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A Brooks type theorem for the maximum local edge connectivity

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Abstract

For a graph $G$, let $\chi(G)$ and $\lambda(G)$ denote the chromatic number of $G$ and the maximum local edge connectivity of $G$, respectively. A result of Dirac [4] implies that every graph $G$ satisfies $\chi(G) \leq \lambda(G) + 1$. In this paper we characterize the graphs $G$ for which $\chi(G) = \lambda(G) + 1$. The case $\lambda(G) = 3$ was already solved by Alboulker et al. [1]. We show that a graph $G$ with $\lambda(G) = k \geq 4$ satisfies $\chi(G) = k + 1$ if and only if $G$ contains a block which can be obtained from copies of $K_{k+1}$ by repeated applications of the Hajós join.

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1 Introduction and main result

The paper deals with the classical vertex coloring problem for graphs. The term graph refers to a finite undirected graph without loops and without multiple edges. The chromatic number of a graph $G$, denoted by $\chi(G)$, is the least number of colors needed to color the vertices of $G$ such that each vertex receives a color and adjacent vertices receive different colors. There

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are several degree bounds for the chromatic number. For a graph $G$, let 
$\delta(G) = \min_{v \in V(G)} d_G(v)$ and $\Delta(G) = \max_{v \in V(G)} d_G(v)$ denote the minimum degree and the maximum degree of $G$, respectively. Furthermore, let 
$$col(G) = 1 + \max_{H \subseteq G} \delta(H)$$
denote the coloring number of $G$, and let 
$$mad(G) = \max_{\emptyset \neq H \subseteq G} \frac{2|E(H)|}{|V(H)|}$$
denote the maximum average degree of $G$. By $H \subseteq G$ we mean that $H$ is a subgraph of $G$. If $G$ is the empty graph, that is, $V(G) = \emptyset$, we briefly write $G = \emptyset$ and define $\delta(G) = \Delta(G) = mad(G) = 0$ and $col(G) = 1$. A simple sequential coloring argument shows that $\chi(G) \leq col(G)$, which implies that every graph $G$ satisfies 
$$\chi(G) \leq col(G) \leq \lfloor mad(G) \rfloor + 1 \leq \Delta(G) + 1.$$ 
These inequalities were discussed in a paper by Jensen and Toft [10]. Brooks’ famous theorem provides a characterization for the class of graphs $G$ satisfying $\chi(G) = \Delta(G) + 1$. Let $k \geq 0$ be an integer. For $k \neq 2$, let $B_k$ denote the class of complete graphs having order $k + 1$; and let $B_2$ denote the class of odd cycles. A graph in $B_k$ has maximum degree $k$ and chromatic number $k + 1$. Brooks’ theorem [2] is as follows.

**Theorem 1.1 (Brooks 1941)** Let $G$ be a non-empty graph. Then $\chi(G) \leq \Delta(G) + 1$ and equality holds if and only if $G$ has a connected component belonging to the class $B_{\Delta(G)}$.

In this paper we are interested in connectivity parameters of graphs. Let $G$ be a graph with at least two vertices. The local connectivity $\kappa_G(v, w)$ of distinct vertices $v$ and $w$ is the maximum number of internally vertex disjoint $v$-$w$ paths of $G$. The local edge connectivity $\lambda_G(v, w)$ of distinct vertices $v$ and $w$ is the maximum number of edge-disjoint $v$-$w$ paths of $G$. The maximum local connectivity of $G$ is 
$$\kappa(G) = \max \{ \kappa_G(v, w) \mid v, w \in V(G), v \neq w \},$$
and the maximum local edge connectivity of $G$ is 
$$\lambda(G) = \max \{ \lambda_G(v, w) \mid v, w \in V(G), v \neq w \}.$$
For a graph $G$ having only one vertex, we define $\kappa(G) = \lambda(G) = 0$. Clearly, the definition implies that $\kappa(G) \leq \lambda(G)$ for every graph $G$. By a result of Mader [11] it follows that $\delta(G) \leq \kappa(G)$. Since $\kappa$ is a monotone graph parameter in the sense that $H \subseteq G$ implies $\kappa(H) \leq \kappa(G)$, it follows that every graph $G$ satisfies $\text{col}(G) \leq \kappa(G) + 1$. Consequently, every graph $G$ satisfies

$$\chi(G) \leq \text{col}(G) \leq \kappa(G) + 1 \leq \lambda(G) + 1 \leq \Delta(G) + 1.$$  \hspace{1cm} (1.1)

Our aim is to characterize the class of graphs $G$ for which $\chi(G) = \lambda(G) + 1$. For such a characterization we use the fact that if we have an optimal coloring of each block of a graph $G$, then we can combine these colorings to an optimal coloring of $G$ by permuting colors in the blocks if necessary. For every non-empty graph $G$, we thus have

$$\chi(G) = \max \{\chi(H) \mid H \text{ is a block of } G\}.$$  \hspace{1cm} (1.2)

We also need a famous construction, first used by Hajós [9]. Let $G_1$ and $G_2$ be two vertex-disjoint graphs and, for $i = 1, 2$, let $e_i = v_i w_i$ be an edge of $G_i$. Let $G$ be the graph obtained from $G_1$ and $G_2$ by deleting the edges $e_1$ and $e_2$ from $G_1$ and $G_2$, respectively, identifying the vertices $v_1$ and $v_2$, and adding the new edge $w_1 w_2$. We then say that $G$ is the Hajós join of $G_1$ and $G_2$ and write $G = (G_1, v_1, w_1) \join (G_2, v_2, w_2)$ or briefly $G = G_1 \join G_2$.

For an integer $k \geq 0$ we define a class $\mathcal{H}_k$ of graphs as follows. If $k \leq 2$, then $\mathcal{H}_k = B_k$. The class $\mathcal{H}_3$ is the smallest class of graphs that contains all odd wheels and is closed under taking Hajós joins. Recall that an odd wheel is a graph obtained from an odd cycle by adding a new vertex and joining this vertex to all vertices of the cycle. If $k \geq 4$, then $\mathcal{H}_k$ is the smallest class of graphs that contains all complete graphs of order $k + 1$ and is closed under taking Hajós joins. Our main result is the following counterpart of Brooks’ theorem. In fact, Brooks’ theorem may easily be deduced from it.

**Theorem 1.2** Let $G$ be a non-empty graph. Then $\chi(G) \leq \lambda(G) + 1$ and equality holds if and only if $G$ has a block belonging to the class $\mathcal{H}_{\lambda(G)}$.

For the proof of this result, let $G$ be a non-empty graph with $\lambda(G) = k$. By (1.1), we obtain $\chi(G) \leq k + 1$. By an observation of Hajós [9] it follows that every graph in $\mathcal{H}_k$ has chromatic number $k + 1$. Hence if some block of $G$ belongs to $\mathcal{H}_k$, then (1.2) implies that $\chi(G) = k + 1$. So it only remains to
show that if \( \chi(G) = k + 1 \), then some block of \( G \) belongs to \( \mathcal{H}_k \). For proving this, we shall use the critical graph method, see [12].

A graph \( G \) is critical if every proper subgraph \( H \) of \( G \) satisfies \( \chi(H) < \chi(G) \). We shall use the following two properties of critical graphs. As an immediate consequence of (1.2) we obtain that if \( G \) is a critical graph, then \( G = \emptyset \) or \( G \) contains no separating vertex, implying that \( G \) is its only block. Furthermore, every graph contains a critical subgraph with the same chromatic number.

Let \( G \) be a non-empty graph with \( \lambda(G) = k \) and \( \chi(G) = k + 1 \). Then \( G \) contains a critical subgraph \( H \) with chromatic number \( k + 1 \), and we obtain that \( \lambda(H) \leq \lambda(G) = k \). So the proof of Theorem 1.2 is complete if we can show that \( H \) is a block of \( G \) which belongs to \( \mathcal{H}_k \). For an integer \( k \geq 0 \), let \( C_k \) denote the class of graphs \( H \) such that \( H \) is a critical graph with chromatic number \( k + 1 \) and with \( \lambda(H) \leq k \). We shall prove that the two classes \( C_k \) and \( \mathcal{H}_k \) are the same.

## 2 Connectivity of critical graphs

In this section we shall review known results about the structure of critical graphs. First we need some notation. Let \( G \) be an arbitrary graph. For an integer \( k \geq 0 \), let \( \mathcal{CO}_k(G) \) denote the set of all colorings of \( G \) with color set \( \{1, 2, \ldots, k\} \). Then a function \( f : V(G) \to \{1, 2, \ldots, k\} \) belongs to \( \mathcal{CO}_k(G) \) if and only if \( f^{-1}(c) \) is an independent vertex set of \( G \) (possibly empty) for every color \( c \in \{1, 2, \ldots, k\} \). A set \( S \subseteq V(G) \cup E(G) \) is called a separating set of \( G \) if \( G - S \) has more components than \( G \). A vertex \( v \) of \( G \) is called a separating vertex of \( G \) if \( \{v\} \) is a separating set of \( G \). An edge \( e \) of \( G \) is called a bridge of \( G \) if \( \{e\} \) is a separating set of \( G \). For a vertex set \( X \subseteq V(G) \), let \( \partial_G(X) \) denote the set of all edges of \( G \) having exactly one end in \( X \). Clearly, if \( G \) is connected and \( \emptyset \neq X \subsetneq V(G) \), then \( F = \partial_G(X) \) is a separating set of edges of \( G \). The converse is not true. However if \( F \) is a minimal separating edge set of a connected graph \( G \), then \( F = \partial_G(X) \) for some vertex set \( X \). As a consequence of Menger’s theorem about edge connectivity, we obtain that if \( v \) and \( w \) are two distinct vertices of \( G \), then

\[
\lambda_G(v, w) = \min\{|\partial_G(X)| \mid X \subseteq V(G), v \in X, w \notin X\}.
\]

Color critical graphs were first introduced and investigated by Dirac in the 1950s. He established the basic properties of critical graphs in a series of
papers [3], [4] and [5]. Some of these basic properties are listed in the next theorem.

**Theorem 2.1 (Dirac 1952)** Let $G$ be a critical graph with chromatic number $k + 1$ for an integer $k \geq 0$. Then the following statements hold:

(a) $\delta(G) \leq k$

(b) If $k = 0, 1$, then $G$ is a complete graph of order $k + 1$; and if $k = 2$, then $G$ is an odd cycle.

(c) No separating vertex set of $G$ is a clique of $G$. As a consequence, $G$ is connected and has no separating vertex, i.e., $G$ is a block.

(d) If $v$ and $w$ are two distinct vertices of $G$, then $\lambda_G(v, w) \geq k$. As a consequence $G$ is $k$-edge-connected.

Theorem 2.1(a) leads to a very natural way of classifying the vertices of a critical graph into two classes. Let $G$ be a critical graph with chromatic number $k + 1$. The vertices of $G$ having degree $k$ in $G$ are called *low vertices* of $G$, and the remaining vertices are called *high vertices* of $G$. So any high vertex of $G$ has degree at least $k + 1$ in $G$. Furthermore, let $G_L$ be the subgraph of $G$ induced by the low vertices of $G$, and let $G_H$ be the subgraph of $G$ induced by the high vertices of $G$. We call $G_L$ the *low vertex subgraph* of $G$ and $G_H$ the *high vertex subgraph* of $G$. This classification is due to Gallai [8] who proved the following theorem. Note that statements (b) and (c) of Gallai’s theorem are simple consequences of statement (a), which is an extension of Brooks’ theorem.

**Theorem 2.2 (Gallai 1963)** Let $G$ be a critical graph with chromatic number $k + 1$ for an integer $k \geq 1$. Then the following statements hold:

(a) Every block of $G_L$ is a complete graph or an odd cycle

(b) If $G_H = \emptyset$, then $G$ is a complete graph of order $k + 1$ if $k \neq 2$, and $G$ is an odd cycle if $k = 2$.

(c) If $|V(G_H)| = 1$, then either $G$ has a separating vertex set of two vertices or $k = 3$ and $G$ is an odd wheel.
As observed by Dirac, a critical graph is connected and contains no separating vertex. Dirac [3] and Gallai [8] characterized critical graphs having a separating vertex set of size two. In particular, they proved the following theorem, which shows how to decompose a critical graph having a separating vertex set of size two into smaller critical graphs.

**Theorem 2.3 (Dirac 1952 and Gallai 1963)** Let $G$ be a critical graph with chromatic number $k + 1$ for an integer $k \geq 3$, and let $S \subseteq V(G)$ be a separating vertex set of $G$ with $|S| \leq 2$. Then $S$ is an independent vertex set of $G$ consisting of two vertices, say $v$ and $w$, and $G - S$ has exactly two components $H_1$ and $H_2$. Moreover, if $G_i = G[V(H_i) \cup S]$ for $i = 1, 2$, we can adjust the notation so that for some coloring $f_1 \in CO_k(G_1)$ we have $f_1(v) = f_1(w)$. Then the following statements hold:

(a) Every coloring $f \in CO_k(G_1)$ satisfies $f(v) = f(w)$ and every coloring $f \in CO_k(G_2)$ satisfies $f(v) \neq f(w)$.

(b) The subgraph $G'_1 = G_1 + vw$ obtained from $G_1$ by adding the edge $vw$ is critical and has chromatic number $k + 1$.

(c) The vertices $v$ and $w$ have no common neighbor in $G_2$ and the subgraph $G'_2 = G_2/S$ obtained from $G_2$ by identifying $v$ and $w$ is critical and has chromatic number $k + 1$.

Dirac [6] and Gallai [8] also proved the converse theorem, that $G$ is critical and has chromatic number $k + 1$ provided that $G'_1$ is critical and has chromatic number $k + 1$ and $G_2$ obtained from the critical graph $G'_2$ with chromatic number $k + 1$ by splitting a vertex into $v$ and $w$ has chromatic number $k$.

Hajós [9] invented his construction to characterize the class of graphs with chromatic number at least $k + 1$. Another advantage of the Hajós join is the well known fact that it not only preserve the chromatic number, but also criticality. It may be viewed as a special case of the Dirac–Gallai construction, described above.

**Theorem 2.4 (Hajós 1961)** Let $G = G_1 \triangledown G_2$ be the Hajós join of two graphs $G_1$ and $G_2$, and let $k \geq 3$ be an integer. Then $G$ is critical and has chromatic number $k + 1$ if and only if both $G_1$ and $G_2$ are critical and have chromatic number $k + 1$.  

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If $G$ is the Hajós join of two graphs that are critical and have chromatic number $k + 1$, where $k \geq 3$, then $G$ is critical and has chromatic number $k+1$. Moreover, $G$ has a separating set consisting of one edge and one vertex. Theorem 2.3 implies that the converse statement also holds.

**Theorem 2.5** Let $G$ be a critical graph graph with chromatic number $k + 1$ for an integer $k \geq 3$. If $G$ has a separating set consisting of one edge and one vertex, then $G$ is the Hajós join of two graphs.

Next we will discuss a decomposition result for critical graphs having chromatic number $k + 1$ and having a separating edge set of size $k$. Let $G$ be an arbitrary graph. By an edge cut of $G$ we mean a triple $(X, Y, F)$ such that $X$ is a non-empty proper subset of $V(G)$, $Y = V(G) \setminus X$, and $F = \partial G(X) = \partial G(Y)$. If $(X, Y, F)$ is an edge cut of $G$, then we denote by $X_F$ (respectively $Y_F$) the set of vertices of $X$ (respectively, $Y$) which are incident to some edge of $F$. An edge cut $(X, Y, F)$ of $G$ is non-trivial if $|X_F| \geq 2$ and $|Y_F| \geq 2$. The following decomposition result was proved independently by T. Gallai and Toft [13].

**Theorem 2.6 (Toft 1970)** Let $G$ be a critical graph with chromatic number $k + 1$ for an integer $k \geq 3$, and let $F \subseteq E(G)$ be a separating edge set of $G$ with $|F| \leq k$. Then $|F| = k$ and there is an edge cut $(X, Y, F)$ of $G$ satisfying the following properties:

(a) Every coloring $f \in CO_k(G[X])$ satisfies $|f(X_F)| = 1$ and every coloring $f \in CO_k(G[Y])$ satisfies $|f(Y_F)| = k$.

(b) The subgraph $G_1$ obtained from $G[X \cup Y_F]$ by adding all edges between the vertices of $Y_F$, so that $Y_F$ becomes a clique of $G_1$, is critical and has chromatic number $k + 1$.

(c) The subgraph $G_2$ obtained from $G[Y]$ by adding a new vertex $v$ and joining $v$ to all vertices of $Y_F$ is critical and has chromatic number $k + 1$.

A particular nice proof of this result is due to T. Gallai (oral communication to the second author). Recall that the **clique number** of a graph $G$, denoted by $\omega(G)$, is the largest cardinality of a clique in $G$. A graph $G$ is **perfect** if every induced subgraph $H$ of $G$ satisfies $\chi(H) = \omega(H)$. For the proof of the next lemma, due to Gallai, we use the fact that complements of bipartite graphs are perfect.
Lemma 2.7 Let $H$ be a graph and let $k \geq 3$ be an integer. Suppose that $(A, B, F')$ is an edge cut of $H$ such that $|F'| \leq k$ and $A$ as well as $B$ are cliques of $H$ with $|A| = |B| = k$. If $\chi(H) \geq k + 1$, then $|F'| = k$ and $F' = \partial_H(\{v\})$ for some vertex $v$ of $H$.

Proof. The graph $H$ is perfect and so $\omega(H) = \chi(H) \geq k + 1$. Consequently, $H$ contains a clique $X$ with $|X| = k + 1$. Let $s = |A \cap X|$ and hence $k + 1 - s = |B \cap X|$. Since $|A| = |B| = k$, this implies that $s \geq 1$ and $k + 1 - s \geq 1$. Since $X$ is a clique of $H$, the set $E'$ of edges of $H$ joining a vertex of $A \cap X$ with a vertex of $B \cap X$ satisfies $E' \subseteq F'$ and $|E'| = s(k + 1 - s)$. Clearly, $g''(s) = -2$, which implies that the function $g(s) = s(k + 1 - s)$ is strictly concave on the real interval $[1, k]$. Since $g(1) = g(k) = k$, we conclude that $g(s) > k$ for all $s \in (1, k)$. Since $g(s) = |E'| \leq |F'| \leq k$, this implies that $s = 1$ or $s = k$. In both cases we obtain that $|E'| = |F'| = k$, and hence $E' = F' = \partial_H(\{v\})$ for some vertex $v$ of $H$. \hfill \Box

Based on Lemma 2.7 it is easy to give a proof of Theorem 2.6, see also the paper by Dirac, Sørensen, and Toft [7]. Theorem 2.6 is a reformulation of a result by Toft in his Ph.D thesis. Toft gave a complete characterization of the class of critical graphs, having chromatic number $k + 1$ and containing a separating edge set of size $k$. The characterization involves critical hypergraphs.

Figure 1 shows three critical graphs with $\chi = 4$. The first graph is an odd wheel and the second graph is the Hajós join of two $K_4$'s; both graphs belong to the class $C_3$. The third graph does not belong to $C_3$; it has an separating edge set of size 3, but $\lambda = 4$.

Figure 1: Three critical graphs with chromatic number $\chi = 4$. 

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3 Proof of the main result

Theorem 3.1 Let \( k \geq 0 \) be an integer. Then the two graph classes \( C_k \) and \( H_k \) coincide.

Proof. That the two classes \( C_k \) and \( H_k \) coincide if \( 0 \leq k \leq 2 \) follows from Theorem 2.1(a). In this case both classes consists of all critical graphs with chromatic number \( k + 1 \). In what follows we therefore assume that \( k \geq 3 \). The proof of the following claim is straightforward and left to the reader.

Claim 1 The odd wheels belong to the class \( C_3 \) and the complete graphs of order \( k + 1 \) belong to the class \( C_k \).

Claim 2 Let \( k \geq 3 \) be an integer, and let \( G = G_1 \triangle G_2 \) the Hajós join of two graphs \( G_1 \) and \( G_2 \). Then \( G \) belongs to the class \( C_k \) if and only if both \( G_1, G_2 \) belong to the class \( C_k \).

Proof: We may assume that \( G = (G_1, v_1, w_1) \triangle (G_2, v_2, w_2) \) and \( v \) is the vertex of \( G \) obtained by identifying \( v_1 \) and \( v_2 \). First suppose that \( G_1, G_2 \in C_k \). From Theorem 2.4 it follows that \( G \) is critical and has chromatic number \( k + 1 \). So it suffices to prove that \( \lambda(G) \leq k \). To this end let \( u \) and \( u' \) be distinct vertices of \( G \) and let \( p = \lambda_G(u, u') \). Then there is a system \( \mathcal{P} \) of \( p \) edge disjoint \( u-u' \) paths in \( G \). If \( u \) and \( u' \) belong both to \( G_1 \), then only one path \( P \) of \( \mathcal{P} \) may contain vertices not in \( G_1 \). In this case \( P \) contains the vertex \( v \) and the edge \( w_1w_2 \). If we replace in \( P \) the subpath \( vPw_1 \) by the edge \( v_1w_1 \), we obtain a system of \( p \) edge disjoint \( u-u' \) paths in \( G_1 \), and hence \( p \leq \lambda_{G_1}(u, u') \leq k \). If \( u \) and \( u' \) belong to \( G_2 \), a similar argument shows that \( p \leq k \). It remains to consider the case that one vertex, say \( u \), belongs to \( G_1 \) and the other vertex \( u' \) belongs to \( G_2 \). By symmetry we may assume that \( u \neq v \). Again at most one path \( P \) of \( \mathcal{P} \) uses the edge \( w_1w_2 \) and the remaining paths of \( \mathcal{P} \) all uses the vertex \( v(= v_1 = v_2) \). If we replace \( P \) by the path \( uPw_1 + w_1v_1 \), then we obtain \( p \) edge disjoint \( u-v_1 \) path in \( G_1 \), and hence \( p \leq \lambda_{G_1}(u, v_1) \leq k \). This shows that \( \lambda(G) \leq k \) and so \( G \in C_k \).

Suppose conversely that \( G \in C_k \). From Theorem 2.4 it follows that \( G_1 \) and \( G_1 \) are critical graphs, both with chromatic number \( k + 1 \). So it suffices to show that \( \lambda(G_i) \leq k \) for \( i = 1, 2 \). By symmetry it suffices to show that \( \lambda(G_1) \leq k \). To this end let \( u \) and \( u' \) be distinct vertices of \( G_1 \) and let \( p = \lambda_{G_1}(u, u') \). Then there is a system \( \mathcal{P} \) of \( p \) edge disjoint \( u-u' \) paths in \( G_1 \). At most one path \( P \) of \( \mathcal{P} \) can contain the edge \( v_1w_1 \). Clearly, there is a
Claim 3 Let \( k \geq 3 \) be an integer. Then the class \( \mathcal{H}_k \) is a subclass of \( \mathcal{C}_k \).

Claim 4 Let \( k \geq 3 \) be an integer, and let \( G \) be a graph belonging to the class \( \mathcal{C}_k \). If \( G \) is 3-connected, then either \( k = 3 \) and \( G \) is an odd wheel, or \( k \geq 4 \) and \( G \) is a complete graph of order \( k + 1 \).

**Proof:** The proof is by contradiction, where we consider a counterexample \( G \) whose order \(|G|\) is minimum. Then \( G \in \mathcal{C}_k \) is a 3-connected graph, and either \( k = 3 \) and \( G \) is not an odd wheel, or \( k \geq 4 \) and \( G \) is not a complete graph of order \( k + 1 \). First we claim that \(|G_H| \geq 2\). If \( G_H = \emptyset \), then Theorem 2.2(b) implies that \( G \) is a complete graph of order \( k + 1 \), a contradiction. If \(|G_H| = 1\), then Theorem 2.2(c) implies that \( k = 3 \) and \( G \) is an odd wheel, a contradiction. This proves the claim that \(|G_H| \geq 2\). Then let \( u \) and \( v \) be distinct high vertices of \( G \). Since \( G \in \mathcal{C}_k \), Theorem 2.1(d) implies that \( \lambda_G(u, v) = k \) and, therefore, \( G \) contains a separating edge set \( F \) of size \( k \) which separates \( u \) and \( v \). From Theorem 2.6 it then follows that there is an edge cut \((X, Y, F)\) satisfying the three properties of that theorem. Since \( F \) separates \( u \) and \( v \), we may assume that \( u \in X \) and \( v \in Y \). By Theorem 2.6(a), \(|Y_F| = k \) and hence each vertex of \( Y_F \) is incident to exactly one edge of \( F \). Since \( Y \) contains the high vertex \( v \), we conclude that \(|Y_F| < |Y|\). Now we consider the graph \( G' \) obtained from \( G[X \cup Y_F] \) by adding all edges between the vertices of \( Y_F \), so that \( Y_F \) becomes a clique of \( G' \). By Theorem 2.6(b), \( G' \) is a critical graph with chromatic number \( k + 1 \). Clearly, every vertex of \( Y_F \) is a low vertex of \( G \) and every vertex of \( X \) has in \( G' \) the same degree as in \( G \). Since \( X \) contains the high vertex \( u \) of \( G \), this implies that \(|X_F| < |X|\). Since \( G \) is 3-connected, we conclude that \(|X_F| \geq 3\) and that \( G' \) is 3-connected.

Now we claim that \( \lambda(G') \leq k \). To prove this, let \( x \) and \( y \) be distinct vertices of \( G' \). If \( x \) or \( y \) is a low vertex of \( G' \), then \( \lambda_{G'}(x, y) \leq k \) and there is nothing to prove. So assume that both \( x \) and \( y \) are high vertices of \( G' \). Then both vertices \( x \) and \( y \) belong to \( X \). Let \( p = \lambda_{G'}(x, y) \) and let \( \mathcal{P} \) be a system of \( p \) edge disjoint \( x \)-\( y \) paths in \( G' \). We may choose \( \mathcal{P} \) such that the number
of edges in \( \mathcal{P} \) is minimum. Let \( \mathcal{P}_1 \) be the paths in \( \mathcal{P} \) which uses edges of \( F \). Since \( |Y_F| = k \) and each vertex of \( Y_F \) is incident with exactly one edge of \( F \), this implies that each path \( P \) in \( \mathcal{P}_1 \) contains exactly two edges of \( F \). Since \( |X_F| < |X| \) and \( |Y_F| < |Y| \), there are vertices \( u' \in X \setminus X_F \) and \( v' \in Y \setminus Y_F \). By Theorem 2.1(d) it follows that \( \lambda_{G'}(u', v') = k \) and, therefore, there are \( k \) edge disjoint \( u'-v' \) paths in \( G \). Since \( |Y_F| = k \), for each vertex \( z \in Y_F \), there is a \( v'-z \) path \( P_z \) in \( G' \) such that these paths are edge disjoint. Now let \( P \) be an arbitrary path in \( \mathcal{P}_1 \). Then \( P \) contains exactly two vertices of \( Y_F \), say \( z \) and \( z' \), and we can replace the edge \( zz' \) of the path \( P \) by a \( z-z' \) path contained in \( P_z \cup P_{z'} \). In this way we obtain a system of \( p \) edge disjoint \( u'-v' \) paths in \( G \), which implies that \( p \leq \lambda_{G'}(x, y) \leq k \). This proves the claim that \( \lambda_{G'} \leq k \). Consequently \( G' \in C_k \). Clearly, \( |G'| < |G| \) and either \( k = 3 \) and \( G' \) is not an odd wheel, or \( k \geq 4 \) and \( G \) is not a complete graph of order \( k + 1 \). This, however, is a contradiction to the choice of \( G \). Thus the claim is proved.

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**Claim 5** Let \( k \geq 3 \) be an integer, and let \( G \) be a graph belonging to the class \( C_k \). If \( G \) has a separating vertex set of size 2, then \( G = G_1 \triangle G_2 \) is the Hajós sum of two graphs \( G_1 \) and \( G_2 \), which both belong to \( C_k \).

**Proof:** If \( G \) has a separating set consisting of one edge and one vertex, then Theorem 2.1 implies that \( G \) is the Hajós join of two graphs \( G_1 \) and \( G_2 \). By Claim 2 it then follows that both \( G_1 \) and \( G_2 \) belong to \( C_k \) and we are done. It remains to consider the case that \( G \) does not contain a separating set consisting of one edge and one vertex. By assumption, there is a separating vertex set of size 2, say \( S = \{u, v\} \). Then Theorem 2.3 implies that \( G - S \) has exactly two components \( H_1 \) and \( H_2 \) such that the graphs \( G_i = G[V(H_i) \cup S] \) with \( i = 1, 2 \) satisfies the three properties of that theorem. In particular, we have that \( G'_1 = G_1 + uv \) is critical and has chromatic number \( k \). By Theorem 2.1(c), it then follows that \( \lambda_{G'_1}(u, v) \geq k \) implying that \( \lambda_{G_2}(u, v) \geq k - 1 \). Since \( G \in C_k \), we then conclude that \( \lambda_{G_2}(u, v) \leq 1 \). Since \( G_2 \) is connected, this implies that \( G_2 \) has a bridge \( e \). Since \( k \geq 3 \), we conclude that \( \{u, e\} \) or \( \{v, e\} \) is a separating set of \( G \), a contradiction.

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As a consequence of Claim 4 and Claim 5 we conclude that the class \( C_k \) is a subclass of the class \( \mathcal{H}_k \). Together with Claim 3 this yields \( \mathcal{H}_k = C_k \) as wanted.
Proof of Theorem 1.2: For the proof of this theorem let $G$ be a non-empty graph with $\lambda(G) = k$. By (1.1) we obtain that $\chi(G) \leq k + 1$. If one block $H$ of $G$ belongs to $\mathcal{H}_k$, then $H \in \mathcal{C}_k$ (by Theorem 3.1) and hence $\chi(G) = k + 1$ (by (1.2)).

Assume conversely that $\chi(G) = k + 1$. Then $G$ contains a subgraph $H$ which is critical and has chromatic number $k + 1$. Clearly, $\lambda(H) \leq \lambda(G) \leq k$, and, therefore, $H \in \mathcal{C}_k$. By Theorem 2.1(b), $H$ contains no separating vertex. We claim that $H$ is a block of $G$. For otherwise, $H$ would be a proper subgraph of a block $G'$ of $G$. This implies that there are distinct vertices $u$ and $v$ in $H$ which are joined by a path $P$ of $G$ with $E(P) \cap E(H) = \emptyset$. Since $\lambda_H(u, v) \geq k$ (by Theorem 2.1(c)), this implies that $\lambda_G(u, v) \geq k + 1$, which is impossible. This proves the claim that $H$ is a block of $G$. By Theorem 3.1, $\mathcal{C}_k = \mathcal{H}_k$ implying that $H \in \mathcal{H}_k$. This completes the proof of the theorem $\blacksquare$.

The case $\lambda = 3$ of Theorem 1.2 was obtained earlier by Alboulker et al. [1]; their proof is similar to our proof. Let $\mathcal{L}_k$ denote the class of graphs $G$ satisfying $\lambda(G) \leq k$. It is well known that membership in $\mathcal{L}_k$ can be tested in polynomial time. It is also easy to show that there is a polynomial-time algorithm that, given a graph $G \in \mathcal{L}_k$, decides whether $G$ or one of its blocks belong to $\mathcal{H}_k$. So it can be tested in polynomial time whether a graph $G \in \mathcal{L}_k$ satisfies $\chi(G) \leq k$. Moreover, the proof of Theorem 1.2 yields a polynomial-time algorithm that, given a graph $G \in \mathcal{L}_k$, finds a coloring of $\mathcal{CO}_k(G)$ when such a coloring exists. This result provides a positive answer to a conjecture made by Alboulker et al. [1, Conjecture 1.8]. The case $k = 3$ was solved by Alboulker et al. [1].

**Theorem 3.2** For fixed $k \geq 1$, there is a polynomial-time algorithm that, given a graph $G \in \mathcal{L}_k$, finds a coloring in $\mathcal{CO}_k(G)$ or a block belonging to $\mathcal{H}_k$.

**Sketch of Proof:** The Theorem is evident if $k = 1, 2$; and the case $k = 3$ was solved by Alboulker et al. [1]. Hence we assume that $k \geq 4$ and $G \in \mathcal{L}_k$. If we find for each block $H$ of $G$ a coloring in $\mathcal{CO}_k(H)$, we can piece these colorings together by permuting colors to obtain a coloring in $\mathcal{CO}_k(G)$. Hence we may assume that $G$ is a block. First, we check whether $G$ has a separating set $S$ consisting of one vertex and one edge. If we find such a set, say $S = \{v, e\}$ with $v \in V(G)$ and $e \in E(G)$. Then $G - e$ is the union of two connected graphs $G_1$ and $G_2$ having only vertex $v$ in common where $e = w_1w_2$ and $w_i \in V(G_i)$ for $i = 1, 2$. Both blocks $G'_1 = G_1 + vw_1$ and $G'_2 = G_2 + vw_2$
belong to $\mathcal{L}_k$. Now we check whether these blocks belong to $\mathcal{H}_k$. If both blocks $G_1'$ and $G_2'$ belong to $\mathcal{H}_k$, then $vw_i \notin E(G_i)$ for $i = 1, 2$, and hence $G$ belongs to $\mathcal{H}_k$ and we are done. If one of the blocks, say $G_1'$ does not belong to $\mathcal{H}_k$, we can construct a coloring $f_1 \in \mathcal{CO}_{k}(G_1')$. Moreover, no block of $G_2$ belongs to $\mathcal{H}_k$, hence we can construct a coloring $f_2 \in \mathcal{CO}_{k}(G_2)$. Then $f_1 \in \mathcal{CO}_{k}(G_1)$ and $f_1(v) \neq f_1(w_1)$. Since $k \geq 4$, we can permute colors in $f_2$ such that $f_1(v) = f_2(v)$ and $f_1(w_1) \neq f_2(w_2)$. Consequently, $f = f_1 \cup f_2$ belongs to $\mathcal{CO}_{k}(G)$ and we are done.

It remains to consider the case that $G$ contains no separating set consisting of one vertex and one edge. Then let $p$ denote the number of vertices of $G$ whose degree is greater than $k$. If $p \leq 1$, then let $v$ be a vertex of maximum degree in $G$. Color $v$ with color 1 and let $L$ be a list assignment for $H = G - v$ satisfying $L(u) = \{2, 3, \ldots, k\}$ if $vu \in E(G)$ and $L(u) = \{1, 2, \ldots, k\}$ otherwise. Then $H$ is connected and $|L(u)| \geq d_H(u)$ for all $u \in V(H)$. Now we can use the degree version of Brooks’ theorem, see [12, Theorem 2.1]. Either we find a coloring $f$ of $H$ such that $f(u) \in L(u)$ for all $u \in V(H)$, yielding a coloring of $\mathcal{CO}_{k}(G)$, or $|L(u)| = d_H(u)$ for all $u \in V(H)$ and each block of $H$ is a complete graph or an odd cycle. In this case, $d_H(u) \in \{k, k-1\}$ for all $u \in V(H)$ and, since $k \geq 4$, each block of $H$ is a $K_k$ or a $K_2$. Since $G$ contains no separating set consisting of one vertex and one edge, this implies that $H = K_k$ and so $G = K_{k+1} \in \mathcal{H}_k$ and we are done. If $p \geq 2$, then we choose two vertices $u$ and $u'$ whose degrees are greater that $k$. Then we construct an edge cut $(X, Y, F)$ with $u \in X$, $u' \in Y$, and $|F| = \lambda_G(u, u')$. We may assume that $a = |X_F|$ and $b = |Y_F|$ satisfies $a \leq b \leq k$. If $b \leq k - 1$, then both graphs $G[X]$ and $G[Y]$ belong to $\mathcal{L}_k$ and there are colorings $f_X \in \mathcal{CO}_{k}(G[X])$ and $f_Y \in \mathcal{CO}_{k}(G[Y])$. Note that no block of these two graphs can belong to $\mathcal{H}_k$. By permuting colors in $f_Y$, we can combine the two colorings $f_X$ and $f_Y$ to obtain a coloring $f \in \mathcal{CO}_{k}(G)$ (by Lemma 2.7). If $a < b = k$, then we consider the graph $G_1$ obtained from $G[X \cup Y_F]$ by adding all edges between the vertices of $Y_F$, so that $Y_F$ becomes a clique of $G_1$. Then $G_1$ belongs to $\mathcal{L}_k$ (see the proof of Claim 4) and, since $G$ contains no separating set consisting of one vertex and one edge, the block $G_1$ does not belong to $\mathcal{H}_k$. Hence there are colorings $f_1 \in \mathcal{CO}_{k}(G_1)$ and $f_Y \in \mathcal{CO}_{k}(G[Y])$. Then the restriction of $f_1$ to $X$ yields a coloring $f_X \in \mathcal{CO}_{k}(G[X])$ such that $|f_X(X)| \geq 2$. By permuting colors in $f_Y$, we can combine the two colorings $f_X$ and $f_Y$ to obtain a coloring $f \in \mathcal{CO}_{k}(G)$ (by Lemma 2.7). It remains to consider the case $a = b = k$. Then let $G_2$ be the graph obtained from $G[Y \cup X_F]$ by adding all edges between the vertices of $X_F$, so that $X_F$ becomes a clique.
of $G_2$. Then we find colorings $f_1 \in CO_k(G_1)$ and $f_2 \in CO_k(G_2)$ and, hence, colorings $f_X \in CO_k(G[X])$ and $f_Y \in CO_k(G[Y])$ such that $|f_X(X)| \geq 2$ and $|f_Y(Y)| \geq 2$. By permuting colors in $f_Y$, we can combine the two colorings $f_X$ and $f_Y$ to obtain a coloring $f \in CO_k(G)$ (by Lemma 2.7). 

References


