then every pair $(\pi, U)$ satisfying equivalent conditions (i)-(iv) in Proposition 8.3 such that $\pi$ is faithful gives rise to a faithful representation $\pi \times U$ of $A \rtimes_{\alpha, L} G^+$.

(ii) If $\mathcal{V}$ is residually topologically free, then the map $J \mapsto J \cap A$ is a homeomorphism from $\mathcal{I}(A \rtimes_{\alpha, L} G^+)$ onto the subspace of $\mathcal{I}(A)$ consisting of invariant ideals for $(A, G^+, \alpha, L)$.

(iii) If $\mathcal{V}$ is residually topologically free, $A$ is separable and $G^+$ is countable then we have a homeomorphism

$$
\text{Prim}(A \rtimes_{\alpha, L} G^+) \cong \mathcal{O}(\text{Prim} A),
$$

where $\mathcal{O}(\text{Prim} A)$ is the quasi-orbit space associated to $\mathcal{V}$. 
Proof. Since \( G \) is amenable we have \( A \rtimes_{\alpha,L} G^+ = C^*(\mathcal{B}) = C^*_r(\mathcal{B}) \) for the associated Fell bundle.

(i) By Lemma 8.13 and Theorem 3.20, \( A \rtimes_{\alpha,L} G^+ = C^*_r(\mathcal{B}) \) has the intersection property. Since \( \ker(\pi \times U) \cap A = \ker \pi = \{0\} \), we conclude that \( \ker(\pi \times U) = \{0\} \).

(ii) Apply Lemma 8.13 and Corollary 3.23.

(iii) Apply Lemma 8.13 and Theorem 6.8.

\[ \square \]

8.4. Purely infinite crossed products for semigroup corner systems. We fix a semigroup corner system \((A, G^+, \alpha, L)\). Let \( \mathcal{B} = \{B_g\}_{g \in G} \) be its associated Fell bundle, and \( \mathcal{V} = \{\mathcal{V}_g\}_{g \in G} \) be its associated group interaction.

**Lemma 8.18.** The following conditions are equivalent

(i) \( \mathcal{B} = \{B_g\}_{g \in G} \) is aperiodic,

(ii) for each \( t \in G^+ \setminus \{0\} \), each \( a \in A \) and every hereditary subalgebra \( D \) of \( A \)

\[ \inf \{\|d \alpha_t(d)\| : d \in D, \ d \geq 0, \ \|d\| = 1\} = 0. \]

(iii) for each \( t \in G^+ \setminus \{0\} \), each \( a \in A \) and every hereditary subalgebra \( D \) of \( A \)

\[ \inf \{\|d L_t((da)^* da)\| : d \in D, \ d \geq 0, \ \|d\| = 1\} = 0. \]

(iv) for each \( g \in G \setminus \{0\} \), each \( a \in A \) and every hereditary subalgebra \( D \) of \( A \)

\[ \inf \{\|d \mathcal{V}_g((da)^* da)\| : d \in D, \ d \geq 0, \ \|d\| = 1\} = 0. \]

Proof. Let \( t \in G^+ \setminus \{0\} \) and \( a, d \in A \) where \( d \geq 0 \). Since \( \|d(\alpha_t(1) \delta_t) d\| = \|d \alpha_t(d) \delta_t\| = \|d \alpha_t(d)\| \) and \( \|d(\alpha_t(1) \delta_t) d\| = \|\alpha_t(d) d \alpha_t(d)\| = \|d \alpha_t(d)\| \) we see that (i) \( \Leftrightarrow \) (ii). Since

\[ \|d \alpha_t(d)\|^2 = \|\alpha_t(d) a^* d \alpha_t(d)\| = \|L_t(\alpha_t(d)(da)^* da)\| = \|d L_t((da)^* da)\| \]

we get (ii) \( \Leftrightarrow \) (iii). The implication (iv) \( \Rightarrow \) (iii) is clear. Moreover, (ii) \( \Leftrightarrow \) (iii) \( \Rightarrow \) (iv) because

\[ \|d \alpha_t((da)^* da)\| = \|d \alpha_t(a^*) \alpha_t(d)\|^2. \]

\[ \square \]

**Definition 8.19.** We say that a semigroup corner system \((A, G^+, \alpha, L)\) is aperiodic if the equivalent conditions in Lemma 8.18 are satisfied. We say \((A, G^+, \alpha, L)\) is residually aperiodic if the quotient system \((A/I, G^+, \alpha^I, L^I)\) is aperiodic for every invariant ideal \( I \) for \((A, G^+, \alpha, L)\).

Now, we formulate notions of residually infiniteness and paradoxicality for corner systems.

**Definition 8.20.** Let \( a \in A^+ \setminus \{0\} \). We say that \( a \) is infinite for \((A, G^+, \alpha, L)\) if there is \( b \in A^+ \setminus \{0\} \) such that for any \( \varepsilon > 0 \) there are elements \( t_1, \ldots, t_{n+m} \in G^+ \), and \( a_{\pm k} \in a A \), \( k = 1, 2, \ldots, n + m \) such that

\[ a \approx_{\varepsilon} \sum_{k=1}^{n} \alpha_{t_k}(a^*_{-k} a_{-k}) + L_{t_k}(a^*_{-k} a_{-k}), \quad b \approx_{\varepsilon} \sum_{k=n+1}^{n+m} \alpha_{t_k}(a^*_{-k} a_{-k}) + L_{t_k}(a^*_{-k} a_{-k}), \]

\[ \|a^*_{k} a_{l}\| < \frac{\varepsilon}{\max\{n^2, m^2\}} \quad \text{for all} \ l, k = \pm 1, \ldots, \pm (n + m), \ k \neq l. \]

If the above conditions hold for \( b \) equal to \( a \) then we say that \( a \) is paradoxical for \((A, G^+, \alpha, L)\).

We say that \( a \) is residually infinite for \((A, G^+, \alpha, L)\) if for every invariant ideal \( I \) for \((A, G^+, \alpha, L)\) either \( a \in I \) or the image of \( a \) in \( A/I \) is infinite for \((A/I, G^+, \alpha^I, L^I)\).
Proposition 8.21. If \( a \in A^+ \setminus \{0\} \) is residually infinite (resp. paradoxical) for \((A, G^+, \alpha, L)\) then it is residually infinite (resp. paradoxical) for the associated Fell bundle.

Proof. Suppose that \( a \in A^+ \setminus \{0\} \) is infinite for \((A, G^+, \alpha, L)\). Let \( \varepsilon > 0 \) and choose \( t_1, ..., t_{n+m} \in G^+ \), and \( a_{\pm k} \in aA, k = 1, 2, ..., n + m \) as in Definition 8.20 but for \( \varepsilon/4 \). We use the ‘dictionary’ from Remark 8.11 and put

\[
b_k := a_k u_{t_k}, \quad b_{n+k} := a_{-k} u_{t_k}^*, \quad \text{for } k = 1, ..., n;
b_{n+k} := a_k u_{t_k}, \quad b_{n+m+k} := a_{-k} u_{t_k}^*, \quad \text{for } k = n + 1, ..., m.
\]

Clearly, \( b_i \in aB_{s_i} \), for \( i = 1, ..., 2(n + m) \), where \( s_i := t_i \), \( s_{n+i} := -t_i \) for \( i = 1, ..., n \); and \( s_{n+i} := t_i, s_{n+m+i} := -t_i \) for \( i = n + 1, ..., m \). Let \( i, j = 1, ..., 2(n + m) \). Assume that \( i \neq j \). Then \( b_i = a_k u_{t_k} \) or \( a_k u_{t_{-k}}^* \) and \( b_j = a_t u_{t_l} \) or \( b_j = a_t u_{t_{-l}}^* \). In any case \( k \neq l \), and thus by (13) we get

\[
\|b_i^* b_j\| \leq \|a_k^* a_l\| < \frac{\varepsilon}{\max\{2n^2, (2m)^2\}}.
\]

On the other hand, we have

\[
\sum_{i=1}^{2n} b_i^* b_i = \sum_{k=1}^{n} (a_k u_{t_k})^* a_k u_{t_k} + \sum_{k=1}^{n} (a_{-k} u_{t_k}^*)^* a_{-k} u_{t_k}^* = \sum_{k=1}^{n} \alpha_k (a_{-k} a_{-k}) + L_{t_k} (a_k a_k),
\]

and similarly \( \sum_{i=2n+1}^{2m} b_i^* b_i = \sum_{k=n+1}^{m} \alpha_k (a_{-k} a_{-k}) + L_{t_k} (a_k a_k) \). Thus using (32) we get

\[
a \approx \varepsilon \sum_{i=1}^{2n} b_i^* b_i, \quad b \approx \varepsilon \sum_{i=2n+1}^{2m} b_i^* b_i.
\]

Hence \( a \) is infinite for \( B = \{ B_g \}_{g \in G} \).

Replacing, in the above argument, \( b \) with \( a \) one obtains that if \( a \) is paradoxical for \((A, G^+, \alpha, L)\) then it paradoxical for \( B \). Using the fact that invariant ideals for \((A, G^+, \alpha, L)\) and \( B \)-invariant ideals coincide, see Lemma 8.13 one gets that if \( a \) is residually infinite for \((A, G^+, \alpha, L)\) then \( a \) is residually \( B \)-infinite.

Now we are ready to state the conditions implying pure infiniteness of \( A \rtimes_{\alpha, L} G^+ \).

Theorem 8.22. Suppose that \((A, G^+, \alpha, L)\) is a residually aperiodic semigroup corner system and one of the following two conditions holds

(i) \( A \) contains finitely many invariant ideals for \((A, G^+, \alpha, L)\) and every element in \( A^+ \setminus \{0\} \) is Cuntz equivalent to a residually infinite element for \((A, G^+, \alpha, L)\).

(ii) \( A \) has the ideal property and every element in \( A^+ \setminus \{0\} \) is Cuntz equivalent to a residually infinite element for \((A, G^+, \alpha, L)\).

(iii) \( A \) is of real rank zero and every non-zero projection in \( A \) is Cuntz equivalent to a residually infinite element for \((A, G^+, \alpha, L)\).

Then \( A \rtimes_{\alpha, L} G^+ \) has the ideal property and is purely infinite.

Proof. Apply Proposition 8.21 and Theorem 5.13.

Remark 8.23. The main result of [39] is a criterion of pure infiniteness of Stacey’s crossed product \( A \rtimes_{\alpha} \mathbb{N} \) of a unital separable \( C^* \)-algebra \( A \) of real rank zero by an injective endomorphism \( \alpha : A \to A \). This result can be deduced from Theorem 8.22. Indeed, the authors of [39] used the fact that \( A \rtimes_{\alpha} \mathbb{N} \) is Morita equivalent to the crossed product \( B \rtimes_{\beta} \mathbb{Z} \) by an automorphism \( \beta : B \to B \), where \( B \) is a separable \( C^* \)-algebra of real rank zero. They assumed that \( \beta \) satisfies the residual Rokhlin* property, cf. [38] Definition 2.1] and \( \alpha \) (and
therefore also $\beta$ residually contracts projections, cf. \cite[Definition 3.2]{39}. By \cite[Corollary 2.22]{18} and \cite[Theorem 10.4]{38}, the former property is equivalent to residual aperiodicity of $(B, N, \{\beta^n\}_{n \in \mathbb{N}}, \{\beta^{-n}\}_{n \in \mathbb{N}})$. The latter property readily implies that every projection in $B$ is residually infinite for $(B, N, \{\beta^n\}_{n \in \mathbb{N}}, \{\beta^{-n}\}_{n \in \mathbb{N}})$. Hence Theorem \cite[8.22(iii)]{22} applies to $(B, N, \{\beta^n\}_{n \in \mathbb{N}}, \{\beta^{-n}\}_{n \in \mathbb{N}})$.

References


